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FREQUENCY-ALGEBRAIC CONDITIONS FOR STABILITY OF PHASE SYSTEMS WITH APPLICATION TO PHASE-LOCKED LOOPS AND SYNCHRONIZATION SYSTEMS

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Abstract

Global asymptotic behavior of control systems with periodic vector nonlinearities and denumerable sets of equilibria is investigated. Multidimensional systems described by ordinary differential equations, distributed systems described by integrodifferential Volterra equations and discrete systems described by difference equations are examined. New kinds of Lyapunov-type functions and Popov-type functionals are offered. New frequency-domain criteria for gradient-like behavior of the systems are obtained. They are applied to stability investigation of phase-locked loops and to the problem of self-synchronization of two rotors.

Key words

Phase system, gradient-like behavior, Lyapunov-type function, Popov-type functional, frequency-domain criterion.

1 Introduction

Nonlinear systems with non-unique equilibria are widespread among control systems, mechanical systems, electrical and radio-engineering systems. The qualitative analysis of various systems with non-unique equilibria generated a number of new stability problems and new Lyapunov-type theorems.

This paper is devoted to systems with denumerable equilibria set and periodic nonlinear functions. They are often called phase systems.

The stability of multidimensional phase systems was for the first time investigated in [Yakubovich,Leonov and Gelig, 2004], where two types of stability characteristics of phase systems are considered. They are Lagrange stability and gradient-like behavior, which means that every solution of the system tends to a certain equilibrium state as the argument-time goes to infinity. In [Yakubovich,Leonov and Gelig, 2004] new classes of Lyapunov functions specially constructed for phase systems were introduced. They gave the opportunity to establish a number of sufficient conditions for Lagrange stability and gradient-like behavior of the systems. These conditions have often the form of frequency-domain inequalities with varying parameters.

In particular in [Yakubovich,Leonov and Gelig, 2004] the method of periodic Lypunov functions which had been introduced in [Bakaev and Guzh, 1965] for the systems of third order, was extended for multidimensional systems. The technique for constructing periodic Lyapunov functions (it is often called Bakaev-Guzh technique) was then developed and generalized subsequently in [Leonov, Ponomarenko and Smirnova, 1996] and in [Perkin, Smirnova and Shepeljavyi, 2009]. It gave a set of frequency-algebraic conditions for gradient-like behavior of phase systems.

By means of special Lyapunov-type sequences all the stability theorems for autonomous multidimensional systems proved before 1996 were extended to discrete phase systems [Leonov and Smirnova, 2000], [Karpichev, Koryakin, Leonov and Shepeljavyi, 1990]. With the help of the method of a priori integral estimates and the Popov-type functionals they were extended to infinite- dimensional phase systems [Leonov, Ponomarenko and Smirnova, 1996]. Periodic Lyapunov-type sequences and Popov-type functionals destined for discrete and distributed phase systems are generated by the same technique as periodic Lyapunov functions for lumped systems.

In this paper a certain modification of Bakaev-Guzh technique is offered and as a result a new frequency-algebraic stability theorem for multidimensional phase systems is proved. By modified periodic Lyapunov type sequences the theorem is spread to discrete systems. By means of appropriate Popov-type functionals this theorem is extended to a class of infinite-dimensional phase systems.

In this paper for infinite-dimensional phase systems an analogue of frequency-algebraic stability criterion from [Perkin, Smirnova and Shepeljavyi, 2009] is also proved.

It is applied to concrete radio-engineering and mechanical systems. The stability regions obtained by this criterion are compared with results of other investigations.

2 Asymptotic Behavior of Multidimensional Phase Systems

Consider an autonomous phase system

$$\begin{aligned}\dot{z} &= Az + Bf(\sigma),\\ \dot{\sigma} &= C^* z + Rf(\sigma),\end{aligned} \qquad (1)$$

where A, B, C, R are real $(m \times m)$, $(m \times l)$, $(m \times l)$ and $(l \times l)$ - matrices respectively and $f(\sigma)$ is a vector-valued function having the property $f(\sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2), \ldots, \varphi_l(\sigma_l))$ for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$. The symbol * is used for Hermitian conjugation. We suppose that the pair (A, B) is controllable, the pair (A, C) is observable and matrix A is a Hurwitz one.

We assume that every component $\varphi_j(\sigma_j)$ is Δ_j-periodic, belongs to C^1 and has two simple zeros on $[0, \Delta_j)$. Assume also that

$$\int_0^{\Delta_j} \varphi_j(\sigma)d\sigma < 0 \qquad (i = 1, \ldots, l). \qquad (2)$$

Let

$$\alpha_{1j} \leq \frac{d\varphi_j}{d\sigma_j} \leq \alpha_{2j} \qquad (3)$$

to all $\sigma_j \in \mathbf{R}$, where $\alpha_{1j} < 0 < \alpha_{j2}$ $(j = 1, \ldots, l)$. Let $A_1 = diag\{\alpha_{11}, \ldots, \alpha_{1l}\}$, $A_2 = diag\{\alpha_{21}, \ldots, \alpha_{2l}\}$.

The transfer matrix for the linear part of (1) from the input f to the output $(-\dot{\sigma})$ has the form

$$K(p) = -R + C^*(A - pE_m)^{-1}B \qquad (p \in \mathbf{C}),$$

where E_m is a unit $m \times m$-matrix.

We shall need the designation

$$\Re e M = \frac{1}{2}(M + M^*)$$

for any $l \times l$-matrix M.

Let us determine the functions

$$\Phi_j(\sigma) = \sqrt{\left(1 - \alpha_{1j}^{-1}\varphi_j'(\sigma)\right)\left(1 - \alpha_{2j}^{-1}\varphi_j'(\sigma)\right)}. \qquad (4)$$

Theorem 1. Suppose there exist such diagonal matrix $æ = diag\{æ_1, \ldots, æ_l\}$, and positive definite diagonal matrices $\varepsilon = diag\{\varepsilon_1, \ldots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \ldots, \tau_l\}$, $\delta = diag\{\delta_1, \ldots, \delta_l\}$ that the following requirements are fulfilled:

1) for all $\omega \geq 0$ the inequality

$$\Re e \Big\{ æK(i\omega) - K^*(i\omega)\varepsilon K(i\omega) - \left(K(i\omega) + A_1^{-1}i\omega\right)^* \tau \cdot \left(K(i\omega) + A_2^{-1}i\omega\right)\Big\} - \delta \geq 0 \qquad (i^2 = -1)$$

is valid;

2)

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{2j}|æ_j \quad (j = 1, \ldots, l), \qquad (5)$$

where

$$\nu_{2j} = \frac{\int_0^{\Delta_j} \varphi_j(\sigma)d\sigma}{\int_0^{\Delta_j} |\varphi_j(\sigma)|\sqrt{1 + \frac{\tau_j}{\varepsilon_j}\Phi_j^2(\sigma)}d\sigma} \qquad (6)$$

Then

$$\lim_{t\to\infty} z(t) = 0, \quad \lim_{t\to\infty} \sigma(t) = c,$$

where $f(c) = 0$.

