Stochastic Chaos and its control in a stochastic Duffing-van der Pol system

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Abstract

The stochastic Double-Well Duffing-van der Pol (SDVP for short) system with bounded random parameter is first transformed into an equivalent deterministic system by the Chebyshev polynomial approximation method, and then the stochastic chaos and its control by noise are investigated. Therefore the problem of controlling stochastic chaos in the SDVP system can be reduced into the problem of controlling deterministic chaos in the equivalent system. The numerical simulations show that the chaos behavior in the SDVP system is by and large similar to that in the equivalent deterministic Duffing-Van der Pol (DVP for short) system. The chaos behavior can be controlled to the steady states by noise, and with the effect of the random parameter and its intensity, there are still some features.

Key words: Chebyshev orthogonal polynomial, stochastic Double-Well Duffing-Van der Pol system, stochastic chaos, chaos control, white noise

1. **Introduction**

The DVP system is a famous nonlinear dynamical system as follow

$$\ddot{x} - \mu(1 - x^2)\dot{x} - \alpha x + \beta x^3 = f\cos\omega t \tag{1}$$

which widely exists in the fields of physics, engineering and biology [1, 2]. And the deterministic DVP systems or the deterministic DVP system under random excitations have already been studied deeply. There are plenty of dynamical behaviors have been found, such as symmetry breaking bifurcation, period-doubling bifurcation, Neimark bifurcation, chaos and so on [3, 4]. However, most of the results are come from the DVP system with deterministic parameters only. The aim of this paper is to explore the stochastic chaos phenomena and its control of SDVP system with random parameter, comparing with the correlative phenomena in deterministic DVP system.

To investigate the stochastic structural dynamical system, there are three basic numerical methods: Monte Carlo method, stochastic perturbation method and orthogonal polynomial approximation method. The third method, which is a useful analytic method [5-7], was introduced in [8, 9] and further developed by Li [10]. Recently, stochastic bifurcation and chaos in some typical dynamic systems are successfully analyzed by using the Chebyshev orthogonal polynomial approximation method [11-13]. In this work, the same strategy is used to explore the SDVP system.

For the stochastic chaos control of SDVP system here, the noise-aided control method [14] has been used, which is a very important control strategy and has already been applied to many fields, such as the social systems, the economical systems, and neural network systems.

2. Chebyshev polynomial approximation for SDVP system

2.1 Chebyshev orthogonal polynomial

If the random parameter follows arch-like distribution [9], the expression of its probability density function (PDF) is

$$p(\xi) = \begin{cases} (2/\pi)\sqrt{1-\xi^2} & |\xi| \le 1, \\ 0 & |\xi| > 1. \end{cases} \tag{2}$$

In the orthogonal polynomial approximation method, choosing the orthogonal polynomial basis depends on the kinds of PDF of the random parameter. As the orthogonal polynomial basis for the arch-like PDF, the only choice is the second

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Chebyshev orthogonal polynomials which can be put as

$$H_n(\xi) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!} (2\xi)^{n-2k} \tag{3}$$

The recurrent formula for the second Chebyshev orthogonal polynomials is

$$\xi H_n(\xi) = \frac{1}{2}[H_{n-1}(\xi) + H_{n+1}(\xi)] \tag{4}$$

The orthogonality of the second Chebyshev orthogonal polynomials can be expressed as

$$\int_{-1}^{1} \frac{2}{\pi}\sqrt{1-\xi^2} H_i(\xi) H_j(\xi) d\xi = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{5}$$

Owing to the orthogonality of Chebyshev orthogonal polynomials, any measurable function $f(\xi) \subset L^2$ can be expanded as $f(\xi) = \sum_{i=0}^{\infty} c_i H_i(\xi)$ where $c_i = \int_{-1}^{1} p(\xi) f(\xi) H_i(\xi) d\xi$. Only a finite number of items can be taken in this series in practice, and then the sum $f(\xi) = \sum_{i=0}^{N} c_i H_i(\xi)$ is merely an approximation with a minimum mean square residual [15].

