CYBERNETICS AND PHYSICS, VOL. 2, NO. 4, 2013 , 199–204

EFFECTS OF GROWTH ON THE SYNCHRONIZATION OF DISCRETE-TIME DYNAMICAL NETWORKS

A. Anzo
IPICyT Research Center,
Applied Mathematics Division
San Luis Potosí, S. L. P., México.
andres.anzo@ipicyt.edu.mx

J. G. Barajas-Ramírez
IPICyT Research Center,
Applied Mathematics Division
San Luis Potosí, S. L. P., México.
jgbarajas@ipicyt.edu.mx

Abstract

Most real-world networks evolve, that is, new nodes and links are attached or removed from the network. In this work we investigate the effects of growth processes on the synchronized behavior of discrete-time dynamical network. In particular, we consider a network composed of identical Logistic Map, we assume that new nodes are attached to the network according to the model proposed by Barabasi and Albert (BA). We propose that nodes are add sufficiently slowly to the network, that is each growth event occurs after sufficient time has passed from the previous, so that the transitory effects have die out. We numerically investigate the synchronizability of the network as the number of nodes increases from an initial size. The synchronization criterion for dynamical networks with fixed structure is used as an indication of the stability of the resulting networks. Our results show that the synchronized solution remains attractive only for a limited number of additional nodes. Furthermore, the number of additional such that synchronization is not lost directly depends on the structure and size of the initial network.

Key words

Dynamical networks, growth, synchronization.

1 Introduction

Dynamical networks (DN) are composed by a set of dynamical systems called nodes, that are coupled together by edges; the pattern of connections is called the network structure. DN are significant for many fields of science mainly due to their potential applications to model systems in nature such as the Internet, the World Wide Web, food webs, among many others [Newman (2010)]. One of the most significant phenomena in DN is synchronization. Recent works focus on establishing synchronization criteria, mainly for fixed structure DN [Boccaletti *et al.*, (2006)]. However, in order to model a more realistic situation, it is important to take into account that real-world networks actually evolve through different processes like the addition or deletion of nodes and links.

Network evolution has been extensively addressed from the framework of graph theory. In this context, an evolutionary model consists of a set rules that describe the structural changes in the network, which are repeated iteratively in order to emulate the network's evolution. One of the first and most significant evolutionary model for networks was proposed by Barabási and Albert, usually called the BA-model [Barabási and Albert (1999)], which argues that real-world networks evolve thought two generic steps, which are: 1) Growth: at each iteration a new node is added to the network, and 2) Preferential attachment: the new node is more likely to connect to a node with high degree that to a less connected one. After repeating these rules a large number of iterations, we get a network where the vast majority of nodes will have few connections while some nodes will have a very large number of connections. Furthermore, this degree distribution feature remains unalter even if the number of nodes increases, this structural effect is called scale-free effect, and consequently the networks are called scale-free networks. The BA model only focuses on the structural features of the network and does not consider the dynamical aspects of its collective behavior. In this sense, alternative growth algorithms have been proposed, e.g. the authors of [Fan *et al.*, (2004)] proposed a synchronization-optimal growth model, where the BA preferential attachment rule is replaced by a rule where each new node is connected to the nodes that optimize the synchornizability of the network. The model in [Fan *et al.*, (2004)] succeeds in constructing a network with the scale-free feature; nevertheless, the selection of where to connect the node requires the investigation of all possible combinations in order to select the new node's connection.

In general, when we model the structural growth in

DN, the stability of its synchronized behavior we can not be determine from the synchronization criteria for a network with fixed structure. This represent a challenging problem that has attracted the attention of current researches, some of which, have been tackle this complication from the framework of switching systems. Using this formalism we can interpret any change process in the network structure as a discrete event that causes a transition from one network structure to another. This hybrid system description of the growth process in DN can be described as a time-driven or an event-driven system. In the first case, the structure changes at specific instant of time. On the second case, the structure changes if the states of the system go to a region of state space.

