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CHANGE IN THE ADIABATIC INVARIANT AT A SEPARATRIX CROSSING IN A NONLINEAR MODEL OF FESHBACH RESONANCE

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Abstract

Mean field approach has recently been used to model narrow Feshbach resonance passage. We investigate the structure of the corresponding classical phase space and calculate change in the action which corresponds to finite-rate efficiency of the sweep.

Key words

Feshbach resonance, adiabatic invariants, separatrix crossing

1 Introduction

Dynamics of Bose-Einstein condensates (BEC) introduce paradigm of nonlinearity into quantum systems. Recently, many models related to BEC physics (like nonlinear Landau-Zener tunnelling, macroscopic quantum self-trapping, etc) were shown to possess nonlinear properties that are common for classical nonlinear systems. One of the conceptual phenomena of classical perturbation theory is destruction of the adiabatic invariance at separatrix crossings, which is encountered in different fields of physics. It is well-known that action is an adiabatic invariant in Hamiltonian systems that depend on a slowly varying parameter. This result is based on the possibility of averaging over fast motion in the unperturbed [frozen at a certain parameter value] system. The situation is different in case when the unperturbed system has separatrices on its phase portrait. As the parameter varies, the separatrices slowly evolve on the phase portrait. A phase trajectory of the exact system may cross the separatrix of the frozen system. On the separatrix, the period of motion tends to infinity. This results in breakdown of the averaging method and, in this case, more accurate study is necessary to describe behavior of the action. It turns out that a quasi-random jump in the value of the adiabatic invariant occurs at the crossing. The asymptotic formula for this jump in a Hamiltonian system depending on a slowly varying parameter was obtained in [Timofeev, 1978; Neishtadt, 1986; Cary et al, 1986]. Later, the theory of adiabatic separatrix crossings was also developed for slow-fast Hamiltonian systems [Arnold, Kozlov, and Neishtadt, 2006], volume preserving systems [Neishtadt and Vasiliev, 1999], and was applied to certain physical problems (see, e.g., [Itin et al, 2000; Vainchtein et al, 1996; Itin et al, 2002]). It was shown in [Itin et al, 2007] that nonlinear Landau-Zener tunnelling models constitute a particular case for which the general theory can be applied.

2 Main equations and phase portraits

We consider the model of [Tikhonenkov et al, 2006], concentrating on the case of non-zero initial molecular fraction. Within the model, change in the action at the resonance passage gives the remnant atomic fraction as a power-law of a sweeping rate parameter (instead of the exponential law [Landau and Lifshitz, 1976] of Landau-Zener linear model). In [Tikhonenkov et al, 2006], in the framework of a two-mode model of the atom-molecular system in the classical field limit, the following equations of motion were obtained:

$$\dot{u} = -\delta(\tau)v, \quad \dot{v} = \delta(\tau)u + \frac{\sqrt{2}}{4}(w-1)(3w+1), \quad \dot{w} = \sqrt{2}v, \tag{1}$$

where u, v are real and imaginary parts of atom-molecule coherence, and w is the atom-molecule population imbalance; the dot denotes time derivative, $\tau = \epsilon t$, $0 < \epsilon \ll 1$ is a small parameter, and $\delta(\tau)$ is a slowly varying parameter corresponding to the (scaled) detuning. System (1) is a so-called volume-preserving system [Arnold, Kozlov, and Neishtadt, 2006], but can be cast into Hamiltonian form, with the corresponding Hamiltonian given by

