FREQUENCY DOMAIN PERFORMANCE ANALYSIS OF MARGINALLY STABLE LTI SYSTEMS WITH SATURATION

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Abstract

In this paper we discuss the frequency domain performance analysis of a marginally stable linear time-invariant (LTI) system with saturation in the feedback loop. We present two methods, both based on the notion of convergent systems, that allow to evaluate the performance of this type of systems in the frequency domain. The first method uses simulation to evaluate performance, the second method is based on describing functions. For both methods we find sufficient conditions under which a frequency domain analysis can be performed. Both methods are practically validated on an electromechanical setup and a simulation model of this setup.

Key words

Frequency domain performance analysis, convergent systems, describing functions, saturation, marginal stability, practical validation

1 Introduction

For linear systems it is common practice to analyse the performance in the frequency domain. Such an analysis provides valuable information on how the system reacts (in terms of gain and phase) to inputs with various frequencies. That is, it provides insight in how good the system can follow a certain periodic reference signal, and how it reacts on disturbances of a certain frequency.

For nonlinear systems, a similar frequency domain analysis would be very useful as well to indicate the performance of the system. However, such a frequency domain analysis is virtually impossible to perform for nonlinear systems *in general*, due to specific nonlinear behavior, such as the existence of multiple steady-state solutions, non-harmonic responses to harmonic inputs, dependence on initial conditions, etcetera. Nonetheless, for *some* nonlinear systems a frequency domain analysis is possible, as will be demonstrated in this paper. Some other recent results in this field can be found for example in (Jönsson *et al.*, 2003).

In this paper we focus on the class of marginally stable LTI system with saturation in the feedback loop, and discuss conditions under which it is possible to perform a frequency domain analysis for these systems. We describe and demonstrate two different approaches, both based on the notion of convergent systems, that can be used to obtain a frequency domain analysis for these nonlinear systems, i.e. a simulation approach and a describing function approach. It is interesting to note that for these marginally stable LTI systems with saturation, it is impossible to compute a finite L_2-gain between input and arbitrary output using a quadratic storage function, while with the two approaches presented in this paper it is possible to find even more detailed input-output results than an L_2-gain.

Both approaches that we discuss are based on the notion of convergent systems. Convergent systems are, roughly speaking, a class of nonlinear systems with inputs that have a unique bounded globally stable limit solution, which is dependent on the input signal. In the past, quadratic/exponential convergence and quadratically/exponentially convergent design of asymptotically stable systems has been investigated in several publication, see e.g. (Pavlov *et al.*, 2004; Pavlov *et al.*, 2007*a*) and references therein. However, these (design) methods are not applicable if the system is marginally stable. It is only recently that *uniform* convergence was proven for a marginally stable system (van den Berg *et al.*, 2006). If a system is uniformly (or exponentially) convergent, this implies that for each periodic input there is a unique periodic output with the same period as the input, which in turn implies that a (nonlinear) frequency response function can be found (Pavlov *et al.*, 2007*b*). In the first approach, this frequency response function is found using simulation. In the second approach the frequency response function is approximated using the ideas of the describing function method, see e.g. (Khalil, 2002; Rosenwasser, 1969). If a harmonic input is applied to the considered LTI system with saturation, then the describing function can be used to compute a linear approximation of the system, which in turn can be evaluated in the frequency domain. Since it is also possible to compute an upper bound on the error between the linear approximation and the original nonlinear system, see (van den Berg *et al.*, 2007), an interval can be indicated within which the frequency response function of the nonlinear system lies. Both approaches are practically validated on an experimental setup (an electromechanical system) and a simulation model of this setup.

The outline of this paper is as follows. Section 2 presents the class of LTI systems with saturation that is considered throughout this paper, and the electromechanical system that is used as a validation case. Furthermore it is demonstrated by means of an example why frequency domain analysis can not be performed for nonlinear systems in general. Section 3 deals with the notion of convergent systems and discusses how for convergent systems the simulation approach can lead to a frequency domain performance analysis. In Section 4 it is discussed how the describing function approach can lead to a frequency domain performance analysis. The results in Sections 3 and 4 are validated using the electromechanical system. Finally, Section 5 concludes the paper.