2.2 Chebyshev orthogonal polynomial approximation for SDVP system

Consider the DVP system (1) with a random parameter μ, $\mu = \overline{\mu} + \sigma\xi$. Where $\overline{\mu}$ and σ are the mean value and the standard deviation of μ, ξ is taken as a random variable on [−1, 1] with an arch-like PDF, so the SDVP system with bounded random parameter can be expressed as follows

$$\ddot{x} - (\overline{\mu} + \sigma\xi)(1 - x^2)\dot{x} - \alpha x + \beta x^3 = f\cos\omega t \tag{8}$$

where α, β are constants, the response of system (8) is a function of time t and random variable ξ, namely $x = x(t,\xi)$ which can be expressed by the following series according to the orthogonal polynomial approximation method

$$x(t,\xi) = \sum_{i=0}^{N} x_i(t) H_i(\xi) \tag{10}$$

where the subscript i runs for the ordinal number of Chebyshev orthogonal polynomials, N represents the largest order of the polynomials we have taken. It is worth noting again that only if $N \to \infty$, $\sum_{i=0}^{N} x_i(t) H_i(\xi)$ are strictly equivalent to the responses of the SDVP system. In this paper we take $N = 4$, and then the expression (10) is the approximate solution with a minimal mean square residual error. Substituting expression (10) into (8), and using both of the recurrent formula and the orthogonal relationship of Chebyshev orthogonal polynomials, we finally get the equivalent nonlinear deterministic DVP system (13) of SDVP system [5]. The ensemble mean response of the stochastic DVP system can be obtained as (14); When $\xi \equiv 0$ or $\sigma = 0$, system (13) is the mean parameter system and its response can be expressed as (15); and SDVP system (8) is simplified as a deterministic DVP system as (16).

$$\begin{cases} \ddot{x}_0(t) - \overline{\mu}(\dot{x}_0(t) - X_{10}(t)) - \frac{\sigma}{2}(\dot{x}_1(t) - X_{11}(t)) - \alpha x_0(t) + \beta X_0(t) = f\cos\omega t \\ \ddot{x}_1(t) - \overline{\mu}(\dot{x}_1(t) - X_{11}(t)) - \frac{\sigma}{2}[(\dot{x}_2(t) + \dot{x}_0(t)) - (X_{12}(t) + X_{10}(t))] - \alpha x_1(t) + \beta X_1(t) = 0 \\ \ddot{x}_2(t) - \overline{\mu}(\dot{x}_2(t) - X_{12}(t)) - \frac{\sigma}{2}[(\dot{x}_3(t) + \dot{x}_1(t)) - (X_{13}(t) + X_{11}(t))] - \alpha x_2(t) + \beta X_2(t) = 0 \\ \ddot{x}_3(t) - \overline{\mu}(\dot{x}_3(t) - X_{13}(t)) - \frac{\sigma}{2}[(\dot{x}_4(t) + \dot{x}_2(t)) - (X_{14}(t) + X_{12}(t))] - \alpha x_3(t) + \beta X_3(t) = 0 \\ \ddot{x}_4(t) - \overline{\mu}(\dot{x}_4(t) - X_{14}(t)) - \frac{\sigma}{2}[(\dot{x}_5(t) + \dot{x}_3(t)) - (X_{15}(t) + X_{13}(t))] - \alpha x_4(t) + \beta X_4(t) = 0 \end{cases} \tag{13}$$

$$E[x(t,\xi)] = \sum_{i=0}^{4} x_i(t) E[H_i(\xi)] = x_0(t) \tag{14}$$

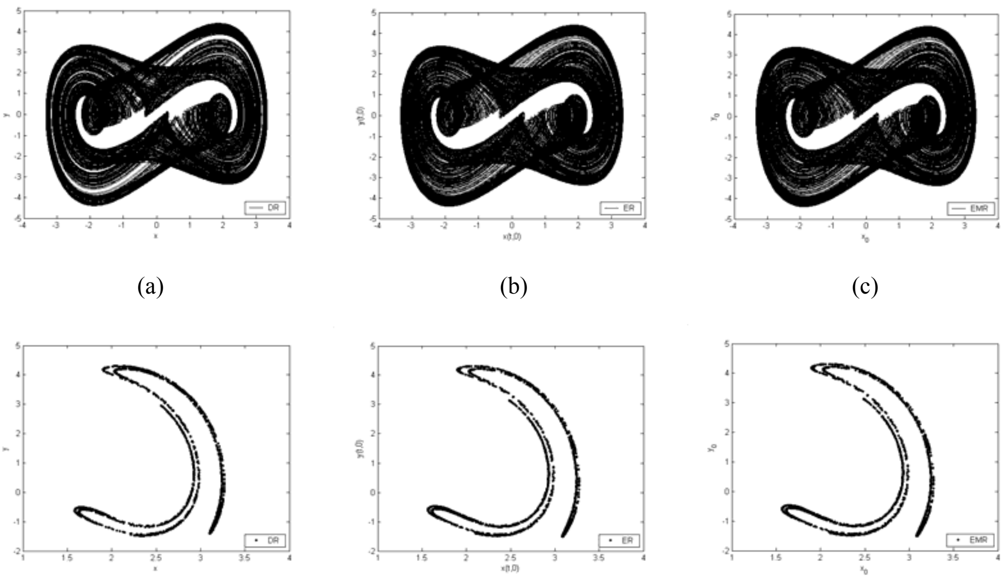
$$x(t,0)=\sum_{i=0}^{4}x_i(t)H_i(0)=x_0(t)-x_2(t)+x_4(t) \tag{15}$$