$$H = \delta(\tau)w + (1-w)\sqrt{1+w}\cos\theta, \tag{2}$$

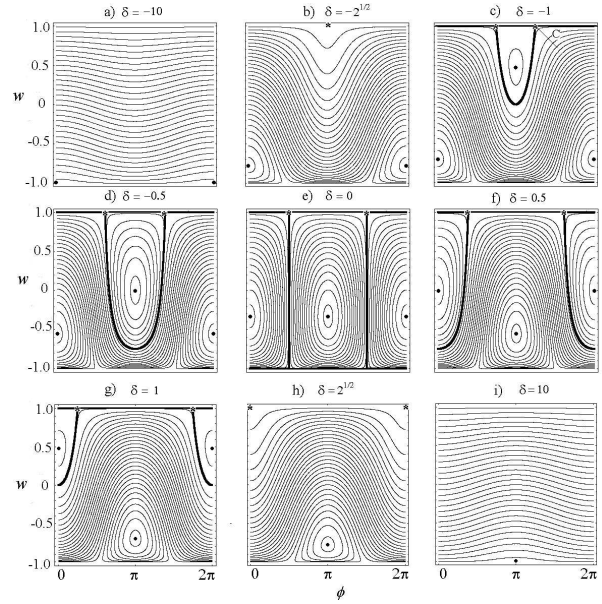

Figure 1. Phase portraits of the system with Hamiltonian (2) frozen at different values of parameter δ.

where w and $\theta = \arctan(v/u)$ are canonically conjugated variables.

Consider the system using the separatrix crossing theory (applicable when ϵ is smaller than initial molecular fraction $1 - w$). More detailed analysis is available in [Itin et al, 2007].

Equations (1) at frozen values of parameter δ possess two first integrals:

$$u^2 + v^2 - (w-1)^2(w+1)/2 = 0, \qquad (3)$$

and $u + (\delta/\sqrt{2})w =$ const. Eq. (3) defines a surface of rotation around $w-$axis with a singular [conical] point at (0,0,1). Equations of motion (1) restricted to surface (3) are equivalent to equations of motion in a Hamiltonian system with Hamiltonian (2). Consider phase portraits of the system with Hamiltonian (2) frozen at different values of parameter δ [see Fig. 1]. In these portraits, points (0,0,1) and (0,0,-1) are represented as segments $w = 1$ and $w = -1$ correspondingly. Point (0,0,1) is always a stable point of (1), but formally speaking this is not true for the points of the segment $w = 1$ on phase portraits of the system with Hamiltonian (2). If δ is negative and $|\delta| > \sqrt{2}$, there is only one stable elliptic point on the phase portrait, at $\theta = 0$ and w not far from -1 [see Fig. 1a].

At $\delta = -\sqrt{2}$ a bifurcation occurs, and, for $-\sqrt{2} < \delta < 0$, the phase portrait looks as shown in Fig. 1c. There are two saddle points at $w = 1, \cos\theta = -\delta/\sqrt{2}$ and a newborn elliptic point at $\theta = \pi$. The trajectory connecting these two saddles corresponds to the singular trajectory on surface (3) (i.e., passing through the singular point $(0, 0, 1)$). On the phase portrait of the system with Hamiltonian (2) it separates rotations and oscillating motions and we call it the separatrix of the frozen system. Below, we consider the "shifted" Hamiltonian $E = H + \delta$; $E = 0$ on the separatrix. At $\delta = 0$ the singular trajectory on (3) passes through (0,0,-1), and correspondingly on the phase portrait the segment $w = -1$ belongs to the separatrix [Fig. 1d]. For $0 < \delta < \sqrt{2}$ the phase portrait looks as shown in Fig. 1e. Finally, at large positive values of δ, again there is only one elliptic stationary point at $\theta = \pi$, and w close to -1. Consider now a phase trajectory on a phase portrait frozen at a certain value of δ. If the trajectory is closed, the area S inside of it is related to the action I of the system as $S = 2\pi I$. If the trajectory is not closed, we define the action as follows. If the area S bounded by the trajectory and lines $w = 1, \theta = 0, \theta = 2\pi$ is smaller than 2π, we still have $S = 2\pi I$. If S is larger than 2π, we put $2\pi I = 4\pi - S$. Defined in this way, I is a continuous function of the coordinates.

