Synchronization in a Distributed System of Moving Chaotic Agents

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***Abstract*— In this paper a system of mobile agents in a two dimensional space, each one carrying a chaotic oscillators, is considered and the related synchronization issues under the framework of time-variant networks are studied. In particular, we show that, if the time scale for the motion of the agents is smaller than that of the associated dynamical systems, the synchronization conditions depend on the agent density on the plane. Numerical results, showing the synchronization properties of a system of $N = 2$ agents, are reported.**

I. Introduction

Collective dynamics emerging from complex networks of simple units has been widely studied in the last decade in different context like bioscience and electronics [1], [2], [3]. In particular the emergence of synchronous behavior of complex dynamics on networks has been widely investigated, in order to find out if and under which conditions it can be obtained.

A case of particular interest is when the nodes of the complex network are dynamical chaotic systems [1] and the edges represent the effective coupling between two chaotic systems. Under this framework, the so-called Master Stability Function (MSF) approach [4], investigating the linearization of the dynamics around the synchronization manifold, provides the conditions under which an arbitrary network of dynamical systems can be synchronized.

In [4] a generic network of N coupled dynamical systems in considered. The dynamics of each node is modelled as $\dot{\mathbf{x}}^i = F(\mathbf{x}^i) - \sigma \sum_j G_{ij} H(\mathbf{x}^j)$ where $i = 1, ..., N$, and $\dot{\mathbf{x}}^i = F(\mathbf{x}^i)$ representing the dynamics of the node. σ is the coupling strength, $H : \mathbf{R}^N \rightarrow \mathbf{R}^N$ the coupling function and $G = [G_{ij}]$ is a zero-row sum matrix modelling network connections.

The synchronization properties of the network are derived by calculating the maximum Lyapunov exponent λ_{max} of the generic variational equation, called the Master Stability Function

$$\dot{\zeta} = [DF - (\alpha + i\beta)DH]\zeta \tag{1}$$

as a function of α and β, where DF and DH represent the Jacobian of $F(\mathbf{x}^i)$ and $H(\mathbf{x}^j)$ computed around the synchronous state. λ_{max}, which does not depend on the specific topology of the network, can be calculated as a function of $\alpha + i\beta$, i.e. $\lambda_{max} = \lambda_{max}(\alpha + i\beta)$, and the stability of the synchronization manifold in the given network can be evaluated verifying that all the eigenmodes related to the eigenvalues γ_h (with $h = 2, \ldots, N$) of the matrix G are stable. Thus, if λ_{max} at the points $\alpha + i\beta = \sigma\gamma_h$ is negative then the synchronous state is stable at the given coupling strength. In fact, we recall that, since G is zero-row sum, the first eigenvalue is $\gamma_1 = 0$ and represents the variational equation of the synchronization manifold. If G is symmetric, it has real eigenvalues, and the MSF can be computed only as function of α.

The functional dependence of λ_{max} on α can give rise to three different cases [1]. The first case (type I) is the case in which λ_{max} is positive $\forall\alpha$ and, thus, the network nodes cannot be synchronized. In the second case (type II) λ_{max} assumes negative values above a threshold value, and then this case always leads to a stable synchronous state for higher enough coupling strength. In the third case (type III), λ_{max} is negative only in a closed interval of values of α: network nodes can be synchronized only if $\sigma\gamma_h$ for $h = 2, \ldots, N$ lie in this interval.

The MSF approach allows the study of the emergence of synchronous behavior in networks with static links. Only recently, the case of complex networks with dynamically changing links has been considered [5], [6], [7], [8]. In particular, commutative graphs have been studied in [8]. The so-called hypothesis of fast switching, introduced in [6], allows to study synchronization when the coupling network switches in short time between all the considered topologies.

In our paper, the case of $N = 2$ mobile agents is considered. The agents and the related oscillators are coupled through a dynamical network on the basis of local interactions. We study the conditions on the model (and in particular on agent motion) for which synchronization of the chaotic oscillator is obtained.

