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NONSTATIONARY CONSENSUS PROBLEM IN NETWORKS WITH IMPERFECT INFORMATION AND DELAY

Natalia Amelina
Department of Mathematics and Mechanics
Saint Petersburg State University
Russia
ngranichina@gmail.com

Alexander Fradkov
Department of Mathematics and Mechanics
Saint Petersburg State University
Russia
Institute of Problems of Mechanical Engineering
Russia
fradkov@mail.ru

Abstract

In this paper stochastic approximation algorithm in networks with incomplete information about the current state of nodes and changing set of communication links are presented. We consider consensus problem in noisy model with switching topology. For the case when the step size of the algorithm does not tend to zero it is proposed to analyze closed loop system by the method of continuous models (ODE approach or Derevitskii-Fradkov-Ljung (DFL) scheme). The simulation results for workload balancing system are presented.

Key words

Consensus problems, graphs, distributed intelligent network systems, measurement noise, stochastic approximation.

1 Introduction

Distributed coordination in networks of dynamic agents has attracted numerous researchers in recent years. It is mostly due to broad applications of multi-agent systems in many areas, formation control [Fax and Murray, 2004], flocking [Toner and Tu, 1998], distributed sensor networks [Cortes and Bullo, 2003], congestion control in communication networks [Paganini, Doyle and Low, 2001], cooperative control of unmanned air vehicles (UAVs), attitude alignment of clusters of satellites, and others. Many of these problems can be reformulated in terms of achieving consensus in multi-agent systems [Jadbabaie, Lin and Morse, 2003; Olfati-Saber and Murray, 2004; Ren and Beard, 2005].

The solutions of such problems are much more complicated in the practical application. On the one hand, it is because of imperfect information exchange, which is, moreover, usually measured with noise. On the other hand, it is due to the effects of quantization effect common to all digital systems [Aysal and Barner, 2003; Xiao, Boyd and Kim, 2007; Schizas, Ribeiro and Giannakis, 2008; Cucker and Mordecki, 2008; Kashyap, Basar and Srikant, 2007].

In [Huang, 2010] the stochastic approximation algorithm for solving consensus problem was proposed and justified for the group of cooperating agents that communicate with imperfect information in discrete time, under condition of switching topology and delay. Stochastic gradient algorithms were used for such problems before [Tsitsiklis, Bertsekas and Athans, 1986; Huang and Manton, 2009; Kar and Moura, 2009; Li and Zhang, 2009]. Stochastic approximation with decreasing step sizes allows each agent to extract state information from its neighbors while reducing the noise effect.

Under dynamic changes of the external agents states(getting new task, etc.), stochastic approximation algorithms with decreasing step size are not applicable. In [Granichin, Vakhitov and Vlasov, 2010; Vakhitov, Granichin and Gurevich, 2009; Granichin, Gurevich and Vakhitov, 2009; Borkar, 2008] the efficiency of stochastic approximation algorithms with constant step size was studied. Their applicability to the problem of workload balancing in centralized network system where noisy information about workload and productivity of nodes was analyzed in [Granichin, Gurevich and Vakhitov, 2009; Vakhitov, Granichin and Panshenskov, 2009].

In [Rajagopal and Wainwright, 2011; Huang and Manton, 2009] authors consider the consensus averaging problem on graphs with noisy measurements of its neighbors states, under general imperfect communications. They use stochastic approximation-type algorithms with decreasing to zero step size.

In this paper we consider the consensus problem in networks with noisy information (for example, about the workload and productivity, switching topology and delay). Such problem is important for control of production networks, multiprocessor or multicomputer networks, etc.

The paper is organized as follows. Section 2 introduces the basic concepts. In Section 3 the consensus algorithm is described. In Section 4 the main results concerning performance of stochastic approximation algorithm is given. Section 5 presents simulation results in workload balancing system.