2 LTI system with saturation

In this section, we first describe the class of systems that is considered throughout this paper. Then, we introduce an electromechanical system within this class of systems, that will be used as a case study to practically validate the theoretical results discussed in this paper. Finally, we show by means of an example that this system can –under certain settings– exhibit rich nonlinear dynamics, which make a frequency domain performance analysis virtually impossible. Based on these observations, we make some statements on the conditions that a nonlinear system should satisfy in order to allow frequency domain analysis. These statements will be elaborated in Sections 3 and 4.

2.1 System description

In this paper we consider the type of systems visualized in Figure 1. Here, the plant dynamics are given by

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p \mathrm{sat}(u) \\ y_p &= C_p x_p \end{aligned}$$

where A_p has one eigenvalue at 0 and the other eigenvalues (if any) in the open left-hand plane, i.e. the plant is marginally stable. The controller dynamics are given

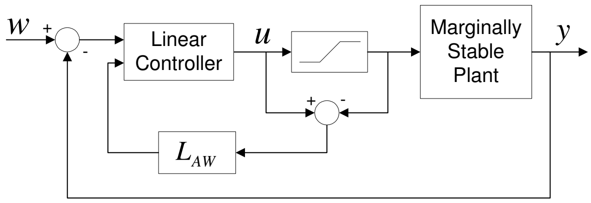

Figure 1. LTI System with marginally stable plant.

by

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c(w - y_p) + L_{AW}\,(\mathrm{sat}(u) - u) \\ u &= C_c x_c + D_c(w - y_p) \end{aligned}$$

in which L_{AW} is a static anti-windup gain, and the saturation function is defined as $\mathrm{sat}(u) = \mathrm{sign}(u)\min(1, |u|)$.

The closed-loop dynamics of this system can be written in Lur'e form

$$\begin{aligned} \dot{x} &= Ax + B\mathrm{sat}(u) + Fw \\ u &= Cx + Dw \\ y &= Hx \end{aligned} \tag{1}$$

with state $x = [\ x_p,\ x_c\ ]^T \in \mathbb{R}^n$, input $w \in \mathbb{R}$, performance output $y \in \mathbb{R}$, matrix H to be defined freely, and

$$\begin{gathered} A = \begin{bmatrix} A_p, & 0 \\ L_{AW} D_c C_p - B_c C_p, & A_c - L_{AW} C_c \end{bmatrix}, \\ B = \begin{bmatrix} B_p \\ L_{AW} \end{bmatrix},\ \ F = \begin{bmatrix} 0 \\ B_c - L_{AW} D_c \end{bmatrix}, \\ C = \begin{bmatrix} -D_c C_p, & C_c \end{bmatrix},\ \ D = \begin{bmatrix} D_c \end{bmatrix}. \end{gathered}$$

Although the theory that we present in Sections 3 and 4 applies to the whole class of systems described by (1), we focus in this paper on a case that has been investigated by means of simulation and real-time experiments in order to validate the theoretical findings. This case is discussed in the following subsection.

2.2 Case: electromechanical system

As a special case of (1), we consider the electromechanical system (see Figure 2) that is shown schematically in Figure 3. The hardware consists of two rotating rigid bodies (masses) connected by an element that has a certain stiffness and damping. The first body is driven by an actuator (brushless DC motor) and the rotation of the second body is measured by a sensor (incremental encoder, 8192 counts/revolution). The hardware is connected (at sample rate: 1 kHz) to a computer with a Matlab Simulink model (Real Time Workshop), which contains a PI controller, a saturation function and a static anti-windup gain as shown in Figure 3. The

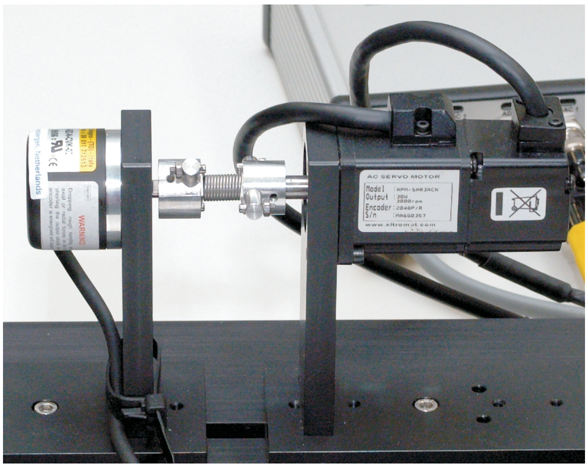

Figure 2. Case: electromechanical system (photo). From left to right: encoder, body 2, spring/damper, body 1, motor+encoder