Proof. We follow here the general scheme for the proof of frequency-algebraic stability theorems, expounded in [Leonov, Ponomarenko and Smirnova, 1996]. First of all we use the transformation of system (1) to the system

$$\frac{dy(t)}{dt} = Qy(t) + L\xi(t), \quad \frac{d\sigma(t)}{dt} = D^* y(t) \tag{7}$$

where

$$Q = \left\| \begin{matrix} A & B \\ O & O \end{matrix} \right\|, \qquad L = \left\| \begin{matrix} O \\ E_l \end{matrix} \right\|, \quad D = \left\| \begin{matrix} C \\ R^* \end{matrix} \right\|,$$

$$y(t) = \left\| \begin{matrix} z(t) \\ f(\sigma(t)) \end{matrix} \right\|, \xi = \frac{d}{dt} f(\sigma(t)),$$

and by O a zero matrix is designated. Next we borrow from [Leonov, Ponomarenko and Smirnova, 1996] the following quadratic form of $y \in \mathbf{R}^{m+l}$, $\xi \in \mathbf{R}^l$:

$$G(y, \xi) = 2y^* H(Qy + L\xi) + y^* D\varepsilon D^* y + y^* L æ D^* y - (D^* y - A_1^{-1}\xi)\tau(A_2^{-1}\xi - D^* y) + y^* L\delta L^* y$$

with a symmetric $(m+l) \times (m+l)$-matrix H and diagonal $l \times l$-matrices ε, æ, τ and δ which are introduced in the text of theorem 1.

It follows from condition 1) of theorem 1 that there exists a real symmetric matrix H, such that the inequality

$$G(y, \xi) \le 0 \qquad (\forall y \in \mathbf{R}^{l+m}, \forall \xi \in \mathbf{R}^l) \tag{8}$$

is true [Leonov, Ponomarenko and Smirnova, 1996].

We are going to use here periodic functions

$$P_j(\sigma) = \sqrt{1 + \frac{\tau_j}{\varepsilon_j} \Phi_j^2(\sigma)}; \tag{9}$$

$$Y_j(\sigma) = \varphi_j(\sigma) - \nu_{2j} |\varphi_j(\sigma)| P_j(\sigma). \tag{10}$$

Note that the parameters ν_{2j} can be rewritten in the form

$$\nu_{2j} = \frac{\int\limits_0^{\Delta_j} \varphi_j(\sigma) d\sigma}{\int\limits_0^{\Delta_j} |\varphi_j(\sigma)| P_j(\sigma) d\sigma} \tag{11}$$

Note also that

$$\int\limits_0^{\Delta_j} Y_j(\sigma) d\sigma = 0. \tag{12}$$

With the help of $Y_i(\sigma)$ we construct a new Lyapunov-type function

$$v(t) = y^*(t) H y(t) + \sum_{k=1}^{l} æ_k \int_{\sigma_k(0)}^{\sigma_k(t)} Y_k(\sigma) d\sigma.$$

Let $\frac{dv}{dt}$ be the derivative of $v(t)$ in virtue of system (7). We have

$$\frac{dv(t)}{dt} = 2y^*(t) H(Qy(t) + L\xi(t)) + \sum_{k=1}^{l} æ_k Y_k(\sigma_k(t)) \dot\sigma_k(t).$$

It follows from (8) that

$$\frac{dv(t)}{dt} \le -\dot\sigma^*(t)\varepsilon\dot\sigma(t) - f^*(\sigma(t)) æ \dot\sigma(t) - f^*(\sigma(t))\delta f(\sigma(t)) + \left(\dot\sigma(t) - A_1^{-1}\xi(t)\right)^* \tau \left(\dot\sigma(t) - A_2^{-1}\xi(t)\right) + \sum_{k=1}^{l} æ_k Y_k(\sigma_k(t)) \dot\sigma_k(t)$$

or

$$\frac{dv(t)}{dt} \le \sum_{k=1}^{l} \big(-\varepsilon_k \dot\sigma_k^2(t) - æ_k \varphi_k(\sigma_k(t)) \dot\sigma_k(t) - \delta_k \varphi_k^2(\sigma_k(t)) - \tau_k \Phi_k^2(\sigma_k(t)) \dot\sigma_k^2(t) + æ_k Y_k(\sigma_k(t))\, \dot\sigma_k(t)\big). \tag{13}$$

Using formulas (9) and (10) we conclude from (13) that

$$\frac{dv(t)}{dt} \le \sum_{k=1}^{l} \big(-\varepsilon_k \dot\sigma_k^2(t) - \delta_k \varphi_k^2(\sigma_k(t)) - \tau_k \Phi_k^2(\sigma_k(t)) \dot\sigma_k^2(t) - æ_k \nu_{2k} P_k(\sigma_k(t)) \left|\varphi_k(\sigma_k)\right| \dot\sigma_k(t)\big). \tag{14}$$

Every term in the right part of inequality (14) is a quadratic form with regard to $|\varphi_k(\sigma_k)|$, $P_k(\sigma_k(t))\dot\sigma_k(t)$. According to condition 3) of theorem 1 every such form is negative definite. So we have

$$\frac{dv(t)}{dt} \le -\sum_{k=1}^{l} \delta_{0k} \varphi_k^2(\sigma_k(t)) \tag{15}$$

with $\delta_{0k} > 0$ $(k = 1, \ldots, l)$. It follows from (15) that

$$v(t) - v(0) \le -\sum_{k=1}^{l} \int_0^t \delta_{0k} \varphi_k^2(\sigma_k(t))\, dt, \; \forall t \ge 0. \tag{16}$$