2 Preliminaries. Consensus problem on graphs

We explain some notation used in this article following [Olfati-Saber and Murray, 2004; Huang, 2010]. For a matrix A, the element at the ith row and the jth column is called the (i, j)-th element and denoted by a^{ij}. For column vectors $Z_1, \dots, Z_l$, $[Z_1; \dots; Z_l]$ denotes the column vector obtained by vertical concatenation of the l vectors.

To describe the network topology we will use the concepts of graph theory. A directed graph (digraph) $G = (N, E)$ consists of a set of nodes $N = \{1, \dots, n\}$ and a set of directed edges E. An edge is denoted by an ordered pair $(i, j) \in N \times N$, where $i \neq j$. A *directed path* (from node i_1 to node i_s) consists of a sequence of nodes $i_1, \dots, i_s$, $s \geq 2$, such that $(i_k, i_{k+1}) \in E$. The digraph G is *strongly connected* if from any node to any other node, there exists a directed path. The *adjacency matrix* of G is an $n \times n$ matrix $A_G = (a^{i,j})_{1 \leq i,j \leq n}$, where $a^{i,j} = 1$ if $(i, j) \in E$, and $a^{i,j} = 0$ otherwise. If G is an undirected graph, each edge is denoted as an unordered pair (i, j), where $i \neq j$.

The dynamic network topology is modeled by a sequence of digraphs $\{G_t = (N, E_t)\}_{t \geq 0}$, where $N = \{1, \dots, n\}$ and $E_t \subset E$ changes with time. The adjacency matrix A_{G_t} is a matrix-valued variable and completely determines E_t. If $(j, i) \in E_t$, node i receives information from node j which is called a neighbor of node i. The neighbor set of node i is $N_t^i = \{j | (j, i) \in E_t\}$. The neighbor set of subset $N_{\overline{N}}$ is defined by

$$N_{\overline{N}} := \bigcup_{i \in \overline{N}} N_t^i = \{j \in N : i \in \overline{N}, (i, j) \in E\}. \quad (1)$$

Let $x_t^i \in \mathbb{R}$ denotes the state of node i at time $t \in \{0, 1, 2, \dots\}$. We refer to $\mathcal{G}_t = (G_t, X_t)$ with $X_t = [x_t^1, \dots, x_t^n]$, $t \geq 0$ as a *network* with the state $X_t \in \mathbb{R}^n$ and *topology* G_t. The state of a node may represent physical quantities including attitude, position, temperature, voltage, and so on. We say that the nodes i and j *agree* in a network if and only if $x_t^i = x_t^j$. We say the nodes of a network have *reached a consensus* if and only if $x_t^i = x_t^j$ $\forall i, j \in N, i \neq j$. Similar definitions are given for continuous time, $t \in [0, \infty)$.

Let nodes of the graph be dynamic agents, described by difference equations:

$$x_{t+1}^i = x_t^i + f(x_t^i, u_t^i), \ \ i \in N \quad (2)$$

or differential equations

$$\dot{x_t^i} = f^i(x_t^i, u_t^i), \ \ i \in N. \quad (3)$$

Then the *dynamic network* is a dynamical system with a state (G_t, X_t) in which the state evolves according to the *network dynamics* $\dot{X_t} = F(X_t, U_t) = [f^1(x_t^1, u_t^1, \dots, f^n(x_t^n, u_t^n]$, where $U_t = [u_t^1, \dots, u_t^n]$ — vector of control variables.

