SYNCHRONIZATION OF COMPLEX NETWORKS WITH NEGATIVE COUPLINGS

Gualberto Solís-Perales
Electronic Department, CUCEI
University of Guadalajara
Guadalajara, México
gualberto.solis@cucei.udg.mx

José L. Zapata
Electronic Department, CUCEI
University of Guadalajara
Guadalajara, México
zapata131@gmail.com

Abstract
In this contribution the synchronization of complex networks with some negative couplings is presented. Generally, a fundamental condition for stability and synchronization of complex networks is that the off-diagonal elements in the connectivity matrix are non-negative. The aim of this contribution is to show by means of a numerical example that the non-negativity of such elements is not a necessary condition for stability of the synchronized regime. We consider a network with constant couplings but with different values and some of them are negative and we propose a particular class of connectivity matrix.

Key words
Negative Couplings, Network synchronization; Synchronizability condition.

1 Introduction

Complex networks are objects that provide information of the collectivity of many dynamical systems. These objects have been widely studied during the last two decades. They can be found in many physical and biological since electrical to neuron in brain networks (for a wide review see [Albert and Barabási, 2002], [Wang and Chen, (203)] and [Boccaletti et.,al., (2006)] and references therein). Dynamical networks pose many challenges extending from the interplay among their dynamical and structural components, with problems ranging from establishing models that capture their key topological features, to determining the stability of their collective behavior. These networks present many challenges, for example, the problem of dynamic networks, that is, the network structure is changing over time and the size of the network can be static or not. The problem of consider nonlinear couplings implies that, the coupling force between nodes is described by a nonlinear relationship of states or it is time-varying; the problem of synchronization of networks with non-identical nodes means that systems in the nodes are strictly different [Strogatz, (2001)]. The problem here studied does not relies in those previously described, however, it is the problem of consider coupling in a general way, this is, that the couplings between nodes are given by constants and even more, some of them are negative. In this sense, many reported results consider that the off-diagonal elements are non negative for stability and synchronization ends, [Belykh, Belykh and Hasler, (2006)], [Belykh, Belykh and Hasler, (2004)], [Wang, Lu and Shi, (2011)] .

The consideration of more general couplings make sense in nonlinear and time varying couplings, since can be considered negative and positive influences between nodes. The previous asseverating can be found in cooperative behaviors in the brain, when the coupling between neurons is still a paradigm. Considering negative couplings a more general networks can be studied.

The fundamental part of our proposal is based on a modification of the summation of the off-elements of the coupling matrix permitting to have negative couplings however, the row sum is still zero. Therefore we provide a relaxed condition on the connectivity and couplings of the network for stability and synchronization.

2 Network description with negative couplings

Let us consider networks of n identical coupled oscillators, with describing equation given by

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{N} \mathcal{C}_{ij}\Gamma x_j \tag{1}$$

where x_i are the state variables of the i-th oscillator on the network; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field representing the dynamics of the i-th system. The inner coupling matrix is of the form $\Gamma = I_{n\times n}$. Respect to the topology of the network it is given by a matrix of the couplings, and it has the following characteristics

(i) The coupling matrix is zero row-sums, $\sum_{j=1}^{N} \mathcal{C}_{i,j} = 0$.

(ii) The sum $\sum_{j=1}^{N} \mathcal{C}_{i,j} \geq 0$ with $j \neq i$ positive sum of the off-diagonal elements.

The characteristic (ii) is the fundamental contribution of the present paper, since it considers that the some couplings between nodes can be negative, moreover, note that if this summation is negative impliyes that there is a positive eigenvalue of the coupling matrix.

In general terms the coupling matrix can be or not symmetric it is only required that it satisfy (i) and (ii). Regardless, these characteristics of the coupling matrix, the strength between two nodes are in general different even more for instance $\mathcal{C}_{i,j} = -\beta\mathcal{C}_{j,i}$ for some real β.

3 Synchronization of networks with negative couplings

Now let us depart from the previous characteristics of the network, now let us consider a network of Lorenz oscillators given by

$$\dot{x}_{1,i} = \sigma_i(x_{2,i} - x_{1,i}) + \sum_{j=1}^{N} \mathcal{C}_{ij}\Gamma x_{1,j} \quad (2)$$

$$\dot{x}_{2,i} = \rho_i x_{1,i} - x_{2,i} - x_{1,i}x_{3,i} + \sum_{j=1}^{N} \mathcal{C}_{ij}\Gamma x_{2,j} \quad (3)$$

$$\dot{x}_{3,i} = -\beta_i x_{3,i} - x_{1,i}x_{3,i} + \sum_{j=1}^{N} \mathcal{C}_{ij}\Gamma x_{3,j} \quad (4)$$

$$y = x_i \quad (5)$$

where σ_i, ρ_i and β_i are the systems parameters and are chosen in such way that the three systems present chaotic behavior for some set of initial conditions $\mathcal{U}_\rangle$. The objective is to reach the synchronization manifold is $x = x_1 = x_2 = ... = x_N$. Now we consider three cases

(1) more negative couplings than positive ones at least in one row with $\sum_{j=1;j\neq i}^{N} \mathcal{C}_{i,j} < 0$

(2) equal number of negative than positives couplings at least in one row with $\sum_{j=1;j\neq i}^{N} \mathcal{C}_{i,j} = 0$

(3) more positive than negative couplings at least in one row with $\sum_{j=1;j\neq i}^{N} \mathcal{C}_{i,j} > 0$.