In [Stilwell *et al.,* (2006)] the authors consider DN that switch their structure among a set of different structures according to a predefined switching law, the DN are restricted to having the same number of identical nodes. An important result is that if the switching is fast enough, then an average model can be used, and the fixed structure results hold. If that is not the case, the stability of synchronized behavior needs to be determine from the hybrid system. In ([Tao *et al.*, (2010)]), under the assumption that some of the admissible structures be synchronizing, if the average dwell time of structure switching is sufficiently slow then synchronization is still possible. However, in these works the number of nodes are always fixed,and in general the number of admissible structures is small, that is, it is no allowed to the DN to grow, which, as mention above, is an intrench aspect of real-world networks.

In this contribution we define as a growth event the addition of a new node to the network. In particular, we consider the case of DN of discrete-time systems called Logistic Maps are added periodically in a sufficiently slow rate and following the BA network growth model. We observe that if the dwelling time between the addition of nodes is large enough, the synchronized behavior can be preserved even when a few nodes are added.

The remainder of the paper is organized as follows: In Section 2, we resume some preliminary results significant to this work. In particular, we review the synchronization criterion for a discrete-time DN with fixed structure and the BA network model. On Section 3, we expose our interpretation of the BA model for DN as a switching systems; and on Section 4 we analyse the limitations of using the synchronization criterion for a DN with fixed structure, to the case a growing DN. Numerical results supporting of claim are shown in Section 5. Finally, in Section 6 we present the conclusions for this work.

2 Preliminaries

2.1 Discrete-Time Dynamical Networks

For a network of N identical discrete-time systems, lineally and bidirectionally coupled with unweighted edges, the dynamical evolution of each node is given by

$$x_i^{k+1} = f(x_i^k) + c\sum_{j=1}^{N} a_{ij} f(x_j^k), \quad i = 1, \ldots, N; \tag{1}$$

where x_i^k is the state variable of the i-th node at the discrete-time instant $k \in \mathbf{Z}$; and the map $f(\cdot)$ describes the dynamics of a single node isolated from the network. We will consider that each node is a Logistic Map:

$$f(x^k) = rx^k(1 - x^k), \tag{2}$$

with $r = 3.9$ and $x_i^k \in \mathbf{R}$. The variable $c \in \mathbf{R}$ is the uniform coupling strength; the coupling matrix $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N\times N}$ describes the network structure as follows: if the i-th and j-th node are connected, the entries $a_{ij} = a_{ji}$ are set to one; if there is no connection between them, the entries are set to zero ($a_{ij} = a_{ji} = 0$). To complete the matrix, the diagonal entries are determined in the following manner:

$$a_{ii} = -\sum_{j=1}^{N} a_{ij} = -\sum_{i=1}^{N} a_{ij} = -d_i, \tag{3}$$

where d_i is the node degree of the i-th node.

By construction, the network structure is diffusive, that is, all sums by row or column of $\mathcal{A}$ are zero. Further, if the network is connected in the sense that no node is isolated from the network, then the coupling matrix is symmetric, irreducible, and its eigenvalues (λ_i) can be ordered as:

$$0 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_N. \tag{4}$$

For a dynamical network, complete synchronization is defined as the phenomena in which the states of all its nodes moves at unison. In an other words, a dynamical network is said to (asymptotically) achieve complete synchronization if as $k \to \infty$ the states of each node in the network tend to the synchronized solution

$$x_1^k = x_2^k = \ldots = x_N^k. \tag{5}$$

The existence of (5) as a solution of (1) is guarantee by the diffusive nature of the network structure and the fact that all nodes are identical. The stability of the synchronized solution can be establish from the linearized dynamics of the network, in this way diverse synchronization criteria have been derived (see for example [Arenas *et al.*, (2008)]). In particular, in ([Li and

Chen (2003)]) it was shown that the synchronized solution (5) is exponentially stable if the uniform coupling strength satisfies the following criterion

$$\frac{1-e^{-h_{max}}}{|\lambda_2|} < c < \frac{1+e^{-h_{max}}}{|\lambda_N|}; \qquad (6)$$

where λ_2 and λ_N are the biggest and smallest nonzero eigenvalue of $\mathcal{A}$, respectively; while h_{max} is the largest Lyapunov exponent of an isolated node, which, for the case of a Logistic Map is $h_{max} = ln(2)$.