We assume that at time t, if $N_t^i \neq \emptyset$, node i receives (possibly outdated) noisy information from its neighbors modeled by

$$y_t^{ik} = x_{t-d_t^{ik}}^k + w_t^{ik}, \ \ k \in N_t^i, \quad (4)$$

where w_t^{ik} is the noise and $d_t^{ik} \geq 0$ is an integer-valued random delay. Since the system starts at $t = 0$, the implicit requirement for the neighbor set is that

$$k \in N_t^i \Rightarrow t - d_t^{ik} \geq 0. \quad (5)$$

Each node uses its own state and its noisy measurements to form its control strategy. We call a feedback on observations

$$u_t^i = k_t^i(y_t^{ij_1}, \dots, y_t^{ij_{m_i}}) \quad (6)$$

a *protocol*(*control algorithm*) with topology G_t. The sets $\{j_1, \dots, j_{m_i}\} \in \overline{N}^{i,t} \subseteq \{i\} \cup N_t^i$ $\forall i$ of nodes $j_1, \dots, j_{m_i} \in N$ satisfy property: $\overline{N}^i \subseteq \{i\} \cup N_t^i$. If $|\overline{N}_i| < n$ $\forall i \in N$ then (6) is called a *distributed protocol*.

Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space and we assume that part or all of the above-defined variables, vectors and matrices are random variables. Denote the maximal set of communication links $E_{\max} = \{(k, i) | sup_{t \geq 0} P((k, i) \in E_t) > 0\}$. For convenience of statistical modeling, we make the convention: w_t^{ik} and d_t^{ik} are defined for all $(k, i) \in E_{\max}$. If (k, i) does not appear in E_t so that (4) does not physically occur, we still introduce w_t^{ik} and d_t^{ik} as dummy random variables. If $(k, i) \notin E_t$, we set $d_t^{ik} = 0$. Let $w_t^{ik} | (k, i) \in E_{\max}$ be listed by a fixed ordering of (k, i) to obtain a noise vector W_t of n_1 dimension.

Definition 1: The n nodes are said to achieve *asymptotic mean square consensus* if $E|x_t^i|^2 < \infty, t \geq 0, 1 \leq i \leq n$, and there exists a random variable x^* such that $\lim_{t \to \infty} E|x_t^i - x^*|^2 = 0$ for $1 \leq i \leq n$.

Definition 2: The n nodes to achieve *asymptotic mean square ε-consensus* if $\mathrm{E}||x_t^i||^2 < \infty, t \geq 0, 1 \leq i \leq n$, and there exists a random variable x^* such that $\overline{\lim}_{t \to \infty} \mathrm{E}||x_t^i - x^*||^2 \leq \varepsilon$ for $1 \leq i \leq n$.

3 Consensus protocols

We use two consensus protocols known from the literature [6-13].

1) For fixed or switching topology and zero communication delay the linear consensus protocol is as follows:

$$u_t^i = \sum_{j\in N_t^i} a^{ij}(x_t^j - x_t^i), \qquad (7)$$

where the set of neighbors N_t^i of node i is variable in networks with switching topology.

2) For fixed topology and communication delay $d_t^{ij} > 0$ corresponding to the edge $i, j \in E$ with the linear time-delayed consensus protocol:

$$u_t^i = \sum_{j\in N_t^i} a^{ij}(x^j_{t-d_t^{ij}} - x^i_{t-d_t^{ij}}). \qquad (8)$$

Define the matrix $B_t = (b_t^{i,k})_{1\le i,k\le n}$ as follows [Huang, 2010]. If $N_t^i = \emptyset$, define

$$b_t^{i,k} = 0 \ \ \forall k \in N. \qquad (9)$$

If $N_t^i \neq \emptyset$, define

$$\begin{cases} b_t^{i,k} \in [\underline{b}, \overline{b}], \quad k \in N_t^i \\ b_t^{i,k} = 0, \ k \notin N_t^i \cup \{i\} \\ b_t^{i,i} = -\sum_{k\in N_t^i} b_t^{i,k} \end{cases}, \qquad (10)$$

where $0 < \underline{b} \le \overline{b} < \infty$ — are two deterministic constants. If the sequence $\{G_t\}_{t\ge0}$ changes randomly in time, $\{B_t\}_{t\ge0}$ is a matrix-valued random process.