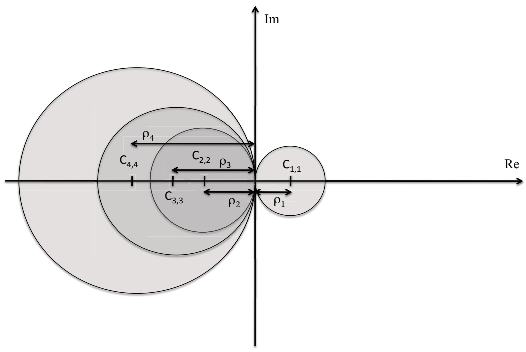

Figure 1. Gerschgoring's circles for case (1).

for all cases we have $\mathcal{C}_{i,i} = -\sum_{j=1;j\neq i}^{N} \mathcal{C}_{i,j}$. It is known that for stability of the network the eigenvalues of the connection matrix or in this case the coupling matrix $\lambda_1 = 0$ and $\lambda_k < 0$ for $k = 2, ...N$.

Now using the Gerschgoring's circles it is easy to check the position of the eigenvalues of the coupling matrix. For case (1) implies that $\mathcal{C}_{i,i} > 0$ for some i, and there is at least one circle on the right hand side of the complex plane as it is shown in Figure 1. From Figure 2 there is a circle centered at the origin and some eigenvalue may lie in the right hand complex plane.

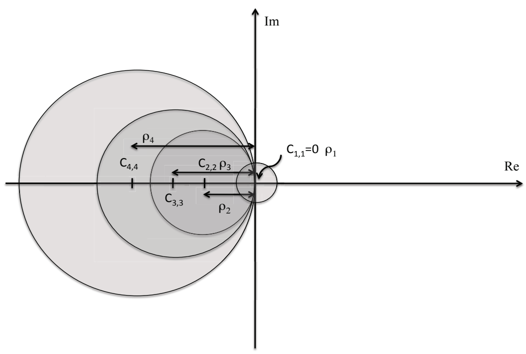

Figure 2. Gerschgoring's circles for case (2).

For the case (3) there is at least one circle centered at the left hand complex plane with radius greater

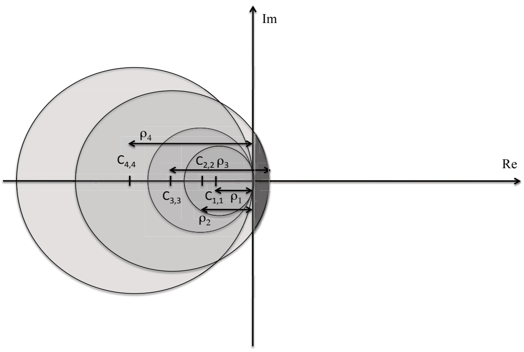

Figure 3. Gerschgoring's circles for case (3).

than the center, this is $|\mathcal{C}_{i,i}| < \sum_{j=1;j\neq i}^{N} |\mathcal{C}_{i,j}|$ therefore there is a region on the right hand complex plane that may contain one eigenvalue as can be seen in Figure 3. It is clear that depending on how negative are the couplings the region is bigger and is more probable that one eigenvalue lies in such a region. Moreover, it can be considered two possible problems, one is when the coupling matrix is symmetric and the other is when the matrix is not symmetric. For simplicity we restrict ourself to the first case. However, when negative couplings are considered the condition for synchronizability given by $c \geq \mid \frac{\bar{d}}{\lambda_2} \mid$ no longer apply as it is, however, it does not implies that the network be unstable or do not reach the synchronization manifold but it means that such a condition can be relaxed and still obtain synchronization, therefore, the condition for synchronization and stability have to be determine in a different way.

A conjecture can be described, the stability and synchronization still depends on the Lyapunov exponent of the dynamical systems $\bar{d}$, but there is not a single coupling factor, and now the eigenvalue λ_2 depends on the couplings and the topology of the network at the same time. Therefore our conjecture is that the synchronizabillity now depends on $\mid \tilde{\lambda}_2 \mid \geq \mid \bar{d} \mid$ where now $\tilde{\lambda}_2$ is the second eigenvalue of the connectivity matrix including the couplings between networks. As can be seen, this condition is almost the same than the usual condition but the eigenvalue now depends only on the coupling matrix.