Since matrix A is a Hurwitz one, function $f(\sigma)$ is bounded, and equalities (12) are true, we can affirm that function $v(t)$ is bounded from bellow. That is why it follows from (16) that

$$\int_0^\infty \varphi_k^2(\sigma_k(t)) dt \le +\infty. \tag{17}$$

It is not difficult to see that as matrix A is a Hurwitz one, functions $z(t)$ and $\dot\sigma(t)$ are bounded on $[0,+\infty)$. Any function $\varphi_k(\sigma_k(t))$ is uniformly continuous on $[0,+\infty)$. Then it follows from (17) according to Barbalat lemma [Popov, 1973] that

$$\varphi_k(\sigma_k(t)) \to 0 \quad \text{as} \quad t \to +\infty \quad (k=1,\dots,l).$$

It is proved in [Leonov, Ponomarenko and Smirnova, 1996] (lemma 2.5.1) that for a continuous Δ_k - periodic function $\varphi_k(\sigma_k)$ with a finite number of zeros on $[0,\Delta_k)$ and a continuous function $\sigma_k(t)$ the latter limit relation implies that

$$\sigma_k(t) \to \hat\sigma_k \quad \text{as} \quad t \to +\infty,$$

where $\varphi(\hat\sigma_k)=0$ $(k=1,\dots,l)$. The first equation of system (1) can be rewritten in the form

$$z(t) = e^{At}z(0) + \int_0^t e^{A(t-\tau)}Bf(\sigma(\tau))d\tau.$$

Every element of matrix e^{At} belongs to $L_2[0,+\infty)$. Then from (17) and the fact that the convolution of two functions from $L_2[0,+\infty)$ tends to 0 as $t\to+\infty$ [Gelig, 1966] we deduce that

$$z(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Theorem 1 is proved.

Let us define the numbers

$$\nu_j = \frac{\int_0^{\Delta_j}\varphi_j(\sigma)d\sigma}{\int_0^{\Delta_j}|\varphi_j(\sigma)|\,d\sigma} \qquad (j=1,\dots,l), \tag{18}$$

$$\nu_{0j} = \frac{\int_0^{\Delta_j}\varphi_j(\sigma)d\sigma}{\int_0^{\Delta_j}|\varphi_j(\sigma)|\sqrt{\left(1-\alpha_{1j}^{-1}\varphi_j'(\sigma)\right)\left(1-\alpha_{2j}^{-1}\varphi_j'(\sigma)\right)}d\sigma} \tag{19}$$

$$(j=1,\dots,l).$$

Theorem 2. Suppose there exist such diagonal matrix $æ = diag\{æ_1,\dots,æ_l\}$, positive definite diagonal matrixes $\varepsilon = diag\{\varepsilon_1,\dots,\varepsilon_l\}$, $\tau = diag\{\tau_1,\dots,\tau_l\}$, $\delta = diag\{\delta_1,\dots,\delta_l\}$ and nonnegative numbers a_k, a_{0k} $(k=1,\dots,l)$ that the following requirements are fulfilled:

1) for all $\omega\ge 0$ the inequality

$$\Re e\,\{æK(i\omega) - K^*(i\omega)\varepsilon K(i\omega) -$$
$$-\left(K(i\omega)+A_1^{-1}i\omega\right)^*\tau\left(K(i\omega)+A_2^{-1}i\omega\right)\Big\} - \delta \ge 0$$
$$(i^2=-1)$$

is valid;
2) $a_k + a_{0k} = 1 \qquad (k=1,\dots,l)$;
3) matrices

$$\left\| \begin{matrix} \varepsilon_k & \frac{æ_k a_k \nu_k}{2} & 0 \\ \frac{æ_k a_k \nu_k}{2} & \delta_k & \frac{æ_k a_{0k}\nu_{0k}}{2} \\ 0 & \frac{æ_k a_{0k}\nu_{0k}}{2} & \tau_k \end{matrix} \right\|$$

are positive definite $(k=1,\dots,l)$.
Then the conclusion of theorem 1 is true.

The proof of theorem 2 is alike that of theorem 1. We introduce the functions

$$F_i(\sigma) = \varphi_i(\sigma) - \nu_i\,|\varphi_i(\sigma)|\,, \tag{20}$$
$$\Psi_i(\sigma) = \varphi_i(\sigma) - \nu_{0i}\Phi_i(\sigma)\,|\varphi_i(\sigma)| \tag{21}$$
$$(i=1,\dots,l).$$

with the properties

$$\int_0^{\Delta_i} F_i(\sigma)d\sigma = 0,\ \int_0^{\Delta_i}\Psi_i(\sigma)d\sigma = 0 \quad (i=1,\dots,l). \tag{22}$$

and use Lyapunov-type function

$$v(t) = y^*(t)Hy(t) + \sum_{k=1}^{l} æ_k\left(a_k\int_{\sigma_k(0)}^{\sigma_k(t)}F_k(\sigma)d\sigma + a_{0k}\int_{\sigma_k(0)}^{\sigma_k(t)}\Psi_k(\sigma)d\sigma\right).$$

Conditions of theorem 2 guarantee that inequality (15) is true.

The full text of the proof can be found in [Perkin, Smirnova and Shepeljavyi, 2009].

3 Asymptotic Behavior of Distributed Systems with Phase Control

Let us consider a control system which is described by a system of Volterra integrodifferential equations

$$\dot\sigma(t) = \alpha(t) + Rf(\sigma(t-h)) - \int_0^t \gamma(t-\tau)f(\sigma(\tau))d\tau. \tag{23}$$

Here $t \geq 0, h \geq 0; \sigma(t) = \|\sigma_j(t)\|_{j=1,\dots,l}$, $\alpha(t) = \|\alpha_j(t)\|_{j=1,\dots,l}$, $f(\sigma) = \|\varphi_j(\sigma_j)\|_{j=1,\dots,l}$ are vector-functions, R is a matrix and $\|\gamma_{kj}(t)\|_{k,j=1,\dots,l}$ is a matrix function. For the system (23) the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \quad (24)$$

is given.