The above defined two types of consensus protocols apply to the networks of discrete-time model (16). In our case only noisy measurement can be used. Therefore we apply stochastic approximation consensus protocol [Huang, 2010; Stankovic, Stankovic and Stipanovic, 2007]

$$u_t^i = \alpha_t \sum_{j\in N_t^i} b^{ij}(y_t^{ij} - x_t^i), \qquad (11)$$

where $\alpha_t > 0$ are step-sizes.

4 Analysis of the closed loop dynamics

In case when the step size α_t does not tend to zero asymptotic consensus is not achieved in general and reasonable goal is to achieve an approximate consensus ($\varepsilon-$consensus). To analyze system dynamics in this case it is proposed to use the so called *method of continuous models* [Derevitskyand and Fradkov, 1974; Derevitskyand and Fradkov, 1981], (also called ODE approach [Ljung, 1977], or Derevitskii-Fradkov-Ljung (DFL) scheme [Gerencser, 2006]). The DFL scheme consists in the approximate replacement of initial discrete-time stochastic equation

$$X_{t+1} = X_t + \alpha_t \Phi(X_t, W_t), \qquad t = 0, 1, 2, \ldots, T \qquad (12)$$

where $X_t \in \mathbb{R}^n$ — state vector, $W_t \in \mathbb{R}^m$ — random disturbance vector, α_t — gain parameter by the ordinary differential equation

$$\frac{dX}{dt} = \bar{\Phi}(X), \qquad (13)$$

where $\bar{\Phi}(X) = E\Phi(X, W_t)$. It was shown in [Derevitskyand and Fradkov, 1974; Derevitskyand and Fradkov, 1981] under small additional regularity assumptions the trajectories $\{X_t\}$ of (12) are close in the mean-square sense to the trajectories of (13) $\{\bar{X}(\tau_t)\}$, where $\tau_t = \alpha_0 + \cdots + \alpha_{t-1}$ namely,

$$E \max_{0\le\tau_t\le\tau_{max}} \|X_t - \bar{X}(\tau_t)\|^2 \le C_1 e^{C_2\tau_{max}}\alpha, \quad (14)$$

where $\alpha = \max_{1\le t\le T} \alpha_t$, $C_1 > 0$, $C_2 > 0$, α — maximal step size. In [Derevitskyand and Fradkov, 1974; Derevitskyand and Fradkov, 1981] it was shown also that if the continuous time model (13) is exponentially stable then solutions of (12) and (13) are close over infinite time interval:

$$E\|X_t - \bar{X}(\tau_t)\|^2 \le C\alpha^\mu \qquad (15)$$

for some $\mu, 0 < \mu < 1$. The same is valid for systems with switches, if the intervals between switches are separated from zero. Based on the results of [Derevitskyand and Fradkov, 1974; Derevitskyand and Fradkov, 1981] we arrive at the following statement.

Theorem 1: Let the number of switches be finite at each bounded time interval and intervals between switches (dwell time) are separated from zero. Graphs G_t are strongly connected for all t and noise vectors W_t, $t \ge 0$ are centered and independent. Then inequalities (14), (15) hold.

Theorem 1 allows to reformulate the problem of study the dynamics of workload balancing as investigation of the continuous model (13) which can be performed either analytically or numerically.

5 Simulation results for workload balancing problem

To illustrate the theoretical results we give an example of simulation for the workload balancing system of computer network.

We consider the system of separation the same type of tasks between different nodes for parallel computing with feedback. Denote $N = \{1, \ldots, n\}$ as a set of intelligent agents (nodes), each of which serves the incoming requests a first-in-first-out queue. We assume that all agents receive the same type tasks, which can be divided into equal in complexity atomic units. Tasks are received at different times and on different nodes randomly. Size of each task, i. e, its complexity, in seconds is assumed known.

At any time t state of agent i, $i = 1, \dots, n$ is described by two characteristics:

1) q_t^i — queue length of the atomic elementary tasks at time t

2) p_t^i — productivity of the node (the number (or percentage) of carried out atomic tasks in the previous time step, subject to full load).

