THIRD ORDER SUPERHARMONIC RESONANCE AND SPATIAL MOTION OF A STRING

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Abstract

This paper theoretically and experimentally deals with the third order superharmonic resonance and the spatial motion occurring in a string. We consider the case when the upper end of the string is fixed and the lower end is harmonically excited in a direction. When the excitation frequency is in the neighborhood of a third the linear natural frequency of the string, the third order superharmonic resonance in the excitation direction occurs. Additionally, the superharmonic resonance(in-plane motion) can produce the resonance(out-of-plane motion) in the direction perpendicular to the excitation through the nonlinear coupling between the excitation direction and the perpendicular direction. First, the equations of motion of the string including the effect of the nonlinear restoring force are shown. Next, the analytical solutions of the equations and the amplitude equations are obtained by the method of multiple scales. Then, the amplitude equations give theoretical frequency response curve. Finally, the experimental results qualitatively confirm the theoretical characteristics.

Key words

string, third order superharmonic resonance, spatial motion, nonlinear restoring force

1 Introduction

Generally, mechanical systems are governed by differencial equations with nonlinear terms which may produce nonlinear phenomena. Because of difficulty of analysing the nonlinear differencial equation, it is very difficult to predict or control the nonlinear phenomenon. In a lot of the mechanical systems, stability and reliability are assured by imposing physical constraints and ignoring the nonlinear terms. However, under such a constraint, high-performance systems cannot be continuously created. Therefore, it is necessary to analyse the differencial equations including the nonlinear terms and to investigate the reliability of the systems.

A string is one of the mechanical systems governed by the nonlinear partial differencial equations. The string is the most fundamental system with infinite degree of freedom and interesting from the engineering point of view. As the nonlinear phenomena in the string, the third order superharmonic resonance and the spatial motion exist. In the string whose upper end is fixed is harmonically excited, a resonance in the direction parallel to the excitation can occur depending on the excitation frequency. It is well known that the excitation frequency in the neighborhood of the integral multiple of the natural frequency produces the resonance. On the other hand, the excitation frequency in the neighborhood of a third of the natural frequency also produces the resonance[1]. This resonance is called the third order superharmonic resonance. Moreover, when the resonance in the direction parallel to the excitation occurs, that in the direction perpendicular to the excitation can occur[2][3]. These resonances in the two direction cause the spatial motion of the string.

This study deals with the third order superharmonic resonance and the spatial motion of the string whose upper end is fixed and lower end is harmonically excited. First, we derive the equations of motion of the string with nonlinear characteristics. Second, we obtain the amplitude equations which govern the amplitude of the analytical solutions and the theoretical frequency responce curve. Finally, to assure the accuracy of the theoretical result, we do an experiment and obtain the experimental frequency resoponce curve.

2 Governing Equations

We consider the string with cross-sectional area A, modulus of longitudinal elasticity E, length l, initial tension T_0 and density ρ, whose upper end ($z = 0$) is fixed and lower end ($z = l$) is harmonically excited as $\delta \cos \nu t$. We put the displacement of the string in the direction parallel to the excitation $\xi(x, t)$ and that in the direction perpendicular to the excitation η. Considering the deflection of a part of the string and setting l to the representative length and l/c_2 representative time, we obtain the dimensionless equations of motion

$$\frac{\partial^2 \xi^*}{\partial t^{*2}} + 2\mu \frac{\partial \xi^*}{\partial t^*} - \frac{\partial^2 \xi^*}{\partial z^{*2}} - \frac{\beta}{2}\int_0^1 \Big\{ \Big(\frac{\partial \xi^*}{\partial z^*}\Big)^2 + \Big(\frac{\partial \eta^*}{\partial z^*}\Big)^2 \Big\} dz^* \cdot \frac{\partial^2 \xi^*}{\partial z^{*2}} = 0, \quad (1)$$

$$\frac{\partial^2 \eta^*}{\partial t^{*2}} + 2\mu \frac{\partial \eta^*}{\partial t^*} - \frac{\partial^2 \eta^*}{\partial z^{*2}} - \frac{\beta}{2}\int_0^1 \Big\{ \Big(\frac{\partial \xi^*}{\partial z^*}\Big)^2 + \Big(\frac{\partial \eta^*}{\partial z^*}\Big)^2 \Big\} dz^* \cdot \frac{\partial^2 \eta^*}{\partial z^{*2}} = 0, \quad (2)$$

where

$$c_1^2 = \frac{E}{\rho}, \quad (3)$$

$$c_2^2 = \frac{T_0}{\rho A}, \quad (4)$$

$$\beta = \frac{c_1^2}{c_2^2} = \frac{EA}{T_0} \quad (5)$$

and μ is damping ratio. The asterisk in the equations means dimensionless quantity. Then, the boundary conditions are written as

$$\begin{cases} \xi^*(0,t^*) = 0,\ \xi^*(1,t^*) = \delta^* \cos \nu^* t^*, \\ \eta^*(0,t^*) = 0,\ \eta^*(1,t^*) = 0. \end{cases} \quad (6)$$

According to Eqs.(1) and (2), ξ^* and η^* couple only by the fourth term in the left hand side of the equation which expresses the nonlinear restoring force of the string. This term produces the third order superharmonic resonance and spatial motion. In the following, we omit the astarisk for simplicity.