3 Variation of the adiabatic invariant at the separatrix crossing

In system with Hamiltonian (2) with $\tau = \epsilon t$, the action I is an adiabatic invariant of motion (e.g., [Arnold, Kozlov, and Neishtadt, 2006]). Far from the separatrix, it undergoes oscillations of order ϵ. Introduce the so called improved adiabatic invariant $J = I + \epsilon f(w, \theta, \tau)$, where [see, e.g., [Neishtadt, 1986]]:

$$f(w, \theta, \tau) = \frac{1}{2\pi}\int_0^T \left(\frac{T}{2} - t\right)\frac{\partial H}{\partial \tau} dt. \qquad (4)$$

The integral in (4) is computed along the unperturbed trajectory passing through the point (w, θ) at the time moment $t = 0$; T is the period of motion on this trajectory. If the trajectory is not closed, T is the time necessary for a phase point on this trajectory to cover the distance of 2π along the θ-axis. Far from the separatrix variation of J along a phase trajectory is of order ϵ^2. As the slow time τ grows, the area bounded by the separatrix $S^*(\tau)$ slowly changes. On the other hand, a value of the adiabatic invariant associated with a certain phase trajectory stays well-preserved. Accordingly, phase trajectories of (2) can cross the separatrix. Let δ be initially large in magnitude and negative, so that the phase portrait is similar to one shown in Fig. 1a. Consider a trajectory passing close to the singular point $(0, 0, 1)$ on (3). Along the corresponding phase trajectory on the plane (θ, w), value of w is close to 1. We assume $1 - w$ on this trajectory to be small, yet finite. Hence, the initial value J_- of improved action J is also small. As the time goes, value of parameter δ grows, and at $\delta = -\sqrt{2}$ the separatrix loop appears. The area $S^*(\tau)$ surrounded by the separatrix grows with time, and the action associated with the phase trajectory stays approximately constant. In the so called adiabatic approximation this action is conserved and equal to J_-, and at the slow time moment $\tau = \tau_*$ such that $S^*(\tau_*) = 2\pi J_-$ the phase trajectory

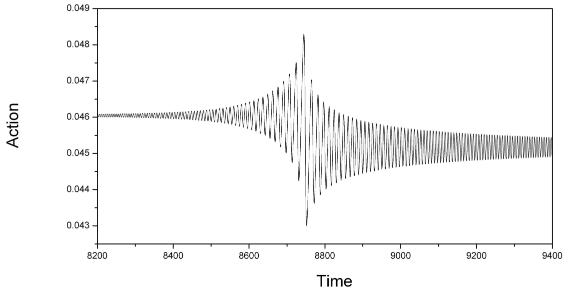

Figure 2. Typical jump of the adiabatic invariant (action) at separatrix.

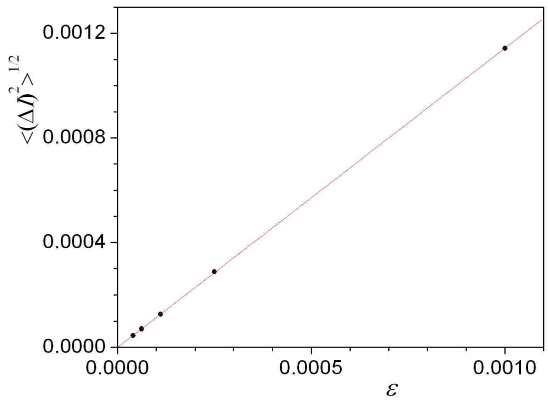

Figure 3. Scaling of jumps in improved adiabatic invariant with ϵ. For each point on the plot, we take a set of 100 trajectories with initial (at $\delta = -10$) values of w being closely distributed around $w = 0.99 = 1 - 10^{-2}$. Final values of improved action were taken far from the separatrix (at $\delta = 0$). The line on the plot is a linear f it to data, and a perfect linear scaling can be seen. The slope of the line gives $\sigma \approx 1.14\epsilon$, and the coefficient is in good agreement with the theoretical value $\sqrt{4/3} \approx 1.15$.