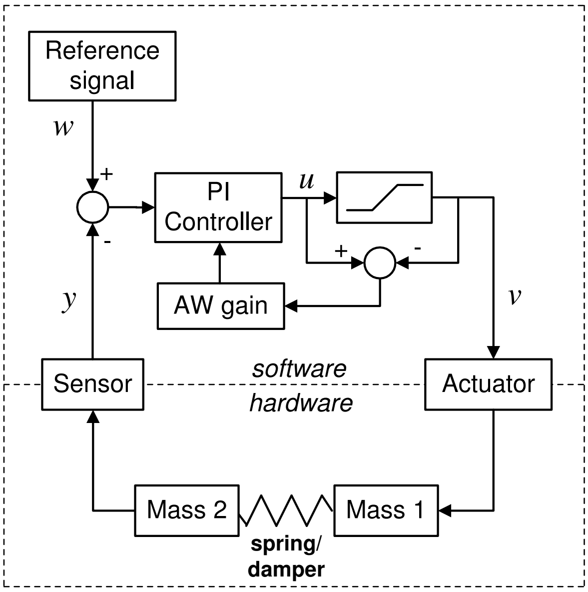

Figure 3. Case: electromechanical system (schematic).

actuator is driven by a velocity controller (not shown in Figure 3), which receives its reference value v from the Simulink model. The settling time of the velocity controller is negligible, so that we can assume that the actuator exactly follows the reference velocity v.

In order to perform also simulations on this case, the parameters of the electromechanical system have been identified and a simulation model has been created. The model is of the form (1) with $x_p = [r_1,\ r_2,\ \dot{r}_2]^T$ (r_i denoting the rotation angle [revolutions] of body i)

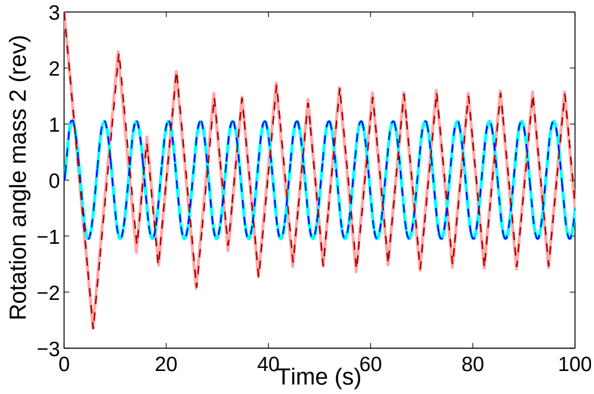

Figure 4. Example 1: multiple steady-state solutions (experiments: dashed lines, simulations: solid lines).

and matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3.9\cdot 10^3 & -3.9\cdot 10^3 & -10.7 & 0 \\ 0 & -1+L_{AW}K_P & 0 & -L_{AW}K_I \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 10.7 \\ L_{AW} \end{bmatrix},\quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1-L_{AW}K_P \end{bmatrix}, \qquad (2)$$

$$C = \begin{bmatrix} 0 & -K_P & 0 & K_I \end{bmatrix},\quad D = \begin{bmatrix} K_P \end{bmatrix},\quad H = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

where K_I, K_P and L_{AW} are controller parameters to be chosen.

2.3 Motivating example: nonlinear behavior

Although it is common practice for linear systems to evaluate the performance in the frequency domain, e.g. steady-state response (gain, phase) to harmonic inputs, for nonlinear systems in general this is not possible. In this subsection two examples are given that clearly indicate what difficulties arise when trying to make a frequency domain analysis of the steady-state response of nonlinear systems.