$$\ddot{x}-\bar{\mu}(1-x^2)\dot{x}-\alpha x+\beta x^3 = f\cos\omega t \tag{16}$$

We will note the responses of deterministic system (16) $x(t)$ as DR; response of the mean parameter system (15) $x(t,0)$ as ER; the ensemble mean response of the equivalent deterministic system (13) $E[x(t,\xi)]$ as EMR. In later chapters we will illustrate the effectiveness of the Chebyshev orthogonal polynomial approximation method by the similarity of DR and ER; research basic nonlinear phenomena of SDVP system through EMR and discuss characters of the SDVP by comparing DR and EMR.

3. Stochastic chaos in SDVP system

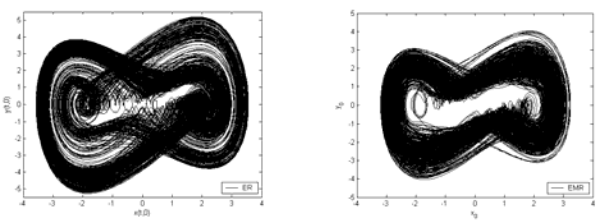
In this section, we analyze the chaos motion of the equivalent deterministic system (13) comparing with the deterministic system (16), take parameters as $\alpha=\beta=0.5, \bar{\mu}=0.1, f=3.5, \omega=0.76$ in system (13) and (16). When $\sigma=0.0$ the phase trajectories and the Poincaré sections of DR, ER and EMR are shown in Fig3.1, from which we can see that three kinds of responses are all chaotic and resemble each other. This phenomenon indicates that the Chebyshev orthogonal polynomial approximation method works very well for the SDVP system. The largest Lyapunov exponent of deterministic system (16) is 0.0916, while the one of the equivalent deterministic system (13) is 0.0671.



(a) (b) (c)

(d) (e) (f)

Fig3.1 When $\sigma=0.0$, (a) phase trajectories of DR; (b) phase trajectories of ER; (c) phase trajectories of EMR; (d) Poincaré sections of DR; (e) Poincaré sections of ER; (f) Poincaré sections of EMR.



(a) (b)

Fig3.2 When $\sigma=0.01$ (a) phase trajectories of ER; (b) phase trajectories of EMR.

When $\sigma \neq 0.0$, take $\sigma = 0.01$ for example, the phase trajectories of ER and EMR are shown in Fig3.2. Comparing with the phase trajectories of DR in Fig3.1 (a), it is obviously that the topological properties of ER nearly keep in step with DR, which shows that the orthogonal polynomial approximation method still works well in this situation. And the effectiveness of orthogonal polynomial approximation method is demonstrated in further steps. In Fig3.2 (b), it is clearly that EMR differs from DR to a certain extent, but the state is still chaotic, and the largest Lyapunov exponent of SDVP system is 0.0714. Thus, SDVP system has its own characteristics under the influence of random factor. Meanwhile, the phenomena of SDVP system still keep the main character of DVP system at the same conditions.

4. Control of stochastic chaos in SDVP system

From the chapter 2 ,we know that system (13) is the equivalent deterministic system of the SDVP system(8), so it is reasonable to reduce the control of stochastic chaos in SDVP system into the control of deterministic chaos in system (13), thus stochastic chaos of the original SDVP system can be controlled. Adding the control term $Q\eta(t)$ to the first equation of system (13), where Q is the intensity of noise, $\eta(t)$ is white Gauss noise which is independent of the random parameter μ, besides, $E\eta(t) = 0, E\eta(t)\eta(t+\tau) = \delta(\tau)$, where $\delta(\tau)$ is the Dirac-Delta function. Then we get the controlled system as follow