II. Model description

We consider a distributed system made of N agents moving in a planar space of size L with periodic boundary conditions. Each individual is a random walker, i.e. it moves with constant module velocity v and the direction of motion, $\theta_i(t)$, is chosen stochastically. Thus, the position and the orientation of each agent are updated according to:

$$\begin{aligned} \mathbf{x}_i(t + \Delta t_M) &= \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t_M \\ \theta_i(t + \Delta t_M) &= \xi_i(t + \Delta t_M) \ , \end{aligned} \tag{2}$$

where $\mathbf{x}_i(t)$ is the position of the i^{th} agent in the plane at time t, and $\xi_i(t)$ are N independent random variables chosen at each time with uniform probability in the interval $[-\pi, \pi]$.

In addition, individuals can move through the bidimensional world performing long-distance jumps. This is accounted for by defining a parameter p_j that quantifies the

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probability for an individual to perform a long-distance jump. In summary, at each time step, each agent evolves following Eqs. (2) (with $\mathbf{v}_i(t) = (v\cos\theta_i(t), v\sin\theta_i(t))$) with probability $1-p_j$ or performs a jump with probability p_j. In the latter case, the agent jumps into a random position in the plane.

Furthermore, each agent is a dynamical chaotic system. It is then characterized also by a state variable vector $\mathbf{y}_i(t) \in \mathbf{R}^n$ which evolves according to a given chaotic law. In this paper, the case of Rössler oscillators [9] is considered. The state dynamics of each agent is described by the following dimensionless equations:

$$\begin{cases} \dot{x}_i = -(y_i + z_i) \\ \dot{y}_i = x_i + a y_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases} \tag{3}$$

with $\mathbf{y}_i(t) = [\ x_i(t) \quad y_i(t) \quad z_i(t)\ ]^T$. The following parameters have been used: $a = 0.2$; $b = 0.2$; $c = 7$.

Each agent interacts at a given time only with those agents located within an interaction radius, defined as r. When two agents interact, the state equations of each agent are changed to include coupling with the other agent, so that, if for instance the i-th agent is coupled to the j-th agent, the diffusive term $K(x_j - x_i)$ is added to the first equation of (3). K is the coupling parameter.

Thus, the dynamic of each agent can be described in terms of the following equations:

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) - K\sum_{j=1}^{N} g_{ij}(t)\mathrm{E}\mathbf{y}_j \tag{4}$$

for $i = 1, \ldots, N$ with $\mathbf{f}(\mathbf{y}) : \mathbf{R}^n \to \mathbf{R}^n$ given by equations (3) and $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $g_{ij}(t)$ are the elements of a time-varying matrix $G(t)$ which defines the neighborhood of each agent at a given time t and depends on the motion of each agent. More in detail, $g_{ij}(t) = g_{ji}(t) = -1$ if the i-th agent and the j-th agent are neighbor at time t; and $g_{ii}(t) = h$ where h is the number of neighbors of the i-th agent at time t. An example is shown in Fig. 1, where two agent trajectories are reported: at $t = t_0$ agents are not neighbor (each one is not within the interaction radius of the other); at $t = t_1$ the two agents are neighbor.

Equations (4) are integrated with a fixed time step defined as Δt_s (in all the simulations $\Delta t_s = 0.001$). In the following we study how the ratio between this parameter and the integration step of motion Δt_M does influence the synchronization properties of the system.

III. Analysis of the model

The behavior of the model described above is influenced by many parameters, and, in particular, it strongly depends on the agents' velocity v and on the motion integration step size Δt_M. In order to clarify this dependence, we applied the concept of fast switching defined in [10]. Stilwell et al. consider N chaotic oscillators connected through a time varying network and describe the synchronization properties of the system by means of the time-average of the coupling matrix $G(t)$, defined as $\bar{G}$; if $\bar{G}$ supports synchronization of the whole system and if the switching between all the possible network configurations is sufficiently fast, then the time-varying network will lead to synchronization.