For the first example, consider the system (1), (2) with $K_I = 20$, $K_P = 8$, $L_{AW} = 0$, and $w = \sin(t)$. We evaluate the solution of this system for two initial conditions, i.e. $x(0) = [0,\ 0,\ 0,\ 0]$ and $x(0) = [3,\ 3,\ 0,\ 0]$, using both the experimental setup (see Figure 3) and simulation. The resulting rotation angle of body 2 as a function of time is given in Figure 4.

For the second example, consider again the system (1), (2) with $K_I = 20$, $K_P = 8$, $L_{AW} = 0$, but now with $w = 5\sin(t)$. In this example we show that different initial conditions can not only lead to different 1-periodic steady-state solutions, but also to multi-periodic steady-state solutions. For four initial conditions the solution of the system is evaluated using both the experimental setup and simulation. The control output u, which clearly displays the multi-periodic solutions, is shown as a function of time in Figure 5.

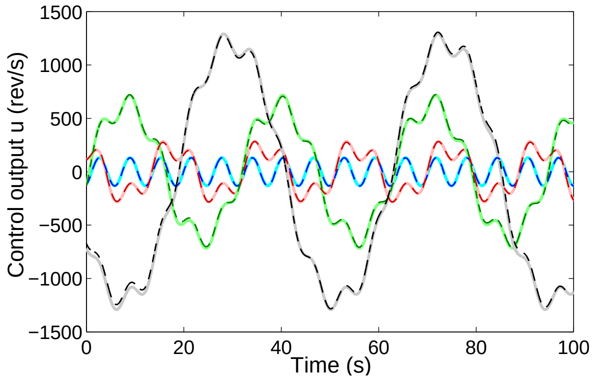

Figure 5. Example 2: multi-periodic steady-state solutions (experiments: dashed lines, simulations: solid lines).

Since frequency domain analysis is based on a one-to-one mapping from input signal (e.g. reference or disturbance) to output signal (e.g. error signal or performance output) of a system, we need to guarantee that for each input signal a unique output signal exists, which has the same period as the input signal and which is independent of the initial conditions. As shown in this subsection, the output signal of a nonlinear system, however, does not necessarily satisfy these conditions, i.e. multiple 1-periodic steady-state solutions (see Figure 4) or multi-periodic steady-state solutions (see Figure 5) may exist. This motivated us to investigate whether there exist conditions under which the steady-state performance of nonlinear system (1) *can* be analyzed in the frequency domain.

Two approaches have been found that allow to find sufficient conditions under which a frequency domain analysis can be performed for system (1), i.e. a simulation-based approach and a describing function approach, which are both based on the notion of convergent systems. These approaches will be discussed in respectively Section 3 and Section 4.

3 Convergent systems and simulation-based frequency domain analysis

In this section, we first give a definition and some properties of convergent systems. Then, we discuss the conditions under which system (1) is uniformly convergent. Finally, we show how to perform a simulation-based frequency domain analysis for the convergent system.

3.1 Convergent systems

Consider the following class of systems,

$$\dot{x}(t) = f(x, w(t)) \tag{3}$$

with state $x \in \mathbb{R}^n$ and input $w \in \overline{\mathbb{PC}}$. Here, $\overline{\mathbb{PC}}$ is the class of bounded piecewise continuous inputs $w(t) : \mathbb{R} \to \mathbb{R}^m$. Furthermore, assume that $f(x, w)$ satisfies some regularity conditions to guarantee the existence of local solutions $x(t, t_0, x_0)$ of system (3) for any input $w \in \overline{\mathbb{PC}}$.

Definition 1. *System* (3) *is said to be* uniformly convergent *for a class of inputs $\mathcal{W} \subset \overline{\mathbb{PC}}$ if for every input $w(t) \in \mathcal{W}$ there is a solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ satisfying the following conditions:*

1. *$\bar{x}(t)$ is defined and bounded for all $t \in (-\infty, +\infty)$,*
2. *$\bar{x}(t)$ is globally uniformly asymptotically stable for every input $w(t) \in \mathcal{W}$.*

The solution $\bar{x}(t)$ is called a *limit solution*. As follows from the above definition, any solution of an uniformly convergent system 'forgets' its initial condition and converges to a limit solution which is independent of the initial conditions. The following statements describe some properties of this limit solution.