We suppose that the following requirements are satisfied:

1. $\alpha_j(t) \in C[0,+\infty) \bigcap L_1[0,+\infty)$, $\alpha_j(t) \to 0$ as $t \to +\infty$ $(j = 1,\dots,l)$;
2. functions γ_{jk} are measurable and $e^{ct}\gamma_{jk}(t) \in L_2[0,+\infty)$ $\quad (k, j = 1, 2,\dots,l)$ for a certain $c > 0$;
3. $\sigma^0(t) \in C^1[-h,0]$;
4. all the properties of $f(\sigma)$ are just the same as in section 2;
5.

$$\int_0^\infty \gamma(t)d\sigma \neq R. \quad (25)$$

System (23) is a phase system. It has a denumerable set of equilibria. The basic characteristic of the linear part of system (23) is the transfer matrix

$$K(p) = -Re^{-ph} + \int_0^\infty \gamma(t)e^{-pt}dt \qquad (p \in \mathbf{C}). \quad (26)$$

Theorem 3. Suppose there exist such positive definite diagonal matrices $æ = diag\{æ_1,\dots,æ_l\}$, $\delta = diag\{\delta_1,\dots,\delta_l\}$, $\varepsilon = diag\{\varepsilon_1,\dots,\varepsilon_l\}$, $\tau = diag\,\{\tau_1,\dots,\tau_l\}$ and such numbers $a_k \in [0,1]$ $(k = 1,\dots,l)$, that the following conditions are satisfied:

1) for all $\omega \in \mathbf{R}$ the inequality

$$\Re e\Big\{æK(i\omega) - K^*(i\omega)\varepsilon K(i\omega) - \big(K(i\omega) + A_1^{-1}i\omega\big)^*\tau\cdot \big(K(i\omega) + A_2^{-1}i\omega\big)\Big\} - \delta > 0 \qquad (i^2 = -1) \quad (27)$$

is true;

2) matrices

$$\left\|\begin{matrix} \varepsilon_k & \frac{æ_k a_k \nu_k}{2} & 0 \\ \frac{æ_k a_k \nu_k}{2} & \delta_k & \frac{æ_k a_{0k}\nu_{0k}}{2} \\ 0 & \frac{æ_k a_{0k}\nu_k}{2} & \tau_k \end{matrix}\right\|$$

where $a_{0k} = 1 - a_k$, are positive definite.

Then

$$\dot\sigma_k \to 0, \quad \sigma_k \to c_k \text{ as } t \to +\infty, \quad (28)$$

where $\varphi_k(c_k) = 0 \quad (k = 1,\dots,l)$.

Proof. Let $\sigma(t)$ be an arbitrary solution of (23) and T be a positive number. Let us introduce the following functions

$$\mu(t) = \begin{Bmatrix} 0 \text{ for } t < 0 \\ t \text{ for } t \in [0,1] \\ 1 \text{ for } t > 1 \end{Bmatrix}, \quad (29)$$

$$\eta(t) = f(\sigma(t)), \quad (30)$$

$$\xi_T(t) = \begin{Bmatrix} \eta(t) & t \leq T \\ \eta(T)e^{\lambda(T-t)} & t > T > 1\,(\lambda > 0) \end{Bmatrix}, \quad (31)$$

$$\eta_T(t) = \mu(t)\xi_T(t), \quad (32)$$

$$\sigma_T(t) = R\eta_T(t-h) - \int_0^t \gamma(t-\tau)\eta_T(\tau)d\tau, \quad (33)$$

$$\begin{gathered}\sigma_0(t) = \alpha(t) + (1-\mu(t-h))R\xi_T(t-h) \\ -\int_0^t (1-\mu(\tau))\gamma(t-\tau)\xi_T(\tau)d\tau.\end{gathered} \quad (34)$$

For $t \in [0,T]$ we have

$$\dot\sigma(t) = \sigma_0(t) + \sigma_T(t) \quad (35)$$

Let $\eta_T(t) = \|\eta_{Tj}\|_{j=1,\dots,l}$, $\eta(t) = \|\eta_j\|_{j=1,\dots,l}$ $\sigma_T(t) = \|\sigma_{Tj}\|_{j=1,\dots,l}$. It follows from the properties of $\eta_T(t)$, $\gamma_{ij}(t)$ that

$$\sigma_{Tj}, \eta_{Tj}, \dot\eta_{Tj} \in L_2[0,+\infty) \quad (j = 1,\dots,l) \quad (36)$$

for each $T > 0$.

Let us consider a one-parameter set of functionals

$$\rho_T = \int_0^\infty \{\sigma_T^*(t)æ\eta_T(t) + \eta_T^*(t)\delta\eta_T(t) + \sigma_T^*(t)\varepsilon\sigma_T(t) + \\ + \big(\sigma_T(t) - A_1^{-1}\dot\eta_T(t)\big)^*\tau\big(\sigma_T(t) - A_2^{-1}\dot\eta_T(t)\big)\Big\}dt. \quad (37)$$

By Parseval equation we have

$$\rho_T = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \{\tilde{\sigma}_T^*(i\omega)æ\tilde{\eta}_T(i\omega) + \tilde{\eta}_T^*(i\omega)\delta\tilde{\eta}_T(i\omega)+ \tilde{\sigma}_T^*(i\omega)\varepsilon\tilde{\sigma}_T(i\omega) + \left(\tilde{\sigma}_T(i\omega) - A_1^{-1}\tilde{\dot{\eta}}_T(i\omega)\right)^* \tau\left(\tilde{\sigma}_T(i\omega) - A_2^{-1}\tilde{\dot{\eta}}_T(i\omega)\right)\}\,d\omega, \tag{38}$$

where by $\tilde{\sigma}_T(i\omega), \tilde{\eta}_T(i\omega), \tilde{\dot{\eta}}_T(i\omega)$ the Fourier transforms of $\sigma_T(t), \eta_T(t), \dot{\eta}_T(t)$ respectively are denoted. By means of equalities

$$\tilde{\sigma}_T(i\omega) = -K(i\omega)\tilde{\eta}_T(i\omega), \qquad \tilde{\dot{\eta}}_T(i\omega) = i\omega\tilde{\eta}_T(i\omega) \tag{39}$$

we obtain that

$$\rho_T = -\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \tilde{\eta}^*(i\omega)\Re e\{æK(i\omega) - \delta - K^*(i\omega)\cdot \cdot\varepsilon K(i\omega) - (K(i\omega) + i\omega A_1^{-1})^*\tau(K(i\omega) + i\omega A_2^{-1})\}\cdot \cdot|\tilde{\eta}(i\omega)|^2 d\omega \tag{40}$$