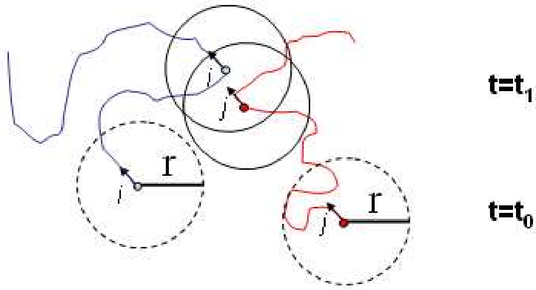

Fig. 1. An example of the trajectories of two random walkers. The interaction radius is shown in dashed line for $t = t_0$ or in solid line for $t = t_1$. During their walk, agents can enter the interaction radius of the other agent (as in $t = t_1$). In this case they are considered neighbor, otherwise the associated chaotic oscillators (for instance, for $t = t_0$) are not coupled.

We discuss here the case of $N = 2$ moving chaotic agents. In this case, two network configurations are only possible at each time t since the two agents can be either neighbor or not. Under the hypothesis of fast switching, we can therefore analyze the synchronization properties of the time-average matrix $\bar{G}$ which is given by:

$$\bar{G} = p_1 G_1 + p_2 G_2 \tag{5}$$

where p_1 is the probability that the two agents are neighbor, while p_2 is the probability that the two agents are not neighbor, and $G_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are the corresponding coupling matrices.

Since $G_2 = 0$, $\bar{G}$ is given by $\bar{G} = p_1 G_1$. Thus, p_1 plays the role of a coupling parameter. The two eigenvalues of $\bar{G}$ are $\lambda_1 = 0$ (since $\bar{G}$ is defined as a zero-row sum matrix) and $\lambda_2 = 2p_1$. Let us consider a typical type III master stability function (i.e., a system for which stability of the synchronization manifold is only guaranteed in a given interval $[\alpha_1, \alpha_2]$) and let us suppose that $K\lambda_2 \in [\alpha_1, \alpha_2]$. Since $0 < p_1 < 1$, this means that there is a critical value of p_1 so that if $p_1 < p_c$ the chaotic dynamics of the two agents will not be synchronized and, on the opposite, if $p_1 > p_c$ the dynamics of the two agents will synchronize. Furthermore, this critical value can be estimated as $p_c = \frac{\alpha_1}{K\lambda_2}$.

In the opposite case, i.e. if $K\lambda_2 > \alpha_2$, the system will behave in a different way: in this case, two critical values of the parameters ($p_1 = \frac{\alpha_1}{K\lambda_2}$ and $p_2 = \frac{\alpha_2}{K\lambda_2}$) can be defined so that the two agents will synchronize only if the parameters are within the range defined by these critical values.

The second point to be discussed is how the hypothesis of fast switching can be assumed in our model. This will also allow us to express the synchronization conditions with respect to the density ρ. The system has two time scales: the motion integration step size Δt_M and the dynamics

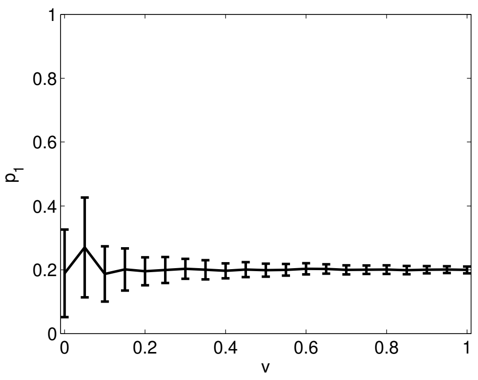

Fig. 2. p_1 vs v. As v increases, the actual value of p_1 tends to its estimation $p_1 = \frac{\pi r^2}{L^2} = 0.2$. Simulation obtained for $\rho = 0.1273$, $N = 2$, $L = \sqrt{\frac{N}{\rho}}$, $r = 1$.

integration step size Δt_s. If the two time scales are chosen to be equal, then at each time step equations (4) will be integrated and a motion step will be performed according to equations (4). Otherwise, if, for instance, Δt_M is chosen to be $\Delta t_M = 10\Delta t_s$, a motion step will be performed each 10 integration step sizes of the dynamics equations (4). In this latter case, the neighborhood of each agent will be updated each 10 integration step sizes Δt_s, and thus the switching between the possible network configurations (G_1 and G_2) will be slower than in the case $\Delta t_M = \Delta t_s$. Fast switching can be therefore implemented acting on the parameter Δt_M.