In what follows we assume p_t^i do not depend on time, i. e. $p_t^i \equiv p^i$. Then the state of the node is defined as $x_t^i = q_t^i$.

Problem for the network of agents is to carry out sequentially received tasks. If all tasks are carried out only by the agent by which they were received. The implementation time of all tasks defined as: $T_m = \max_i q_0^i/p^i$. To minimize the implementation time of all tasks redistribution of tasks among agents should be done.

The new tasks can appear in network over time. New tasks can come directly to any of the n nodes. Each node i at time t can "see" only neighbors of the set N_t^i. Moreover, we assume that graph is strongly connected, i. e., from each node there is a directed path to any other. We assume that the tasks are sent and received at discrete time instants: $t = 0, 1, 2, \dots$. Then the dynamics of each node are described by the following equations:

$$q_{t+1}^{i} = q_t^i - p^i + u_t^i \;\; i = 1, \dots, n, \; t = 0, 1, 2, \dots. \tag{16}$$

At each time t a node i can receive from its "visible" neighbors $j \in N_t^i$ following information:

1) observations about the loading y_t^{ij};

2) p_t^j — productivity of the node.

The problem is to make a protocol of communication between agents, providing that all nodes are equally loaded. It means that if the system will not receive new tasks so all nodes should finish working at the same time. Since q_t^i are random variables the problem is to achieve asymptotic consensus which will be understood in the mean square sense, see Definition 1.

Fig. 1 shows a computing network of the six agents, indicating the possible communication links, some of which may be "closed" and "open up" over time.

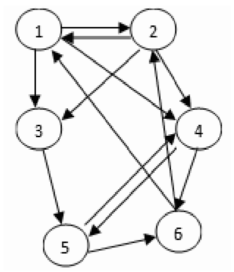

Fig. 1. Maximal set of communication links.

We use stochastic approximation algorithm for consensus problem in the following form:

$$x_{t+1}^i = x_t^i - p^i + \alpha_t \sum_{j \in N_t^i} (y_t^j/p^j - x_t^i/p^i), \tag{17}$$

where α_t is a sequence of positive step sizes. x_t^i denote the state of node i at time t. p^i is a productivity of the node i.

We carry out simulation for the system shown in Fig. 1, consisting of 6 computing blocks. The following initial node workloads were chosen: $x_0^1 = 5000$, $x_0^2 = 3500$, $x_0^3 = 2300$, $x_0^4 = 3150$, $x_0^5 = 7400$, $x_0^6 = 1100$. Productivity of the nodes: $p^1 = 2.5$, $p^2 = 0.5$, $p^3 = 1.7$, $p^4 = 1.1$, $p^5 = 2.7$, $p^6 = 4.2$ and they are not changing in time.

The initial network topology is shown in Fig. 2A. The network topology changes twice over time: Fig. 2B and Fig. 2C.

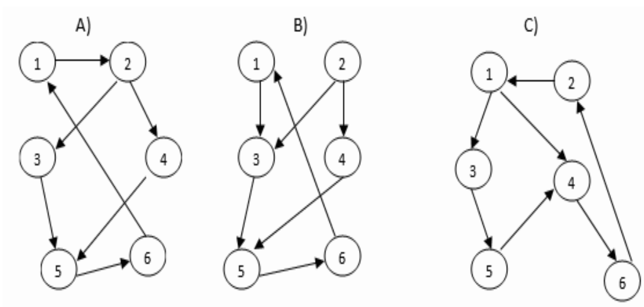

Fig. 2. Network topology.

The system receives new orders in different nodes while working (when $t = 150, t = 450, t = 550$). We use constant step size $\alpha_t = 0.1$. The states of nodes x_t^i are shown in Fig. 3. The network topology changes twice — at time $t = 100$ and in $t = 700$.