crosses the separatrix. After the separatrix crossing, the phase point rotates around the elliptic point inside the separatrix loop and slowly drifts with this point to the bottom of the phase portrait. If $\delta(\tau)$ is a monotonous function, the phase trajectory will never again cross the separatrix. Calculating change in the improved action due to the separatrix crossing according to the general method, we f ind in the main approximation

$$2\pi\Delta J = -2\frac{\epsilon\Theta}{\sqrt{2-\delta_*^2}}\ln(2\sin\pi\xi), \qquad (5)$$

where $\Theta = dS/d\tau|_{\tau=\tau_*}$; $\xi = |h_0/(\epsilon\Theta)|$, h_0 is the value of the Hamiltonian at the time of the last crossing of the vertex bisecting the angle between incoming and outgoing separatrices of the saddle point C outside the separatrix loop (see Fig. 1c) before crossing the separatrix; $\delta_* = \delta(\tau_*)$. The value of ξ strongly depends on initial conditions: a small, of order ϵ, variations of initial conditions result in order 1 changes in ξ. Thus, ξ can be considered as a random value (e.g., when one considers a bunch of trajectories with initial conditions distributed over a certain volume in the phase space). As a result, the value of a jump of the adiabatic invariant at separatrix crossing also strongly depends on initial conditions and is quasi-random. According to (5), its mean value is zero. Formula (5) is valid provided $k\sqrt{\epsilon} < \xi < 1 - k\sqrt{\epsilon}$, where k is a positive constant (i.e., not for all trajectories from the bunch, but for most of them; for details, see [Neishtadt, 1986]). According to [Cary et al, 1986; Neishtadt, 1986], the error of formula (5) is $O(\epsilon^{3/2}|\ln\epsilon|)$. A typical jump in the action is shown in Fig.2.

After the separatrix crossing, the phase point rotates around the elliptic point inside the separatrix loop and slowly drifts with this point to the bottom of the phase portrait. If $\delta(\tau)$ is a monotonous function, the phase trajectory will never again cross the separatrix. Assume that $\delta(\tau)$ is a smooth function and $\delta = \delta_- = $ const at $\tau < \tau_-$, $\delta = \delta_+ = $ const at $\tau > \tau_+$. [It is assumed that $\delta_- < 0$, $\delta_+ > 0$.] In other words, parameter δ is slowly monotonically varying between two border values. Then $\partial E/\partial\tau = 0$ at $\tau = \tau_\pm$, and action I coincides with J. Hence, formula (5) gives the variation in action I. Let the magnitudes of δ_- and δ_+ be large enough. If the initial value of $w = w_-$ is close enough to 1, the corresponding unperturbed phase trajectory is an almost straight line (cf. Fig. 1a). Hence, $I_- \approx (1 - w_-)$. Similarly, $I_+ \approx (-1 - w_+)$. Thus, variation in action I corresponds to the remnant atomic fraction. In the adiabatic limit, value of atom-molecular imbalance is reversed at the passage, while the change in the action produce non-adiabatic correction to this result.

Formula (5) can be simplified to take the following form:

$$\Delta J = -\frac{4\epsilon\delta'_*}{\pi}\ln(2\sin\pi\xi). \qquad (6)$$

Dispersion of jumps in the action can be predicted using formula (5):

$$\sigma^2 = 16\epsilon^2(\delta'_*)^2\pi^{-2}\int_0^1 \ln^2(2\sin\pi\xi)\,d\xi = \frac{4\epsilon^2(\delta'_*)^2}{3} \qquad (7)$$

To check numerically the scaling of the jumps with ϵ and the dispersion, we calculated bunches of trajectories with close initial conditions (see Fig.3). For the numerical calculations, we used linear sweeping with $\delta' = 1$, therefore the predicted value of dispersion is $\sigma^2 = (4/3)\epsilon^2$. Numerically found coefficient is equal to 1.30, which is in reasonable (2% accuracy) agreement with theoretical prediction $4/3 = 1.3333$.

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