An important question related to the stability of the synchronized solution is whether or not there is a positive coupling function such that the criterion in (6) is satisfied. To this end, an alternative version of the criterion can be used. Consider the ratio $R = \frac{-\lambda_2}{\lambda_2-\lambda_N}$, which measure the normalized distance of the eigenspectrum of $\mathcal{A}$. Then a positive coupling strength exist if the ratio satisfies

$$\frac{1}{R} < \frac{2e^{-h_{max}}}{1-e^{-h_{max}}}. \qquad (7)$$

For the dynamical network of Logistic Maps (1), the condition becomes $\frac{1}{R} < 2$.

Notice that the synchronization criteria (6) and (7) are only valid for networks with static (fixed) structure, and in general, can not be consider a valid criteria if the structure of the network changes over time. In particular, in the case of network growth, this is further complicated by the increment in the dimension of $\mathcal{A}$, and the change in its eigenvalues after each growth event. However, this result can be used as an indication of the behavior of synchronized solution for the growing network. We assume that after each growth event the network of identical nodes retains a diffusive structure; therefore as the nodes are added the same synchronization solution remains valid and additionally the eigenvalues of the resulting coupling matrix still can be organized as in (4). Further, we assume that the time between growth events is large enough as to allow for the network dynamics to reach their steady-state behavior. Under these conditions, one can argue that the synchronization conditions (6) and (7) can in fact be use as an indication of the stability of the synchronized solution on the resulting network.

We let the network growth be described by the model proposed in [Barabási and Albert (1999)], as described in the following subsection.

2.2 The BA Model of Network Growth

The network model proposed by Barabasi and Albert states that as the network grows, it does so following a preferential attachment rule, that is, a new node in the network is more likely to connect to an important node that to a less connected one. In what follows we briefly describe the BA model network as a construction algorithm.

The BA model consists of two steps:
The first step is simply called *Growth*.

1. Starting with a small number (m_0) of nodes. At every iteration (or growth event) a new node is added to the network by connecting it m ($\leq m_0$) nodes in the network.

The second step tells us to which of node, already existing in the network, the new node will be connected. The choice is made to favor nodes with a large number of connections, for that reason the process is called *Preferential attachment*. More precisely, the second step can be expressed as follows.

2. For each new node, say the q-th node, the m nodes to which it will be connected are selected from the nodes already in the network through a uniformly random process where the probability that the new node connects to the j-th node is given by

$$\Pi_{q\leftrightarrow j} = \frac{d_j}{\sum_{l=0}^{N} d_l}. \qquad (8)$$

The algorithm is repeated until the network has grown to the desired number of nodes, say N, with $N = m_0 + \sigma$ where σ is the number of iteration of the construction algorithm, in other words, the number of growth events that lead to a network of size N.

A particularly significant aspect of the BA network model is that for a sufficiently large number of nodes, the probability distribution of node degrees in the resulting network is well approximated by a power-law of the form $P(d) \sim d^{-3}$, which remains practically unchanged for larger number nodes, in other words, this feature of the topology is independent of size; this is known as the scale-free effect.

In this contribution, we investigate the effect of the growth events as described by the BA model on the stability of the synchronized state of the resulting DN. In the following section, we propose an interpretation of the BA model as a switching system, where the growth events result on changes in the dynamical description of the network. Namely, the dimension of the coupling matrix increases. Then, we claim that the synchronization solution of the growing DN preserves it stability if the resulting network satisfies the synchronization condition for the corresponding fixed structure network after each growth event.