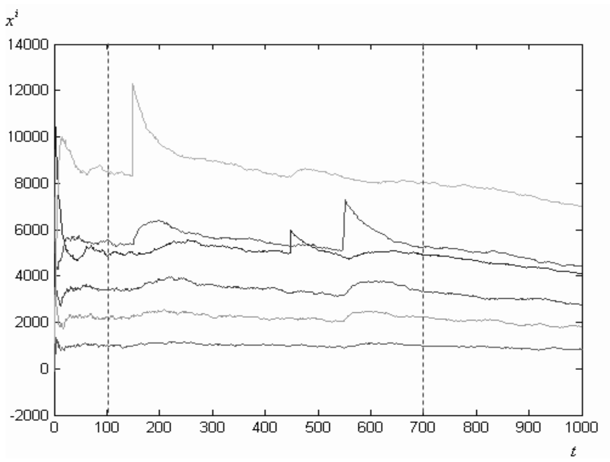

Fig. 3. States of nodes in nonstationary case

In Fig. 4 we see normed state variables for nonstationary case.

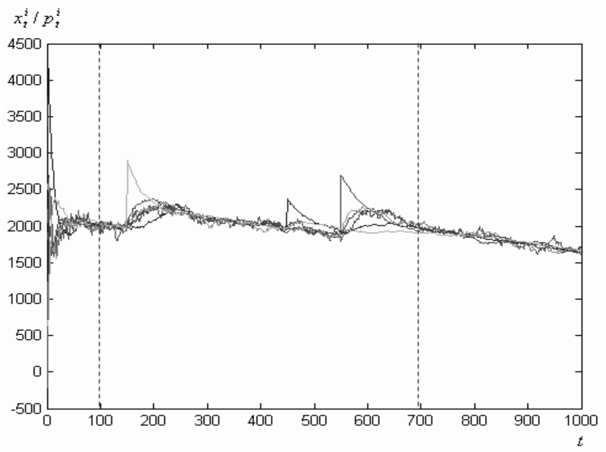

Fig. 4. Normed state variables in nonstationary case

For estimating the asymptotic mean square convergence to consensus of algorithm we denote $Err = \sum_i \sqrt{\frac{(x_t^i/p_t^i - x^*)^2}{n}}$ — characteristic of convergence rate of the algorithm (17). It is shown in Fig. 6.

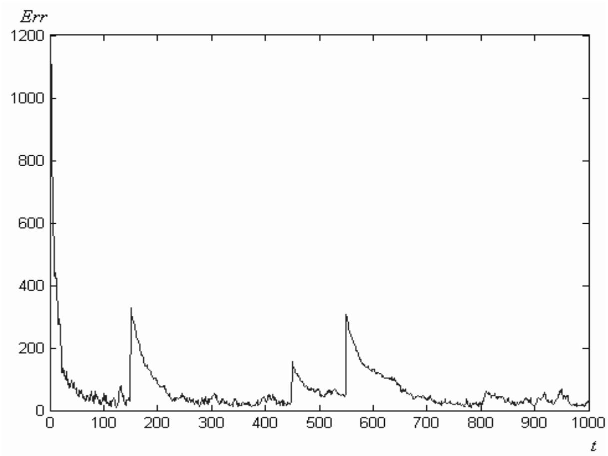

Fig. 6. Asymptotic mean square convergence to consensus

6 Conclusion

In the paper the decentralized workload balancing problem with incomplete information for networks with switching topology is considered.

Stochastic approximation algorithm is used to achieve asymptotic mean square consensus. In the case when the step size of the algorithm does not tend to zero asymptotic consensus is not achieved in general and we considered approximate consensus ($\varepsilon-$consensus). To analyze system dynamics in this case we used the method of continuous models (ODE approach or Derevitskii-Fradkov-Ljung (DFL) scheme).

The simulation results for decentralized workload balancing of computing network system demonstrate good performance of the algorithm. After receiving each new order the algorithm converges in about 100-150 steps.

In future work it would be of interest to analyze the algorithm under the influence of different types of noise. Also attempts will be made to improve the algorithm for use in the case of biased measurement errors.

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