Property 1. *(Pavlov* et al.*, 2007*a*) For a uniformly convergent system, the limit solution $\bar{x}(t)$ is unique, i.e. it is the only solution bounded for all $t \in (-\infty, +\infty)$.*

Property 2. *(Pavlov* et al.*, 2006) Suppose system* (3) *is uniformly convergent. Then, if the input $w(t)$ is constant, the corresponding limit solution $\bar{x}(t)$ is also constant. If the input $w(t)$ is periodic with period T, then the corresponding limit solution $\bar{x}(t)$ is also periodic with the same period T.*

Finally note that a system is called *exponentially convergent* for a class of inputs $\mathcal{W} \subset \overline{\mathbb{PC}}$ if it is uniformly convergent and $\bar{x}(t)$ is globally exponentially stable for every input $w(t) \in \mathcal{W}$.

3.2 Convergent system design

Consider again the system in Figure 1 with the marginally stable plant, as described by (1). Theorem 1 provides conditions under which this system is uniformly convergent.

Theorem 1. *Assume the following conditions hold:*

1. *$A_c - L_{AW}C_c$ is Hurwitz*
2. *$v_l BC v_r < 0$, where v_l and v_r are respectively the left- and right eigenvector of A corresponding to the eigenvalue $\lambda = 0$*
3. *A Lyapunov matrix $P = P^T > 0$ exists for which:*
   $PA + A^TP \leq 0$
   and
   $P(A + BC) + (A + BC)^TP < 0$

Then, system (1) *is uniformly convergent for all $w \in \mathcal{W}$, where $\mathcal{W}$ is the class of uniformly continuous bounded inputs. Furthermore, for any compact set Ω, if the initial condition $x(0) \in \Omega$ then the system has an exponential convergence rate for all $w \in \mathcal{W}$.*

Proof. See (van den Berg, 2008).

Note that if there exists a Lyapunov matrix $P = P^T > 0$ such that $PA + A^TP < 0$ and $P(A + BC) + (A +$

$BC)^T P < 0$ hold (instead of condition 3), then the corresponding system can be proven to be exponentially convergent for all $w \in \overline{\mathbb{PC}}$, and conditions 1 and 2 of Theorem 1 and uniform continuity of w are not even required. However, the system we consider has a marginally stable plant thus $PA + A^T P < 0$ can not be satisfied.

3.3 Performance analysis in frequency domain

Under the conditions of Theorem 1 system (1) is uniformly convergent, which implies that for any input $w \in \mathcal{W}$ the system has a unique limit solution $\bar{x}$ and thus a unique output $\bar{y}$. That is, if we apply a harmonic input signal with period T to the system, then the limit output $\bar{y}$ is unique (i.e. independent of initial conditions) and has period T. Thus, we can find a kind of frequency response function if we evaluate the input-output behavior for a range of frequencies. Since the output signal is not necessarily harmonic, however, we can not obtain a typical gain and phase plot (Bode plot) as for linear systems. Instead, we determine a nonlinear frequency response function, as discussed in (Pavlov *et al.*, 2007*b*), i.e. for the convergent system we determine the gain between the RMS (root mean square) value of the input signal and the RMS value of the limit output signal. As phase is not defined for nonlinear systems, only the gain as discussed above will be considered in our frequency domain analysis.

Note that the periodic output $\bar{y}$ can easily be determined using *simulation* (or real-time experiments). Since the limit solution of a convergent system only depends on the input and is independent of the initial conditions, a single simulation run (experiment) suffices to determine the limit solution of the system. This is a major difference with 'non-convergent' nonlinear systems, for which in principle all initial conditions should be evaluated (i.e. an infinite amount of simulations) to obtain a reliable analysis.

This simulation-based frequency domain analysis is now demonstrated for system (1), (2). By choosing $K_I = 20$, $K_P = 8$, and $L_{AW} = 0.5$ all conditions of Theorem 1 are met, and thus the system is uniformly convergent for all inputs $w \in \mathcal{W}$. Evaluating the solution for the inputs signals $w = b\sin(\omega t)$ for $b = 1$ and $\omega \in [10^{-1}, 10^2]$, and computing the 'complementary sensitivity' gain ($RMS_{\bar{y}(t)}$ / $RMS_{w(t)}$) results in the frequency response function shown in Figure 6. Any other desired frequency response function can be computed in a similar way.