4 Numerical Example

In this section we consider a class of a coupling matrix that have one zero eigenvalue and the others are negative. The matrix have some negative couplings between nodes, this is, some entries of the matrix are negative. In order to show the effects of the negative couplings we consider the Lorenz system oscillators, and we consider a network with N=3. The coupling matrix is as follows

$$\mathcal{C} = \begin{bmatrix} -a & n+a & -n \\ n+a & -(n^2+(a+1)n+a) & (n+a)n \\ -n & (n+a)n & -(n^2+(a-1)n) \end{bmatrix} \tag{6}$$

where $n > 0$ and $a > 1$, this matrix has the following eigenvalues

$$\lambda_1 = 0 \tag{7}$$

$$\lambda_2 = -a - an - \beta \tag{8}$$

$$\lambda_3 = -a - an + \beta \tag{9}$$

with the common factor

$$\beta = \sqrt{a^2(n^2-n+1) + a(2n^3-n^2+3n) + n^4 + 3n^2} \tag{10}$$

from where it is clear that λ_2 and λ_3 are always negative. Note that this matrix define a general class of 3×3 coupling matrix with some negative couplings that satisfies the zero sum in the rows and columns.

For a simple example we consider $a = 5$ and $n = 1$ from where the coupling weighted matrix is given by

$$\mathcal{C} = \begin{bmatrix} -5 & 6 & -1 \\ 6 & -12 & 6 \\ -1 & 6 & -5 \end{bmatrix} \tag{11}$$

with eigenvalues $\Lambda(\mathcal{C}) = diag[0, -4, -18]$ note that for this matrix there is two circles centered at -5 of ratio $r = 7$ and one centered at -12 of ratio $r = 12$ and it is clear that we are in case (3) where there are more positives couplings that negtives. The matrix defines a weigthed network as illustrated in Figure 4. in this network there is a negative weight between two nodes however, the weights can be changed following (6), this is the negative coupling $c_{1,3} = -1$ can take other negative values greater that 1 or can be fixed.

For this case we consider the Lorenz systems with parameters $\sigma = 10$, $\rho = 28$ and $\beta = -8/3$ and the network was coupled at $t = 50$ and the synchronization is achieved and the systems are stable as can be observed in Figure 5. The synchronization is obtained in spite of some of the couplings are negative, the network was connected at $t = 10$. On the other hand, this indicates that the closed loop system is stable and also indicates that from the Gerschgorin circles the eigenvalues are on the left half complex plane. It is important to stress that the condition for synchronization given in the conjecture is not demonstrated, therefore the results in this contribution is still under study, however it is important to show that synchronization of complex network with negative couplings can be achieved and can be used to explain many physical phenomena.

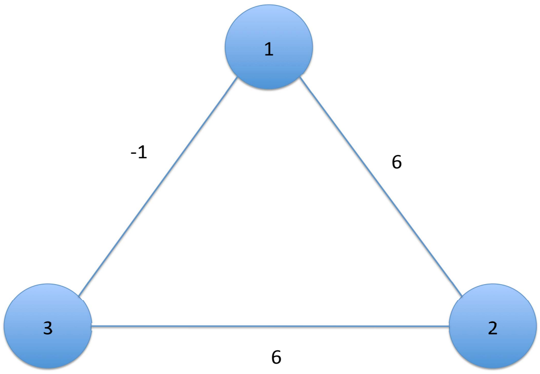

Figure 4. Network with negative couplings.

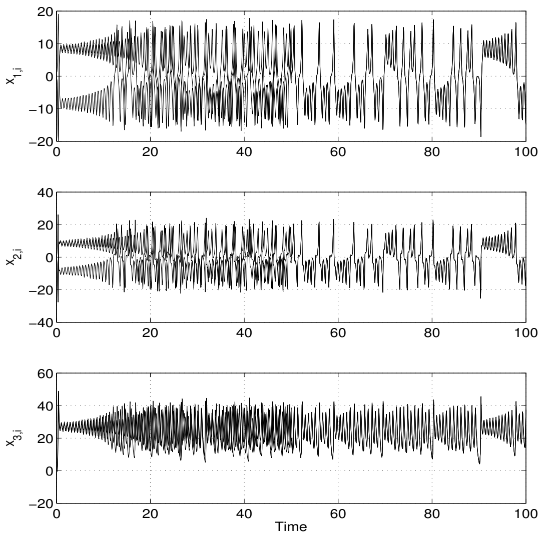

Figure 5. Synchronization of systems in the network with negative couplings.

5 Conclusion

In this contribution we present the problem of synchronization of complex networks with negative couplings. The main idea was to present the problem and a possible conjecture for the study of the synchronization and stability. The condition is mainly based on the commparation between the eigenvalues of both the systems in the network and the second greatest eigenvalue of the coupling matrix. In the case of negative couplings the coupling matrix is or not symmetric and the analysis as usual no longer applies. On the other hand, we have presented a more general problem in terms of the couplings, since negative feedback can be given between nodes. This proposal can be used to explain more generally some physical phenomena, nevertheless, the condition here launched is still under study and a formal demonstration is required and will be publish shortly.

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