3 An Interpretation of Dynamical Network Growth

We start with a network composed by a small number (m_0) of Logistic Maps connected in a fully coupled structure. Rewriting the dynamical description (1) of our initial network in vector form we have

$$\mathbf{X}^{k+1,0} = F^0(\mathbf{X}^{k,0}) + c\mathcal{A}_0 F^0(\mathbf{X}^{k,0}), \qquad (9)$$

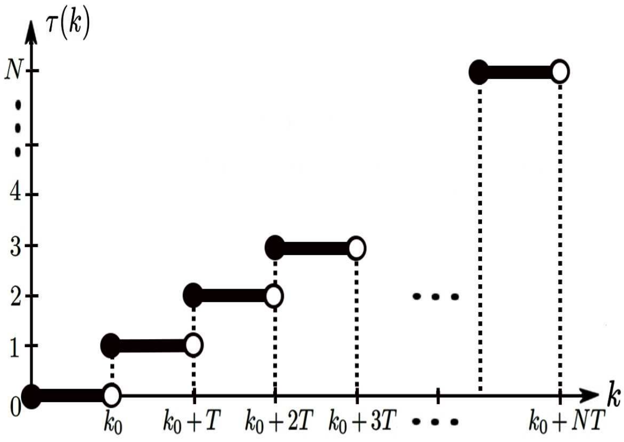

Figure 1. The change of the growth event index for a dynamical network that grows according to the BA model. The first growth event ($\tau(k) = 1$) is present at $k = k_0$, and the subsequent growth events occur at equal time intervals T.

where $\mathbf{X}^{k,0} = [x_1^{k,0}, \ldots, x_{m_0}^{k,0}]^\top \in \mathbf{R}^{m_0}$; $F^0(\mathbf{X}^{k,0}) = [f(x_1^{k,0}), \ldots, f(x_{m_0}^{k,0})]^\top \in \mathbf{R}^{m_0}$; and the initial coupling matrix has the form

$$\mathcal{A}_0 = \begin{pmatrix} -m_0+1 & 1 & \ldots & 1 \\ 1 & -m_0+1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & -m_0+1 \end{pmatrix}. \quad (10)$$

By construction, the eigenspectrum of $\mathcal{A}_0$ is $\lambda_1 = 0$, and $\lambda_j = -m_0$ for $j = 2, ..., m_0$. In order to ensure that our initial network synchronizes, we select a coupling strength c such that the criterion (6) is satisfied.

We assume that the first growth event ($\tau(k) = 1$) occur after sufficient time has passed ($k = k_0$) such that the transitory behaviors have die out. Further we assume that all subsequent growth events occur periodically, that is

$$\tau(k) = \begin{cases} 0, & \text{if } 0 \le k < k_0 \\ n, & \text{if } k_0 + (n-1)T \le k < k_0 + nT \end{cases} \quad (11)$$

for $n = 1, 2, ..., N$; where $\tau(k)$ is the growth event index and T is the time period between growth events. Figure (1) shows an example of $\tau(k)$.

The initial conditions for our initial network ($\mathbf{X}^{0,0}$) are randomly selected from $[0, 1]$. As discrete-time moves along the k index, the event index $\tau(k)$ moves according to (11). Then, at the time instant in which the first growth event occurs, the dynamical description of the network changes to:

$$\mathbf{X}^{k+1,1} = F^1(\mathbf{X}^{k,1}) + c\mathcal{A}_1 F^1(\mathbf{X}^{k,1});$$

Since a new node is added, the vector of state variables becomes

$$\mathbf{X}^{k,1} = [\mathbf{X}^{k,0}, x_{m_0+1}^{k,1}]^\top \in \mathbf{R}^{m_0+1}.$$

The initial condition of the added Logistic Map is a value randomly selected from $[0, 1]$. In a similar manner, $F^1(\mathbf{X}^{k,1})$ is the previous vector function appended with the dynamics of the added node

$$F^1(\mathbf{X}^{k,1}) = [F^1(\mathbf{X}^{k,0}), f(x_{m_0+1}^{k,1})]^\top \in \mathbf{R}^{m_0+1}.$$

The coupling matrix of the network with an added node becomes

$$\tilde{A}_0 = \phi_1(\mathcal{A}_0) = \begin{pmatrix} \mathcal{A}_0 & v_1 \\ v_1^\top & 0 \end{pmatrix},$$

with $v_1 \in \mathbf{R}^{m_0}$ a zero vector. The preferential attachment of the new node to the network is done by randomly selecting m entries of v_1 and change them from 0 to 1. Then we have

$$\hat{A}_0 = \phi_2(\tilde{A}_0) = \begin{pmatrix} \mathcal{A}_0 & \hat{v}_1 \\ \hat{v}_1^\top & 0 \end{pmatrix};$$

where $\hat{v}_1$ is the zero vector v_1 with m randomly selected entries as ones.