Another important parameter of the system is the motion velocity v. At the beginning of the simulation, the two agents have random initial positions. Thus, the probability that they are neighbor (i.e. agent 2 is within the interaction radius r of agent 1) is given by $p_1 = \frac{\pi r^2}{L^2}$. If v is small, the two agents will perform small movements during the whole simulation, thus they will remain either close each other or far each other depending on their initial positions. However, when v is large, one expects that the two agents will move much more, so that one may expect that $p_1 = \frac{\pi r^2}{L^2}$ is a good approximation of their probability of being neighbor. Thus, one expects that v influences the value of the probability p_1 in terms of fluctuations: large values of v lead to small fluctuations (as shown in Fig. 2). This allows us to consider for each simulation a small number of runs.

Furthermore, for large v, by taking into account that $\rho = \frac{N}{L^2}$, it derives that p_1 depends on the density ρ as $p_1 = \frac{\pi r^2 \rho}{N}$. Since p_1 is varied as ρ changes, the existence of p_c (in the case of $\alpha_2 < K\lambda < \alpha_2$) also implies the existence of a critical value of the density, defined as ρ_c under which the two agents will not synchronize. The same argument applies when $K\lambda > \alpha_2$.

The velocity v also acts on the switching between the network configurations. As v increases, the distance covered at each motion step by the agents increases and the switching between possible network topologies becomes faster. This leads to the conclusion that for large values of v and low values of Δt_s we expect that the theoretical considerations discussed above correctly predict the synchronization regions of the system.

IV. Numerical results

We numerically studied the system of $N = 2$ mobile agents with $p_j = 0$, in both the cases of identical systems and non-identical systems. In the latter case, we fixed $c = c_1 = 7$ (first agent) and $c = c_2 = 7.1$ (second agent) in equations (3).

For the analysis of the simulations, we defined the following synchronization error [8]:

$$\delta(t) = \frac{|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|}{3} \tag{6}$$

and we defined $\langle\delta\rangle = \langle\delta(t)\rangle$ as synchronization index, where the average is performed on the interval $[4/5T, T]$ ($T = 500s$ is the total length of the simulation).

First of all, we discuss the behavior with respect to ρ. The analysis of the Master Stability Function of the Rössler oscillator coupled through the x variable leads to $\alpha_1 \simeq 0.22$ and $\alpha_2 \simeq 4.08$. In the analysis that follows we define the synchronization region as that region for which $\langle\delta\rangle = 0$ holds. We use α_1 and α_2 to define the expected critical values of the parameters. For instance, if $K = 1$, we expect that the system has only a threshold parameter, so that the two agents synchronize if $\rho > \rho_c$ with $\rho_c = \frac{\alpha_1 N}{\lambda_2 K \pi r^2} = 0.07$, while, when $K = 10$, we expect that synchronization only occurs for $\rho_1 < \rho < \rho_2$ with $\rho_1 = \frac{\alpha_1 N}{\lambda_2 K \pi r^2} = 0.007$ and $\rho_2 = \frac{\alpha_2 N}{\lambda_2 K \pi r^2} = 0.1299$.

In Fig. 3 the behavior of the system with $K = 10$ and with $K = 1$ (inset) is shown. As it will be clarified below, the behavior of the system depends on Δt_M and v: this graph has been obtained by fixing all the parameters of the simulation so that the fast switching hypothesis holds ($\Delta t_M = 10^{-3}$ and $v = 1$). From data we estimate for the case with $K = 1$ $\rho_c = 0.07$ and for the case with $K = 10$ $\rho_1 = 0.008$ and $\rho_2 = 0.1273$, which agree with the theoretical predictions. Thus, depending on the choice of K, two qualitatively different behaviors arise: for $K = 10$ the two agents synchronize if $0.008 = \rho_1 < \rho < \rho_2 = 0.1273$, while when $K = 1$ the two agents synchronize for $\rho > \rho_c = 0.07$.