For $\omega > 10$ rad/s, the experiments and simulation give different results. Due to the relatively high frequency in combination with the saturation function, the amplitude of the motion of the rigid bodies becomes so small that nonlinear behavior of the experimental setup becomes significant, which in turn results in a different RMS-gain. However, since dealing with the undesired nonlinear behavior of the experimental setup lies outside the scope of this paper, it will not be discussed further here. In the remainder of this paper, we will focus on the dynamics as described by the simulation model.

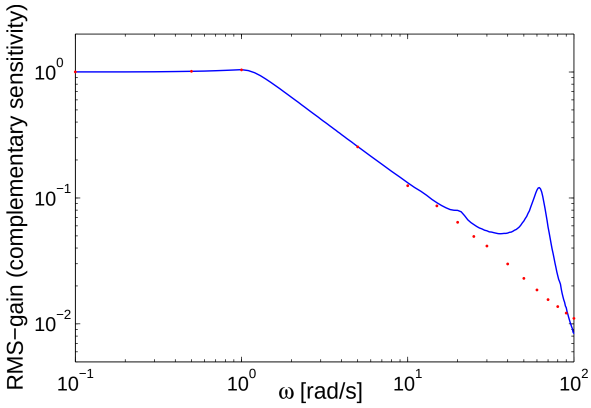

Figure 6. Nonlinear frequency response function (experiments: dots, simulations: solid line).

Figure 6 provides valuable information on the frequency domain performance of the system. It clearly shows how the limit output $\bar{y}$ behaves under input signals for respectively low and high frequencies. A similar plot can be made for the 'sensitivity' gain ($RMS_{w(t)-\bar{y}(t)}$ / $RMS_{w(t)}$) to investigate for example tracking behavior.

Note, however, that the computed frequency response function in Figure 6 is only valid for harmonic input signals with amplitude $b = 1$. For other input amplitudes the frequency response function can be computed as well, but will be different since the limit solution $\bar{y}$ does not only depend on frequency ω but also on amplitude b. For the same reason, the superposition principle does not hold. On the other hand, computing the frequency response function for any multi-harmonic input signal is as simple as computing this function for a harmonic input signal, so the steady-state response to any periodic input can be obtained by this approach.

Furthermore, note that even if we were able to find a finite L_2-gain for this marginally stable system, this would only be a horizontal line in Figure 6, i.e. an upper bound for the frequency domain performance. Our approach based on convergence and simulation provides more detailed information on the frequency domain behavior of the system.

Although this approach leads to an exact performance analysis in the frequency domain, it can be very time-consuming, since the limit solution $\bar{x}(t)$ has to be obtained by simulation (or a real-time experiment): transient behavior should be ruled out by simulating long enough, and the simulation should be performed with high accuracy.

In the following section we will consider another approach, based on the describing function method, which is much more time-efficient, but at the cost of accuracy, i.e. instead of an exact solution, an upper- and lower bound on the performance are given. Also, this approach can only be used for harmonic input signals.

4 Frequency domain analysis based on describing function approach

In this section, we first give a short overview of the describing function method and explain how this theory can be expanded for application to convergent nonlinear systems with harmonic inputs. Then, we discuss a theorem which gives sufficient conditions for computation of a linear approximation and upper- and lower bound of the error of this approximation. Finally, we apply the theory on the system (1), (2).

4.1 Describing function method

Following the describing function method, see e.g. (Khalil, 2002; Rosenwasser, 1969), the limit solution $\bar{x}$ of system (1) is approximated by a periodic limit solution $\bar{\xi}$ of the linear system

$$\begin{cases} \dot{\xi} = A\xi + BK\zeta + Fw \\ \zeta = C\xi + Dw \\ \eta = H\xi \end{cases} \quad (4)$$

in which gain K is to be determined. If the matrix $A + BKC$ does not have eigenvalues on the imaginary axis then for a periodic input $w(t) = b\sin(\omega t)$ the system has a unique periodic limit solution $\bar{\xi}(t)$, and thus a unique periodic limit output $\bar{\zeta}(t)$, which can be described by