From the condition 1) of the theorem it follows that

$$\rho_T < 0. \tag{41}$$

Let us represent the functional ρ_T as follows:

$$\rho_T = I_T + \rho_1 + \rho_{2T} + \rho_{3T} + \rho_{4T}, \tag{42}$$

where

$$I_T = \int\limits_0^T \{\dot{\sigma}^*æ\eta + \eta^*\delta\eta + \dot{\sigma}^*\varepsilon\dot{\sigma}^* + \left(\dot{\sigma} - A_1^{-1}\dot{\eta}\right)^*\tau\left(\dot{\sigma} - A_2^{-1}\dot{\eta}\right)\Big\}\,dt, \tag{43}$$

$$\rho_1 = \int\limits_0^1 \big\{(1-t)\dot{\sigma}^*æ\eta + (1-t^2)\eta^*\delta\eta + \dot{\eta}^*A_1^{-1}\tau A_2^{-1}\dot{\eta} - (\dot{\widehat{t\eta}})^*A_1^{-1}\tau A_2^{-1}\dot{\widehat{t\eta}} + \dot{\sigma}^*\left(A_1^{-1} + A_2^{-1}\right)\tau(\dot{\eta}_T - \dot{\widehat{t\eta}})\Big\}\,dt, \tag{44}$$

$$\rho_{2T} = \int\limits_0^T (-\sigma_0^*æ\eta_T - 2\dot{\sigma}^*(\varepsilon+\tau)\sigma_0 + \sigma_0^*(\varepsilon+\tau)\sigma_0 - \sigma_0^*\left(A_1^{-1} + A_2^{-1}\right)\tau\dot{\eta}_T)\,dt, \tag{45}$$

$$\rho_{3T} = \int\limits_T^\infty \sigma_T^*(t)(\varepsilon+\tau)\sigma_T(t)dt, \tag{46}$$

$$\rho_{4T} = \int\limits_T^\infty \left(-\dot{\eta}_T^*A_1^{-1}\tau\sigma_T + \sigma_T^*æ\eta_T + \eta_T^*\delta\eta_T - \sigma_T^*\tau A_2^{-1}\dot{\eta}_T + \dot{\eta}_T^*A_1^{-1}\tau A_2^{-1}\dot{\eta}_T\right)dt. \tag{47}$$

It follows from the properties of $a_j(t)$, $\gamma_{kj}(t)$, $\eta_{Tj}(t)$ $(k, j = 1, \ldots, l)$ that

$$\rho_{kT} < C_k \quad (k = 2, 4), \tag{48}$$

where C_k does not depend on T. From (41), (42), (48) with regard to the positiveness of ρ_{3T} it arises that

$$I_T < C_5, \tag{49}$$

where C_5 does not depend on T.

The functional I_T can be represented in the following way

$$I_T = \sum_{j=1}^{l}\int\limits_0^T \big\{æ_j\varphi_j(\sigma_j(t))\dot{\sigma}_j(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \varepsilon\dot{\sigma}_j^2(t) + \tau_j\Phi_j^2(\sigma_j(t))\dot{\sigma}_j^2(t)\big\}\,dt. \tag{50}$$

Let us use the functions F_i and Ψ_i defined in theorem 2. From the definition of F_i and Ψ_i it follows that

$$I_T = \sum_{j=1}^{l}\int\limits_0^T \{æ_ja_j\nu_j|\varphi_j(\sigma)|\dot{\sigma}_j(t) + æ_ja_{0j}\nu_{0j}|\varphi_j(\sigma)|\cdot \cdot\Phi_j(\sigma_j)\dot{\sigma}_j(t) + \varepsilon\dot{\sigma}_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \tau_j\Phi_j^2(\sigma_j(t))\cdot \cdot\dot{\sigma}_j^2(t)\big\}\,dt + \sum_{j=1}^{l}\left[\int\limits_0^T æ_ja_jF_j(\sigma_j(t))\dot{\sigma}_j(t)dt + \int\limits_0^T æ_ja_{0j}\nu_{0j}\Psi_{0j}(\sigma_j(t))\dot{\sigma}_j(t)dt\right]. \tag{51}$$

It follows from (22) that all integrals $\int\limits_0^T F_j(\sigma_j(t))\dot{\sigma}_j(t)dt$ and $\int\limits_0^T \Psi_j(\sigma_j(t))\dot{\sigma}_j(t)dt$ are bounded by constants which do not depend on T.

This assumption together with (49) implies that

$$\sum_{j=1}^{l}\int\limits_0^T \{æ_ja_j\nu_j|\varphi_j(\sigma)|\dot{\sigma}_j(t) + æ_ja_{0j}\nu_{0j}|\varphi_j(\sigma_j)|\cdot \cdot\Phi_j(\sigma_j)\dot{\sigma}_j(t) + \varepsilon\dot{\sigma}_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \tau_j\Phi_j^2(\sigma_j(t))\dot{\sigma}_j^2(t)\big\}\,dt < C_6, \tag{52}$$

where C_6 does not depend on T. By virtue of the condition 2 of the theorem every sum which stands under the integral sign in the left part of (52) is a positive definite quadratic form of $\dot{\sigma}_j$, $|\varphi_j(\sigma_j)|$, $\Phi_j(\sigma_j)\dot{\sigma}_j$. Then it follows from (52) that

$$\int\limits_0^{+\infty} \varphi_j^2(\sigma_j(t))dt < +\infty, \tag{53}$$

$$\int\limits_0^{+\infty} \dot{\sigma}_j^2(t)dt < +\infty. \tag{54}$$

Let us now repeat the argument of [Leonov, Ponomarenko and Smirnova, 1996]. Any $\varphi_j(\sigma_j)$ is uniformly continuous. It is easy to see that $\sigma_j(t)$ is uniformly continuous as well. Then it follows from (53) and (54) according to Barbalat lemma [Popov, 1973] that $\varphi_j(\sigma_j(t))$ and $\dot\sigma_j(t)$ tend to zero as t tends to $+\infty$. This property of $\varphi_j(\sigma_j(t))$ implies that $\sigma_j(t)$ tends to a zero of $\varphi_j(\sigma_j)$ as t tends to $+\infty$. Theorem 3 is proved.