3 Analysis by the Method of Multiple Scales

Considering mechanism of the resonance, we can evaluate the excitation amplitude and damping ratio as follows:

$$\delta = \epsilon \hat{\delta}, \quad (7)$$

$$\mu = \epsilon^2 \hat{\mu}, \quad (8)$$

where ϵ is minute parameter and the caret means $O(1)$. Then, we make an assumption about the solutions

$$\xi = \epsilon \xi_1 + \epsilon^3 \xi_3, \quad (9)$$

$$\eta = \epsilon \eta_1 + \epsilon^3 \eta_3, \quad (10)$$

and introduce the multiple time scales

$$t_0 = t, \quad (11)$$

$$t_2 = \epsilon^2 t. \quad (12)$$

Applying the above to Eqs.(1) and (2), we obtain the equations of each order as follows:
$O(\epsilon)$:

$$D_0^2 \xi_1 - \xi_1'' = 0, \quad (13)$$

$$D_0^2 \eta_1 - \eta_1'' = 0, \quad (14)$$

$$\begin{cases} \xi_1(0,t) = 0,\ \xi_1(1,t) = \hat{\delta} \cos \nu t, \\ \eta_1(0,t) = 0,\ \eta_1(1,t) = 0, \end{cases} \quad (15)$$

$O(\epsilon^3)$:

$$D_0^2 \xi_3 - \xi_3'' = -2D_0 D_2 \xi_1 - 2\hat{\mu} D_0 \xi_1, + \frac{\beta}{2}\int_0^1 \Big\{ \Big(\frac{\partial \xi_1}{\partial z}\Big)^2 + \Big(\frac{\partial \eta_1}{\partial z}\Big)^2 \Big\} dz \cdot \xi_1'', \quad (16)$$

$$D_0^2 \eta_3 - \eta_3'' = -2D_0 D_2 \eta_1 - 2\hat{\mu} D_0 \eta_1, + \frac{\beta}{2}\int_0^1 \Big\{ \Big(\frac{\partial \xi_1}{\partial z}\Big)^2 + \Big(\frac{\partial \eta_1}{\partial z}\Big)^2 \Big\} dz \cdot \eta_1'', \quad (17)$$

$$\begin{cases} \xi_3(0,t) = 0,\ \xi_3(1,t) = 0, \\ \eta_3(0,t) = 0,\ \eta_3(1,t) = 0, \end{cases} \quad (18)$$

where $D_0 = \partial/\partial t_0$ and $D_2 = \partial/\partial t_2$. In the following, we focus on the case when the excitation frequency ν is in the neighborhood of a third of the first order natural frequency of the string ω_1. Using the detuning $\sigma = \epsilon^2 \hat{\sigma}$, it is expressed as

$$\nu = \frac{1}{3}\omega_1 + \epsilon^2 \hat{\sigma}. \quad (19)$$

4 Stability Analysis

The above equations of $O(\epsilon)$ give the analytical solution. The vibration of the string in the direction parallel to the excitation includes the frequency component of the excitation frequency and that of three times the excitation frequency. On the other hand, the vibration in the direction perpendicular to the excitation includes the frecuency component of three times the excitation frequency. The amplitude equations obtained from the above equations of $O(\epsilon^3)$ govern the amplitude and the phase difference. Analysing the equations of $O(\epsilon^3)$, we can obtain the steady states and can investigate their stability. Then, the analytical result give the theoretical frequency responce curve which expresses the reration between the steady state amplitude and the detuning of the excitation frequency.

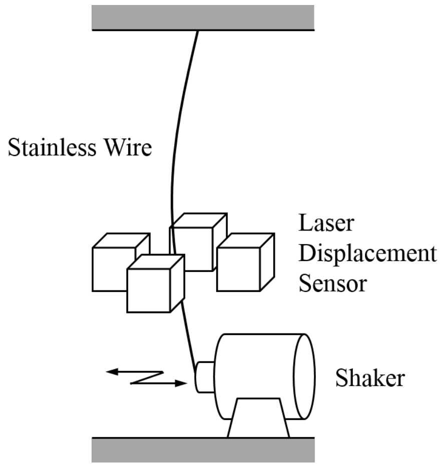

Figure 1. Experimental apparatus

5 Experiment

To inspect appropriateness of the theoretical frequency responce curve, we did experiments. Figure 1 is the experimental apparatus. The upper end of the stainless wire is fixed to the ceiling and the lower end of it is excited by the shaker. The laser displacement sensors mesure the displacement of the wire in the direction parallel to the excitation and that in the direction perpendicular to the excitation. Analysing the mesurement result by fast Fourier transform, we obtain the experimental frequency responce curve. The frequency responce curve obtained in the experiment showed the characteristics of the theoretical frequency responce curve.

6 Conclusion

We analysed the third order superharmonic resonance and the spatial motion in the string whose upper end is fixed and lower end is harmonically excited. First, we showed the equations of motion of the string taking into account the effect of the nonlinear restoring force. Next, we obtained the analytical solutions of the equations and the amplitude equations by the method of multiple scales. Then, by using the amplitude equations, we obtained the theoretical frequency responce curve. Finally, we did the experiment and confirmed appropriateness of the theoretical analytical result.

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