$$\bar{\zeta}(t) = a\sin(\omega t + \psi), \quad (5)$$

for some amplitude $a > 0$ and phase ψ. In the process of harmonic linearization gain K is chosen to minimize the following criterion

$$J := \frac{1}{T}\int_0^T [\mathrm{sat}(\bar{\zeta}(t)) - K\bar{\zeta}(t)]^2 dt,$$

where $\bar{\zeta}(t) = C\bar{\xi}(t) + Dw(t)$ and T is the period of input w. The optimal gain K can be found by solving the condition

$$\frac{dJ}{dK} = 0$$

and is given by

$$K = \left(\int_0^T \bar{\zeta}^2(t)dt\right)^{-1}\int_0^T \mathrm{sat}(\bar{\zeta}(t))\bar{\zeta}(t)dt.$$

Applying the fact that the saturation nonlinearity is an odd function and filling in (5), this simplifies to

$$K(a) = \frac{1}{\pi a}\int_0^{2\pi} \mathrm{sat}(a\sin\theta)\sin\theta d\theta,$$

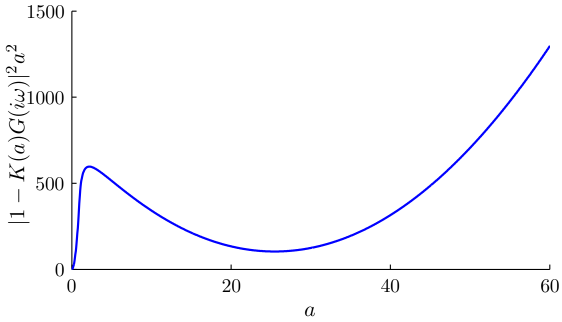

Figure 7. Left hand side of (6) for system (1), (2) with $K_I = 20, K_P = 8, L_{AW} = 0, \omega = 1$.

which leads to the describing function:

$$K(a) = \begin{cases} 1, & a \le 1 \\ \frac{2}{\pi}\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{1-\frac{1}{a^2}}\right), & a > 1 \end{cases}$$

Under the assumption that A does not have eigenvalues $\pm i\omega$, the value of amplitude a can be determined by solving the so-called harmonic balance equation, which for system (4) is given by

$$|1 - K(a)G(i\omega)|^2a^2 = |C(i\omega I_n - A)^{-1}F + D|^2b^2 \quad (6)$$

with $G(i\omega) = C(i\omega I_n - A)^{-1}B$. Note that the left-hand side of (6) is a nonlinear function of a, and the value of the right-hand side of (6) depends on the input amplitude b and input frequency ω. Therefore, if we want to solve this equation for a, there may exist multiple solutions of a for one pair of (b,ω), see e.g. Figure 7. In this Figure we plotted the left-hand side of (6) as a function of a, and if for example $|1-K(a)G(i\omega)|^2a^2 = |C(i\omega I_n - A)^{-1}F + D|^2b^2 = 300$ for some pair of (b,ω), then multiple solution of a exist. If, on the other hand, there *is* a unique positive real solution $a(b,\omega)$ for a given pair of (b,ω), we can easily compute the limit solution $\bar{\xi}(t)$ of (4) by filling in $K(a(b,\omega))$, and compute how accurate this solution $\bar{\xi}(t)$ approximates $\bar{x}(t)$. However, if the solution $a(b,\omega)$ is not unique positive and real, e.g. there are multiple solutions for a, then this approach is not applicable.

Denote

$$\rho_1 := \sup_{k=3,5,\ldots} |C(ik\omega I_n - A - \frac{1}{2}BC)^{-1}B|$$

$$\rho_2 := \sup_{k=3,5,\ldots} |H(ik\omega I_n - A - \frac{1}{2}BC)^{-1}B|$$

$$\gamma = \frac{2\rho_2}{2-\rho_1}$$

$$v(a) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{2}{\pi} \int_0^{\pi} \mathrm{sat}(a \sin\theta) \sin\theta d\theta \cdot \sin\vartheta \right.\right.$$
$$\left.\left. - \mathrm{sat}(a \sin\vartheta) \right]^2 d\vartheta \right)^{\frac{1}{2}}.$$

to be used in the following theorem.