Theorem 4. Suppose there exist such positive diagonal matrices $æ = diag\{æ_1,\dots,æ_l\}$, $\delta = diag\{\delta_1,\dots,\delta_l\}$, $\varepsilon = diag\{\varepsilon_1,\dots,\varepsilon_l\}$, $\tau = diag\{\tau_1,\dots,\tau_l\}$, that for all $\omega \geq 0$ the frequency-domain inequality (27) is fulfilled. Suppose also that for varying parameters ε_j, δ_j, $æ_j$ the inequalities

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{2j}|æ_j \quad (j = 1,\dots,l), \tag{55}$$

where ν_{2j} is defined in theorem 1, are valid. Then the conclusion of theorem 1 is true.

Proof. Let us repeat the first part of the proof of theorem 3 and prove that the inequality (49), where C_5 does not depend on T, is true.

Let us then consider the function which stands under the integral sign in the functional I_T and transform it. We shall use the functions P_j and Y_j, introduced in text of the proof of theorem 1. Note that

$$\begin{aligned}&æ_j\dot\sigma_j\eta_j + \delta_j\eta_j^2 + \tau_j(\dot\sigma_j - \alpha_{1j}^{-1}\dot\eta_j)(\dot\sigma_j - \alpha_{2j}^{-1}\dot\eta_j)+\\ &+\varepsilon_j\dot\sigma_j^2 = æ_j\dot\sigma_j(Y_j(\sigma_j) + \nu_{2j}|\eta_j|P_j(\sigma_j))+\\ &+\delta_j\eta_j^2 + \tau_j\Phi_j^2(\sigma_j)\dot\sigma_j^2 + \varepsilon_j\dot\sigma_j^2 = æ_j\dot\sigma_jY_j(\sigma_j)+\\ &+\left(\delta_j\eta_j^2 + æ_j\nu_{2j}|\eta_j|\dot\sigma_jP_j(\sigma_j) + \varepsilon_j\dot\sigma_j^2P_j^2(\sigma_j)\right).\end{aligned} \tag{56}$$

So

$$\begin{aligned}I_T = &\sum_{j=1}^{l}(æ_j\int_0^T\dot\sigma_j(t)Y_j(\sigma_j(t))dt+\\ &+\int_0^T\left(\delta_j\eta_j^2(t) + \nu_{2j}æ_j|\eta_j(t)|\dot\sigma_j(t)P_j(\sigma_j(t))+\right.\\ &\left.+\varepsilon_j\dot\sigma_j^2P_j^2(\sigma(t))\right)dt).\end{aligned} \tag{57}$$

By virtue of (12) we affirm that

$$\int_0^T\dot\sigma_j(t)Y_j(\sigma_j(t))dt = \int_{\sigma_j(0)}^{\sigma_j(T)}Y_j(\sigma_j)d\sigma_j < C_{10}, \tag{58}$$

where C_{10} does not depend on T. On the other hand by virtue of (55) the quadratic forms

$$\delta_j\eta_j^2(t)+\nu_{2j}æ_j|\eta_j(t)|\dot\sigma_j(t)P_j(\sigma_j(t))+\varepsilon_j\dot\sigma_j^2P_j^2(\sigma_j(t))) \tag{59}$$

are positive definite.

So it follows from (49) and (58) that

$$\int_0^T\eta_j^2(t)dt < C_{12}, \quad \int_0^T\dot\sigma_j^2(t)dt < C_{11}, \tag{60}$$

where C_{12} and C_{11} do not depend on T. Now we can use the concluding part of the proof of theorem 1.

4 Gradient-Like Behavior of Radio-Engineering and Mechanical Systems

1) Theorem 3 was applied to stability investigation of a second order phase-locked loop with proportional-integrating filter and time delay in the loop. In this case $m = l = 1$ and the transfer function has the form

$$K(p) = T\frac{1+\beta Tp}{1+Tp}e^{-phT}, \ (T > 0, h > 0, \beta\in(0,1)). \tag{61}$$

For $\varphi(\sigma) = \sin\sigma - \gamma$ $(\gamma\in(0,1))$; $\beta = 0,2$; $h = 0,01; 0,1; 1$ the estimates for the boundaries of lock-in ranges on the plane $\{T^2,\gamma\}$ were obtained. These estimates were compared with the lock-in ranges obtained in [Belyustina, Kinyapina and Fishman, 1990] by qualitative-numerical methods. It turned out that the ranges received by means of theorem 1 have the same structure as those in [Belyustina, Kinyapina and Fishman, 1990]. For $T^2 < h^{-1}$ the ranges obtained by theorem 1 are 15-25% smaller than the ranges received in [Belyustina, Kinyapina and Fishman, 1990].

2) Theorem 3 was also applied to one of problems of vibrational mechanics [Blekhman, 2000; Blekhman, 2012; Pena-Ramirez, Fey and Nijmeijer, 2012]. It is the problem of self-synchronization of two rotors on a vibrator with one degree of freedom [Blekhman, 1988]. The equations describing the change of the slowly variable components $\Theta_s(t)$ $(s = 1,2)$ of the phase of the rotor motion are

$$\left\{\begin{array}{l}I_1\ddot\Theta_1 + K_1\dot\Theta_1 + A\sin(\Theta_1-\Theta_2) - \beta = 0,\\ I_2\ddot\Theta_2 + K_2\dot\Theta_2 - A\sin(\Theta_1-\Theta_2) + \beta = 0\end{array}\right\} \tag{62}$$

where I_1, I_2, K_1, K_2, β are positive parameters [Sperling, Merten and Duckstein, 1997]. The self-synchronization of the rotors means that the difference $\sigma = \Theta_1 - \Theta_2$ tends to a zero of $\varphi(\sigma) = \sin\sigma - \beta/A$ as $t\to+\infty$. The system (62) can be reduced to (23) with $l = 1, R = 0$ and the transfer function

$$K(p) = A\left(\frac{1}{I_1p+K_1} + \frac{1}{I_2p+K_2}\right). \tag{63}$$

In monograph [Leonov and Smirnova, 2000] various requirements on the coefficients of (62) are given which guarantee that the relations (28) are true. These requirements are such that the conditions of theorem 1 are satisfied in case $a_1 = 1(a_{01} = 0)$. Varying the parameter a_1 in theorem 3 we can weaken these requirements. Let us introduce the parameter

$$y = \frac{K_1K_2(K_1I_2 + K_2I_1)}{AI_1I_2(K_1 + K_2)}.$$

Suppose that

$$A > \sqrt{\frac{K_1K_2\left(K_1^2I_2^2 + K_2^2I_1^2\right)}{2\left(K_1 + K_2\right)\left(K_1I_2^2 + K_2I_1^2\right)}} \cdot \frac{I_1K_2 + I_2K_1}{I_1 \cdot I_2}. \tag{64}$$