The importance of the fast switching hypothesis is studied by considering how the behavior of the system changes when Δt_M is varied. For sake of simplicity we focus on the case with $K = 1$, in which, as discussed above, $\rho_c = 0.07$. Two cases are shown in Fig. 4 and Fig. 5 which take into account both the cases of identical or non identical agents. The two cases are $\rho = \bar{\rho} = 0.03$ and $\rho = \tilde{\rho} = 0.12$, which correspond to $\bar{\rho} < \rho_c$ and $\tilde{\rho} > \rho_c$. The analysis carried on in the previous section predicts that the two agents will synchronize for sufficiently fast switching. As it can be observed in Fig. 4, this occurs only for low values of Δt_M. In the case with $\rho < \rho_c$, the two agents do not synchronize even if Δt_M is decreased to its minimum value (i.e. $\Delta t_M = \Delta t_s$). In Fig. 4 and Fig. 5 the role of v is also shown. It can be observed that, if $\rho > \rho_c$, for higher values of v the

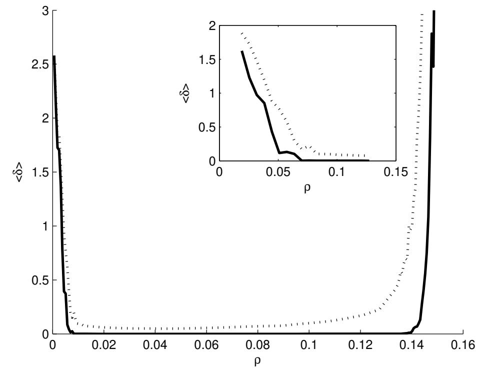

Fig. 3. Synchronization index vs density ρ of agents. When $K = 10$ the two agents synchronize if $0.008 = \rho_1 < \rho < \rho_2 = 0.1273$, while when $K = 1$ (inset) the two agents synchronize for $\rho > \rho_c = 0.07$. The other parameters have been chosen as follows: $\Delta t_M = 10^{-3}$, $v = 1$, $r = 1$, $N = 2$.

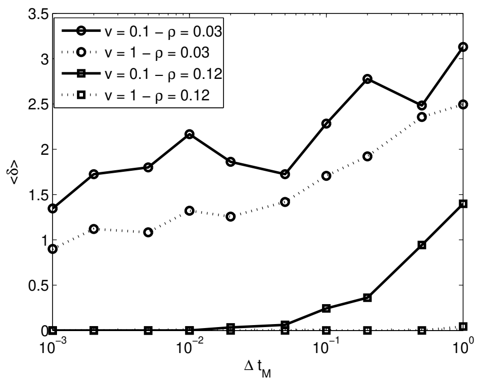

Fig. 4. Synchronization index vs. Δt_M for different values of v and ρ. Curves are related to two identical systems.

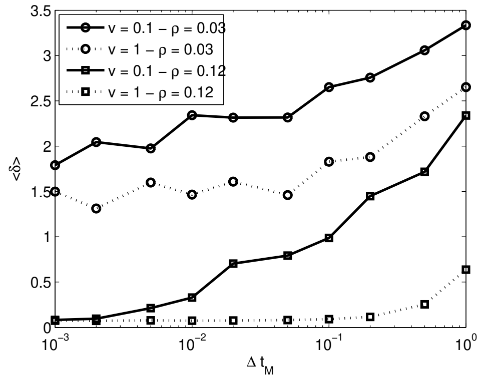

Fig. 5. Synchronization index vs. Δt_M for different values of v and ρ. Curves are related to two non identical systems.

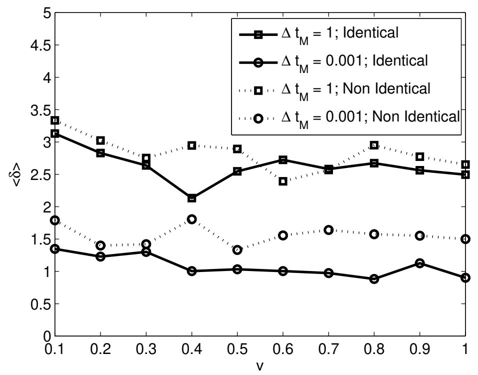

Fig. 6. Synchronization index vs. v for different values of Δt_m and $\rho < \rho_c$, $K = 1$.

synchronization index $\langle\delta\rangle = 0$ even if the integration step of the motion Δt_M increases. Furthermore, in the case $\rho < \rho_c$, higher values of agent velocity lead to smaller values of $\langle\delta\rangle$. When non-identical systems are considered (Fig. 5), a similar effect on the synchronization index due to the higher velocity can be observed, though complete synchronization is never achieved.