Theorem 2. *Consider system* (1) *with periodic input* $w(t) = b\sin(\omega t)$ *and assume the following conditions are met*

1. (A, B) *is controllable,* (A, C) *is observable,*
2. *the harmonic balance equation* (6) *has a unique positive real solution* $a(b, \omega)$*,*
3. $\rho_1 < 2$*,*
4. *for the linear system* (4) *with* $K = K(a(b, \omega))$ *where* $a(b, \omega)$ *is the unique positive real solution of* (6)*, the matrix* $A + BKC$ *does not have eigenvalues on the imaginary axis.*

Then system (1) *has a unique* $2\pi/\omega$*-periodic solution* $\bar{x}(t)$ *and the error between the limit output* $\bar{y}(t)$ *and* $\bar{\eta}(t)$ *is bounded by:*

$$\left(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left[\bar{y}(t) - \bar{\eta}(t) \right]^2 dt \right)^{\frac{1}{2}} \leq \gamma v(a(b, \omega)). \tag{7}$$

Proof. See (van den Berg *et al.*, 2007).

Note that under the conditions in Theorem 2, the system is not necessarily convergent. Although the system is guaranteed to have a unique $2\pi/\omega$-periodic solution $\bar{x}(t)$ under the given conditions, periodic solutions with a different period may exist. An example in which there is a unique positive real solution to the harmonic balance equation, but multiple steady-state solutions exist, is given in Figure 5. In order to make sure that the linear approximation (4) and error bounds (7) actually describe the only solution of the nonlinear system, one needs to prove convergence of the system first.

4.2 Performance analysis example

To demonstrate the use of Theorem 2 consider again system (1), (2), with $K_I = 20$, $K_P = 8$, $L_{AW} = 0.5$, and $w(t) = b\sin(\omega t)$ with $b = 1$ and $\omega \in [10^{-1}, 10^2]$. From Section 3 we know that this system is convergent. Instead of performing many time-consuming simulations, we now simply compute the linearization and error bounds for the given range of frequencies using the approach given in Subsection 4.1. The result is given in Figure 8. For comparison, the results of the simulation approach, and the gain of the linear system, i.e. system (1) with $\mathrm{sat}(u) = u$, are plotted as well.

As one can see, the exact results as obtained with the simulation approach lie well within the error bounds of the approximation obtained by the describing function approach. Since the error bounds are relatively small for this case, the describing function approach gives a quite accurate description of the frequency domain behavior of nonlinear system (1), (2). It can also be clearly seen that the frequency domain behavior of the system with saturation substantially differs from the system without saturation.

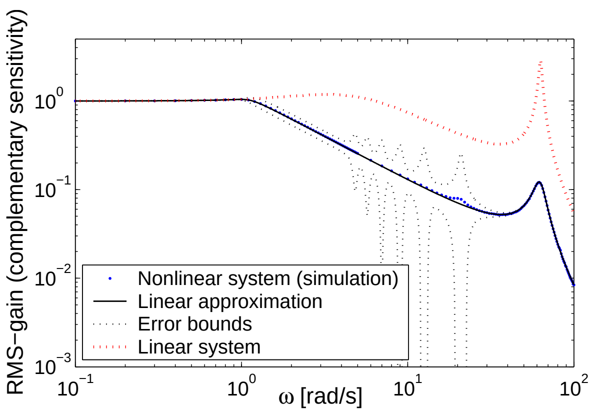

Figure 8. Frequency domain results of describing function approach in comparison with simulation approach and linear system.

5 Conclusion

Two approaches have been described that can be used to obtain a frequency domain performance analysis for a class of marginally stable LTI systems with saturation. The first step in both approaches is to prove/obtain convergence of the system. Then, the simulation approach leads to an exact performance analysis, but can be time-consuming. The other approach, based on the describing function method, is computationally much faster, but at the cost of some accuracy: only an upper- and lower bound can given on the performance of the nonlinear system, although these bounds can be very close. An electromechanical system has been used as a case to demonstrate and practically validate both approaches.

Acknowledgements

This work was partially supported by the Dutch-Russian program "DyCoHyMS" (NWO grant 047.017.018).

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