In order to have a diffusive connection in the resulting network, the diagonal entries of $\mathcal{A}_1$ are calculated from $\hat{A}_0$ as $a_{ii} = -\sum_{j=1}^{N} a_{ij}$. Then, finally we have

$$\mathcal{A}_1 = \phi_3(\hat{A}_0).$$

Summarizing in our interpretation of the growth of a dynamical network implies a three part process: first the previous coupling matrix is appended with zero vector ($\phi_1(\mathcal{A}_0)$); then, the new node is randomly coupled to m nodes nodes ($\phi_2 \circ \phi_1(\mathcal{A}_0)$), and finally, the diagonal entries are recalculated ($\phi_3 \circ \phi_2 \circ \phi_1(\mathcal{A}_0)$).

The dynamical description of the network including growth events is

$$\mathbf{X}^{k+1,\tau(k)} = F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}) + c\mathcal{A}_{\tau(k)} F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}); \quad (12)$$

where $\mathbf{X}^{k,\tau(k)} = [\mathbf{X}^{k,\tau(k)-1}, x_{m_0+\tau(k)}^{k,\tau(k)}]^\top \in \mathbf{R}^{m_0+\tau(k)}$; $F^{\tau(k)}(\mathbf{X}^{k,\tau(k)}) = [F^{\tau(k)-1}(\mathbf{X}^{k,\tau(k)-1}), f(x_{m_0+\tau(k)}^{k,\tau(k)})]^\top \in \mathbf{R}^{m_0+\tau(k)}$; and $\mathcal{A}^{\tau(k)} = \phi_3 \circ \phi_2 \circ \phi_1(\mathcal{A}_{\tau(k)-1})$; with $\tau(k)$ given by equation (11).

Notice that when a growth event occurs, lets say at $k = \bar{k}$, $\tau(k)$ increases by one and the structure of the network changes with the inclusion of the new node as described above. However, the dynamical evolution of the nodes continues along the discrete-time index

k without change. This means that after the growth event ($k = \bar{k}+1$), the dynamical network continues its evolution with the corresponding new structure until a new growth event occurs ($k = \bar{k} + T$), then the structure changes again, and the growth process continues in that way until the network has grown to the desired N nodes.

4 A Synchronization Criterion for Growing Dynamical Networks

Following the same basic ideas presented in Subsection *2.1*, we define synchronization on a growing dynamical network as the phenomenon in which the nodes existing in the network move at unison. That is, a growing dynamical network is said to be synchronize if the solution

$$x_1^{k,\tau(k)} = x_2^{k,\tau(k)} = \ldots = x_{m_0+\tau(k)}^{k,\tau(k)} \qquad (13)$$

is a stable solution of the resulting network. Considering the case of identical nodes under diffusive coupling as the network grows. All the resulting networks will have the same synchronized solution. Moreover, if prior to the growth event the synchronized solution (13) was stable, after the growth event, the network solution is perturbed by dynamics of the new node. Then, under the assumption that resulting network also satisfies the synchronization criterion, that perturbation will vanish into the synchronization manifold. On the other hand, if for the resulting network does not satisfy the synchronization criterion, the perturbation will not vanish making the synchronized solution unstable. In that sense, when the growth event does not make the synchronized solution unstable, we say that the network preserved synchronization, on the other case, the growth event desynchronizes the network.