$$\begin{cases} \ddot{x}_0(t) - \bar{\mu}(\dot{x}_0(t) - X_{10}(t)) - \frac{\sigma}{2}(\dot{x}_1(t) - X_{11}(t)) - \alpha x_0(t) + \beta X_0(t) = f \cos \omega t + Q\eta(t) \\ \ddot{x}_1(t) - \bar{\mu}(\dot{x}_1(t) - X_{11}(t)) - \frac{\sigma}{2}[(\dot{x}_2(t) + \dot{x}_0(t)) - (X_{12}(t) + X_{10}(t))] - \alpha x_1(t) + \beta X_1(t) = 0 \\ \ddot{x}_2(t) - \bar{\mu}(\dot{x}_2(t) - X_{12}(t)) - \frac{\sigma}{2}[(\dot{x}_3(t) + \dot{x}_1(t)) - (X_{13}(t) + X_{11}(t))] - \alpha x_2(t) + \beta X_2(t) = 0 \\ \ddot{x}_3(t) - \bar{\mu}(\dot{x}_3(t) - X_{13}(t)) - \frac{\sigma}{2}[(\dot{x}_4(t) + \dot{x}_2(t)) - (X_{14}(t) + X_{12}(t))] - \alpha x_3(t) + \beta X_3(t) = 0 \\ \ddot{x}_4(t) - \bar{\mu}(\dot{x}_4(t) - X_{14}(t)) - \frac{\sigma}{2}[(\dot{x}_5(t) + \dot{x}_3(t)) - (X_{15}(t) + X_{13}(t))] - \alpha x_4(t) + \beta X_4(t) = 0 \end{cases} \tag{17}$$

4.1 Stochastic chaos control for $\sigma = 0.01$

In this section, we investigate stochastic chaos control of SDVP system when the intensity of random parameter is invariant. Take $\sigma = 0.01$ for example, choose parameters of the controlled system (17) as the same as parameters in section 3. When $Q = 0.0$, EMR is chaotic and the largest Lyapunov exponent of system (17) is 0.13. Increasing the value of Q, from the largest Lyapunov exponents diagram Fig.4.1, we can see that the original stochastic chaos is controlled by the increasing of noise intensity. Furthermore, with the different noise intensities Q, it can be controlled to different steady states as shown in Fig4.2.

From the research above, we can say that control of stochastic chaos in SDVP system is realistic, and the system can be controlled to different steady states with some different noise intensities during the control process.

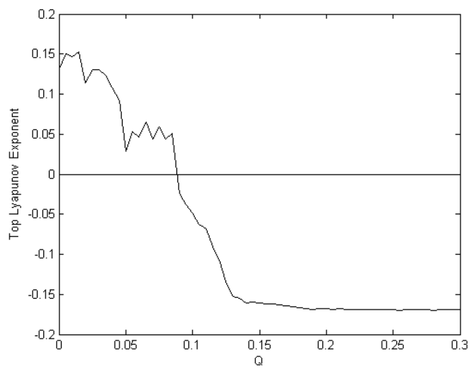
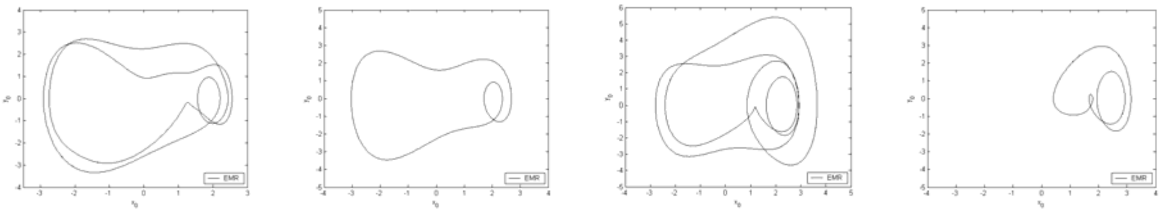


Fig 4.1 The largest Lyapunov exponents diagram of the controlled system (17), when $\sigma = 0.01, Q \in [0.0,0.3]$.

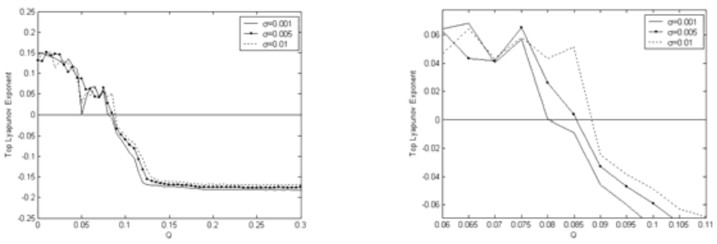


(a) (b) (c) (d)

Fig4.2 Phase portraits of EMR of the controlled system (17) (a) period-2 for $Q = 0.092$; (b) period-1 for $Q = 0.15$; (c) period-2 for $Q = 0.40$; (d) period-1 for $Q = 0.70$.