In this case for $A = 2\beta$ theorem 3 guarantees (28) if $y > 0,97$ and theorems of [Leonov and Smirnova, 2000] give $y > 1.13$

5 Discrete Systems

Consider a discrete phase system

$$\begin{array}{l} z(n+1) = Az(n) + Bf(\sigma(n)), \\ \sigma(n+1) = \sigma(n) + C^*z(n) + Rf(\sigma(n)) \\ (n = 0, 1, 2, \ldots), \end{array} \tag{65}$$

where A, B, C, R are described in section 2. We suppose that the pair (A, B) is controllable, the pair (A, C) is observable and all eigenvalues of matrix A are situated inside the open unit circle. All the properties of $f(\sigma)$ are just the same as in section 2. The transfer matrix $K(p)$ for the linear part of system (65) has the form

$$K(p) = -R + C^*(A - pE_m)^{-1}B \quad (p \in \mathbf{C}). \tag{66}$$

We shall present in this section certain analogues of theorems 1 and 2. We shall need numbers $k_{1j} = 2\alpha_{1j} - \alpha_{2j}$ and $k_{2j} = 2\alpha_{2j} - \alpha_{1j}$ and diagonal matrices $K_1 = diag(k_{11}, \ldots, k_{1l})$ and $K_2 = diag(k_{21}, \ldots, k_{2l})$.

Theorem 5. Suppose there exist such positive definite diagonal matrices $\varepsilon = diag\{\varepsilon_1, \ldots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \ldots, \tau_l\}$, $\delta = diag\{\delta_1, \ldots, \delta_l\}$, a diagonal matrix $æ = diag\{æ_1, \ldots, æ_l\}$ that the following requirements are fulfilled:

1) for all $p \in \mathbf{C}$, $|p| = 1$ the inequality

$$\begin{array}{l} \Re e\left\{æK(p) - \left(K(p) + (p-1)K_1^{-1}\right)^* \right. \\ \left. \tau\left(K(p) + (p-1)K_2^{-1}\right)\right\} - K^*(p)\varepsilon K(p) - \delta \geq 0 \end{array} \tag{67}$$

is valid;

2) the inequalities

$$(1 - \frac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}})\varepsilon_k >$$

$$> \frac{æ_k\alpha_{0k}}{2}(1 - \nu_{2k}\sqrt{1 + \frac{\tau_k}{\varepsilon_k}\frac{(\alpha_{2k} - \alpha_{1k})^2}{|\alpha_{1k}|\alpha_{2k}}}), \tag{68}$$

where $\alpha_{0k} = \alpha_{2k}$ if $æ_k > 0$, and $\alpha_{0k} = \alpha_{1k}$ if $æ_k < 0$, and

$$2\sqrt{\varepsilon_k\delta_k\frac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}} > |\nu_{2k}æ_k|(k = 1, 2, \ldots, l) \tag{69}$$

are true.

Then

$$\begin{array}{l} \lim\limits_{n\to\infty} f(\sigma(n)) = 0, \\ \lim\limits_{n\to\infty} z(n) = 0, \\ \lim\limits_{n\to\infty} (\sigma(n+1) - \sigma(n)) = 0, \\ \lim\limits_{n\to\infty} (\sigma(n)) = c, \end{array}$$

where $f(c) = 0$.

Proof. The proof is based on the proof of theorem 5.4.1 from [Leonov and Smirnova, 2000]. Its first step is the extension of the state space of the system. For the purpose we introduce the notations

$$y = \left\| \begin{array}{c} z \\ f(\sigma) \end{array} \right\|, \qquad P = \left\| \begin{array}{cc} A & B \\ O & E_l \end{array} \right\|,$$

$$L = \left\| \begin{array}{c} O \\ E_l \end{array} \right\|, \qquad D^* = \left\| C^*, R \right\|,$$

$$\xi_1(n) = f(\sigma(n+1)) - f(\sigma(n)).$$

Then system (65) can be represented as

$$\begin{array}{l} y(n+1) = Py(n) + L\xi_1(n), \\ \sigma(n+1) = \sigma(n) + D^*y(n), \\ (n = 0, 1, 2, \ldots). \end{array} \tag{70}$$

The second step is to determine the quadratic form of $y \in \mathbf{R}^{m+l}$ and $\xi_1 \in \mathbf{R}^l$

$$\begin{array}{l} M(y, \xi_1) = (Py + L\xi_1)^*H(Py + L\xi_1)- \\ y^*Hy + y^*LæDy + y^*D\varepsilon D^*y + y^*L\delta L^*y- \\ (D^*y - K_1^{-1}\xi_1)^*\tau(K_2^{-1}\xi_1 - D^*y), \end{array} \tag{71}$$

where H is a symmetric $(m+l)\times(m+l)$ matrix and ε, $æ$, δ, τ are diagonal matrices from the text of theorem 5.

It follows from [Leonov and Smirnova, 2000] that if the condition 1) of theorem 5 is true then there exists matrix $H = H^*$ such that for all $y \in \mathbf{R}^{m+l}$ and $\xi_1 \in \mathbf{R}^l$

$$M(y, \xi_1) \leq 0. \tag{72}$$

Since all the eigenvalues of matrix A are situated inside the unit circle and function $f(\sigma)$ is bounded we can affirm that sequence $W(n) = y^*(n)Hy(n)$, where $y(n)$ satisfies (70) is bounded as well.