We further investigate the dependence of the synchronization index on the velocity of the agents by considering two different cases: $\rho < \rho_c$ and $\rho > \rho_c$. When $\rho < \rho_c$, increasing v has small effects on the synchronization properties of the system. The two agents do not synchronize even for relatively large values of v as shown in Fig. 6.

On the opposite, if $\rho > \rho_c$, the velocity of the two agents can play an important role. If the switching between the two different networks (G_1 and G_2) is sufficiently fast, i.e. if Δt_M is low, then synchronization occurs independently of the value of v. However, if Δt_M is high, the agents synchronize only if their velocity is sufficiently high. This is shown in Fig. 7 for both the cases of identical and non-identical agents. It can be observed that when the two systems are not identical, even if perfect synchronization is not achieved, increasing values of v (when $\Delta t_M = 1$) lead to lower values of the synchronization index.

When the agents may perform long-distance jumps (i.e. for $p_j \neq 0$), the behavior of the system changes. Long-distance jumps may help synchronization as shown in Fig. 8 which reports the synchronization index $\langle\delta\rangle$ vs Δt_M for different values of p_j. In the considered case the theoretical considerations predict synchronization. As discussed above, this occurs under the hypothesis of fast switching and thus only for low values of Δt_M. As it can be observed, when $p_j = 0.1$ synchronization may be achieved at higher values of Δt_M with respect to the case of $p_j = 0$. This is more evident for $v = 0.1$ than for $v = 1$, since the switching rate also depends on v. This is an effect similar to that observed in other systems where the presence of either long-distance jumps [11] or long-distance communications [12] improves communication between agents.

V. Conclusions

In this paper, we have studied a system of $N = 2$ moving agents, each one associated with a chaotic oscillator, where the couplings between the agents vary in time according to their motion. In particular, at a given time instant the two agents are coupled if their positions lie in the interaction

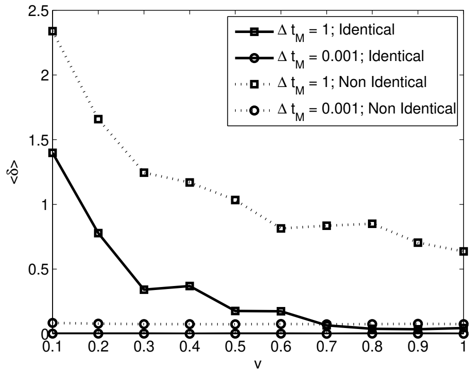

Fig. 7. Synchronization index vs. v for different values of Δt_m and $\rho > \rho_c$, $K = 1$.

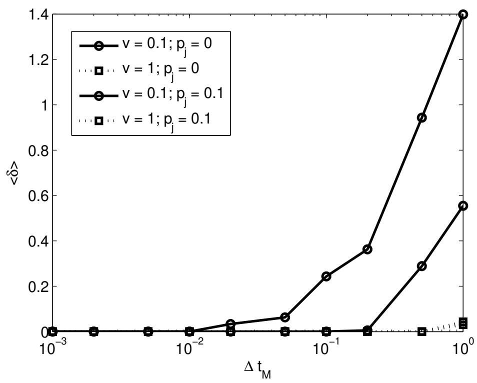

Fig. 8. Synchronization index vs. Δt_M for different values of p_j and v (identical systems).

radius of the other agent. The resulting system can be therefore described by a dynamical network with time-variant links. By applying the fast switching hypothesis, conditions under which synchronization occurs can be derived. We have shown that a critical parameter is the density of the agents. Depending on the value of the coupling parameter two different cases occur: synchronization is achieved for density values greater than a threshold value or for a given interval of values. Increasing the agent velocity v or introducing the possibility of long-distance jumps make faster the switching between the two configurations (coupled agents, not coupled agents). We plan to extend our study to the general case of $N > 2$.

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