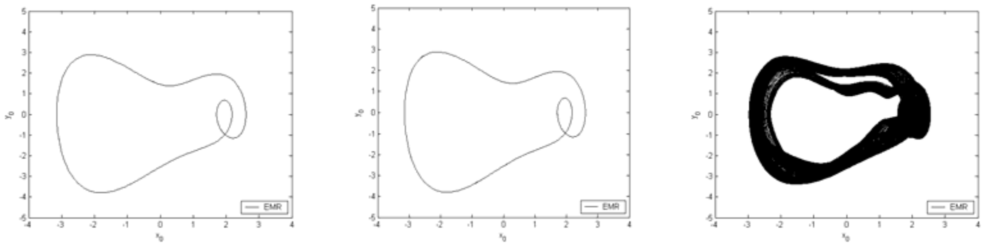
4.2 The influences of the change of σ on the effectiveness of stochastic chaos control of SDVP system

In section 4.1 we discussed chaos control by the white noise when $\sigma = 0.01$. Obviously, the intensity of random parameter σ is variable; therefore, it is necessary to analyze the features of stochastic chaos control with various values of σ except for the certain intensity.



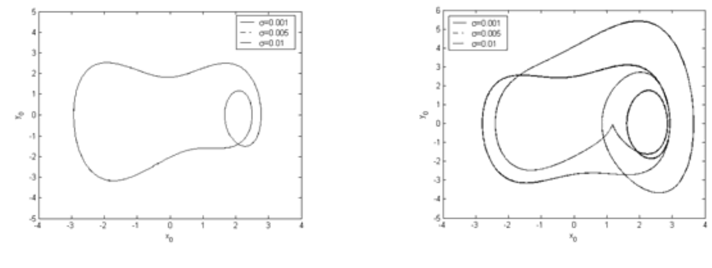
(a) (b)

Fig4.3 The largest Lyapunov exponent diagrams of the controlled system (17) (a) $Q \in [0,0.3]$;(b) $Q \in [0.06,0.11]$.



(a) (b) (c)

Fig4.4 When $Q = 0.087$, EMR of the controlled system (17), (a) $\sigma = 0.001$; (b) $\sigma = 0.005$; (c) $\sigma = 0.01$.



(a) (b)

Fig4.5 The phase portraits of the controlled system (17): (a) period-1 for $Q = 0.20$; (b) period-2 for $Q = 0.40$.

Take $\sigma = 0.001$, $\sigma = 0.005$ and $\sigma = 0.01$ for example, by analyzing the variety of the largest Lyapunov exponent in these three situations under the control of white noise, as shown in Fig4.3 (a), we can know the influence of random parameter more clearly. Under the influence of white noise, for different σ, the chaotic movements are all controlled, and the Lyapunov exponent reaches a plateau under the control when $Q > 0.15$. This phenomenon illustrates that the noise-aided control method works well when the intensity is variable too. From Fig. 4.3 (a), we see that the controlled behavior has some recognizable feature in the light of different σ. Fig4.3 (b) is the zoom in figure of Fig4.3 (a), from which we can see that, in order to control the chaotic motion, the noise intensity must be larger and larger with the growing σ. In other words, the threshold value Q changes with σ. In Fig4.4, it shows different states with diverse σ when the noise intensity fixed. Then take small values of σ, such as $\sigma = 0.001$ and $\sigma = 0.005$, the system can be controlled into period-1 state while let $\sigma = 0.1$, the system is still chaotic. Phase portrait and time history diagrams of the system with certain noise intensity are shown in Fig4.5, It is clear that by the same value of Q, with diverse intensities of random parameter, the system (17) will be controlled to similar periodic states.

5. Conclusion

In this paper, the stochastic chaos of SDVP system with bounded random parameter and its control by noise-aided control method are investigated. First of all, the SDVP system is reduced into its equivalent deterministic system by using the Chebyshev orthogonal polynomial approximation method, so that the problem of controlling stochastic chaos is reduced to the problem of controlling deterministic chaos of the equivalent system. Therefore, the Lyapunov exponent can be used to explore chaos behavior and its control of the equivalent deterministic system. Numerical results show that chaotic behavior of SDVP system is by and large similar to that of the deterministic DVP system, which attest to that the orthogonal polynomial approximation method is effective. Meanwhile, SDVP system has its own features under the effect of random parameter. Furthermore, chaos of equivalent deterministic system with different intensities of random parameter is controlled by noise-aided control method, and numerical results confirm the validity of this control method. Namely, stochastic chaos of the original SDVP system has been controlled. Furthermore, we also find that the threshold value of the intensity of white noise increases following the increment of the intensity of the random parameter.

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