As mention before, we consider a growing network where the growth events occur only after a sufficiently large time has passed, such that all transient behaviors have died out. This dwell time restriction, along with the claim provided in the previous paragraph, allows us to establish the stability of the resulting synchronized solution (13) using the criteria (6)-(7) at each growth event. That is, the corresponding synchronized solution will be stable if the uniform coupling strength satisfies

$$\frac{1-e^{-h_{max}}}{|\lambda_2^{\tau(k)}|} < c < \frac{1+e^{-h_{max}}}{|\lambda_N^{\tau(k)}|}; \qquad (14)$$

where $\lambda_i^{\tau(k)}$ is the i-th eigenvalue of the coupling matrix $\mathcal{A}^{\tau(k)}$. In particular, for our Logistic map network, stability of the synchronized solution is determine by the criteria

$$\frac{0.5}{|\lambda_2^{\tau(k)}|} < c < \frac{1.5}{|\lambda_N^{\tau(k)}|}. \qquad (15)$$

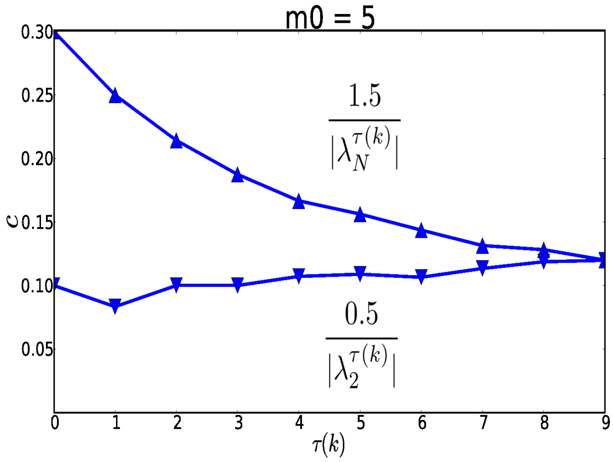

Figure 2. Synchronization area for a dynamical network of $m_0 = 5$ Logistic Maps which structure evolves according to the BA model.

5 Numerical Results

According to the criterion (15), for a network of coupled Logistic Maps connected in a fully coupled structure, as the number of initial nodes (m_0) becomes larger, the range of values for the coupling strength (c) such that synchronization is achieved becomes smaller. That is, if we start with very few nodes, the synchronization will be preserved for a larger number of growth events. As an illustration, we take a network of $m_0 = 5$ fully connected Logistic maps, that become synchronized for a coupling strength in the interval $c \in [0.1, 0.3]$. As we can see on Figure (2), for $c = 0.1$ synchronization is preserved up to seven growth events (with $m = m_0$).

The time series of the nodes dynamics as the network grows are shown in Figure (3), which is plotted in terms of the error between their dynamical states. In order to avoid the transitory behavior of the first nodes, we let the nodes to evolve until $k_0 = 100$ time steps, which, as we can seen in Figure (3.a), is enough to allow the error between the dynamical states of the m_0 nodes to died out (≈ 0), i.e, the nodes are practically synchronized. After this k_0 time steps, the network start to growth with each new node added after $T = 100$ iterations. In Figure (3.a) we can see that at $k = 100$, the second node arrives and begins to evolve, the perturbation provided by the node vanishes rapidly, and the new node synchronized with network. Note that the second growth event does not affect the dynamical evolution of the first m_0 nodes. Again, in Figure (3.b), we observe that at $k = 200$ and $k = 300$ two new events occurs with the addition of two nodes, which alter very little the synchronized behavior. The same is seen again in Figure (3.c) with two next events, where the synchronized behavior is significantly altered. Finally, on Figure, we observe that when the last node is added, the nodes are no longer synchronized, as for the resulting network the synchronization criterion is not satisfied, that is, this growth event desynchronizes the network.

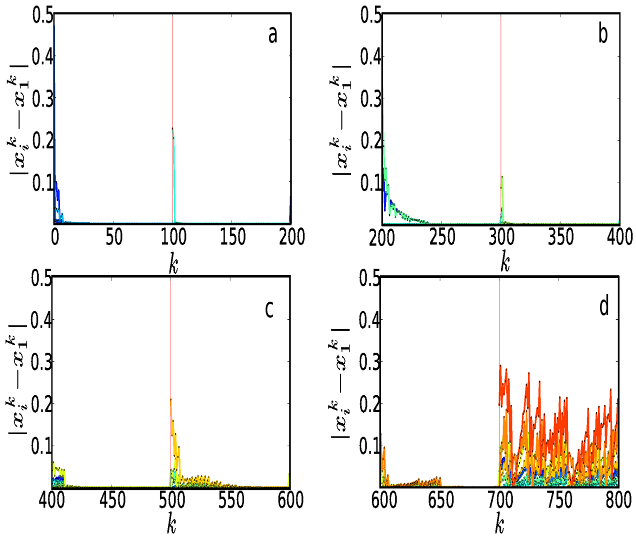

Figure 3. Errors between the nodes dynamical states. The first growth event occurs at the time step $k_0 = 100$, and the interval between two growth events is $T = 100$.

6 Conclusions

In this contribution we investigate the effects of network growth in the synchronized state of DN. We propose that the synchronization criterion for a dynamical network with fixed structure can be used to establish if the stability of the synchronized solution is preserved as the network grows. Our claim is based on our interpretation of the growth process as a switching system, where each growth event consist on the addition of a new node to a network that is practically synchronized. The growth process, that is assumed to occur only after a sufficiently long time has passed, results on a diffusive network with the same synchronized solution; which experience the additional node's dynamics as a perturbation. Then, if the resulting network satisfies the synchronization criterion, the perturbation will vanish and synchronization will be preserved; otherwise, its affect will desynchronize the collective dynamics of the network.

A next step in the study of evolving dynamical network is to analyze processes that can be improved the synchronizability of the resulting network; possibly through additional process of change a part from growth, such as deletion of nodes or edges, rewiring the connections, or adjusting coupling strengths. Results on these directions of research will be publish elsewhere.

Acknowledgements

We appreciate the financial support of the Institute for Scientific and Technological Research of San Luis Potosi, IPICyT, and the National Council for Science and Technology of México, CONACyT.

References

Arenas, A., Diaz-Guilera, A., Kurths, J., Moreno, Y., and Zhou, C. (2008). Synchronization in complex networks, *Physics Reports* **469**(3), pp. 93–153.

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., and Hwang, D. (2006). Complex networks: Structure and dynamics, *Physics reports* **424**(4), pp. 175–308.

Barabási, A.-L. and Albert, R. (1999). Emergence of scaling in random networks, *Science* **285**, pp. 509–512.

Li, X and Chen, G. (2003) Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint, *IEEE Trans. Circ. & Syst. I* **50**(11), pp. 1381–1390.

Wang, X. F. and Chen, G. (2002), Synchronization in scale-free dynamical networks: robustness and fragility, *IEEE Trans. on Circuits and Systems I* **49**(1), pp. 54–62.

Fan, J., Li, X., and Wang, X. F. (2004). On Synchronous preference of complex dynamical networks, *Physica A: Statistical Mechanics and its Applications* **355**(2-4), pp. 657–666.

Fan, J. and Wang, X. F. (2005). On synchronization in scale-free dynamical networks, *Physica A: Statistical Mechanics and its Applications* **349**(3-4), pp. 443–451.

Wang, Y., Zeng, A., Di, Z., and Fan, Y. (2011). Enhancing synchronization in growing networks, *EPL* **96**(2011), 58007.

Newman, M. E. J. (2010). *Networks: An Introduction*, Oxford University Press, USA.

Tao, L., Zhao, J., and Hill, D. J. (2010). Exponential synchronization of complex delayed dynamical networks with switching topology, *IEEE Trans. on Circuits and Systems* **57-I**(11), pp. 2502–2511.

Stilwell, D.J., Bollt, E.M., and Roberson, D.G. (2006). Sufficient conditions for fast switching synchronization in time-varying network topologies, *SIAM J. Applied Dynam. Syst.* **5**(1), pp. 140–156.

C. G. Cassandras and S. Lafortune (2008). *Introduction to Discrete Event Systems*, Springer 2nd edition.

Liberzon, D. (2003). *Switching in Systems and Control*, Birkhauser, USA.

Sun, Z. and Ge, S. (2005). *Switched Linear Systems, Control and Design* Springer, London.