Let us use functions $P_j(\sigma)$ and $Y_j(\sigma)$ which were introduced in the proof of theorem 1 and define a Lyapunov-type sequence

$$V(n) = W(n) + \sum_{k=1}^{l} æ_k \int_{\sigma_k(0)}^{\sigma_k(n)} Y_k(\sigma)d\sigma. \qquad (73)$$

Let us consider the difference

$$V(n+1) - V(n) = W(n+1) - W(n) + \sum_{k=1}^{l} æ_k \int_{\sigma_k(n)}^{\sigma_k(n+1)} Y_k(\sigma)d\sigma. \qquad (74)$$

It follows from (72) that

$$\begin{aligned} &W(n+1) - W(n) \leq \\ &\sum_{k=1}^{l} \{-æ_k\varphi_k(\sigma_k(n))(\sigma_k(n+1) - \sigma_k(n)) - \\ &\varepsilon_k(\sigma_k(n+1) - \sigma_k(n))^2 - \delta_k\varphi_k^2(\sigma_k(n)) - \\ &\tau_k[k_{1k}^{-1}(\varphi_k(\sigma_k(n+1)) - \varphi_k(\sigma_k(n))) - \\ &(\sigma_k(n+1) - \sigma_k(n))][k_{2k}^{-1}(\varphi_k(\sigma_k(n+1)) - \\ &\varphi_k(\sigma_k(n))) - (\sigma_k(n+1) - \sigma_k(n))]. \end{aligned} \qquad (75)$$

On the other hand we can establish the estimate [Leonov and Smirnova, 2000]

$$\begin{aligned} &æ_k \int_{\sigma_k(n)}^{\sigma_k(n+1)} Y_k(\sigma)d\sigma \leq æ_k(\varphi_k(\sigma_k(n)) + \\ &\Theta_k|\varphi_k(\sigma_k(n))|)(\sigma_k(n+1) - \\ &\sigma_k(n)) + æ_k\frac{\alpha_{0k}}{2}(1+\Theta_k)(\sigma_k(n+1) - \sigma_k(n))^2 \end{aligned} \qquad (76)$$

where

$$\Theta_k = |\nu_{2k}P_k(\sigma'_{kn})| \qquad (77)$$

and

$$\sigma_k(n) \gtrless \sigma'_{kn} \gtrless \sigma_k(n+1). \qquad (78)$$

Note that

$$\Phi_k(\sigma) < \frac{\alpha_{2k} - \alpha_{1k}}{\sqrt{|\alpha_{1k}|\alpha_{2k}}}. \qquad (79)$$

Hence

$$P_k(\sigma'_{kn}) < \sqrt{1 + \frac{\tau_j(\alpha_{2k} - \alpha_{1k})^2}{\varepsilon_j\alpha_{2k}|\alpha_{1k}|}} \qquad (80)$$

It is established in [Smirnova and Shepeljavyi, 2007] that

$$\begin{aligned} &[k_{2k}^{-1}(\varphi_k(\sigma_k(n+1)) - \varphi_k(\sigma_k(n))) - \\ &-(\sigma_k(n+1) - \sigma_k(n))] \\ &[k_{1k}^{-1}(\varphi_k(\sigma_k(n+1)) - \varphi_k(\sigma_k(n))) - \\ &-(\sigma_k(n+1) - \sigma_k(n))] \\ &\geq \frac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}\Phi_k^2(\sigma'_{kn})(\sigma_k(n+1) - \sigma_k(n))^2 = \\ &= \frac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}(P^2(\sigma'_{kn}) - 1)\frac{\varepsilon_k}{\tau_k}(\sigma_k(n+1) - \sigma_k(n))^2. \end{aligned} \qquad (81)$$

Formulae (74)-(81) imply that

$$V(n+1) - V(n) \leq \sum_{k=1}^{l} Z_k(n), \qquad (82)$$

where

$$\begin{aligned} &Z_k(n) = -\Bigg(\varepsilon_k - \frac{æ_k\alpha_{0k}}{2}(1 + \\ &|\nu_{2k}|\sqrt{1 + \frac{(\alpha_{2k} - \alpha_{1k})^2\tau_k}{|\alpha_{1k}|\alpha_{2k}\varepsilon_k}}\Bigg) - \\ &\frac{\varepsilon_k\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}}\Bigg)(\sigma_k(n+1) - \sigma_k(n))^2 - \delta_k\varphi_k^2(\sigma_k(n)) - \\ &-\varepsilon_k\frac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}}P_k^2(\sigma'_{kn})(\sigma_k(n+1) - \sigma_k(n))^2 \\ &+æ_k|\nu_{2k}\varphi_k(\sigma_k(n))P_k(\sigma'_{kn})|(\sigma_k(n+1) - \sigma_k(n)). \end{aligned} \qquad (83)$$

By virtue of condition 2) of the theorem we have that

$$V(n+1) - V(n) \leq -\delta_0\,|f(\sigma(n))|^2 \quad (\delta_0 > 0), \qquad (84)$$

where by $|f|$ the Euclidian norm of vector f is designated. Since sequence $W(n)$ $(n = 0, 1, 2, \ldots)$ is bounded and functions $Y_k(\sigma)$ $(k = 1, 2, \ldots, l)$ satisfy (12) we can affirm that sequence $V(n)$ $(n = 0, 1, 2, \ldots)$ is bounded as well. Then is follows from (84) that the series $\sum_{n=1}^{+\infty} |f(\sigma(n))|^2$converges. Hence

$$\lim_{n\to+\infty} |f(\sigma(n))| = 0 \qquad (85)$$

and consequently as soon as all eigenvalues of A are situated inside the unit circle we can affirm that $\lim_{n\to+\infty} z(n) = 0$.

Then from (70) it follows that $\sigma(n+1) - \sigma(n) \to 0$ as $n \to +\infty$.From this fact and (85) it follows that $\sigma(n) \to \hat{\sigma}$ as $n \to +\infty$,with $f(\hat{\sigma}) = 0$. Theorem 5 is proved.

Theorem 6. Suppose there exist such positive definite diagonal matrices $\varepsilon = diag\{\varepsilon_1, \ldots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \ldots, \tau_l\}$, $\delta = diag\{\delta_1, \ldots, \delta_l\}$, a diagonal matrix $æ = diag\{æ_1, \ldots, æ_l\}$ and numbers $a_k \in$

$[0,1]$ $(k=1,\dots,l)$ that the requirement 1) from Theorem 5 is fulfilled and matrices

$$\left\| \begin{matrix} \varepsilon_k - \frac{æ_k \alpha_{0k}}{2}\left(a_k(1+|\nu_k|)+ a_{0k}\left(1-\frac{\alpha_{2k}-\alpha_{1k}}{\sqrt{|\alpha_{1k}|\alpha_{2k}}}\right)\right) & \dfrac{æ_k \nu_k a_k}{2} & 0 \\ \dfrac{æ_k \nu_k a_k}{2} & \delta_k & \dfrac{æ_k a_{0k} \nu_{0k}}{2} \\ 0 & \dfrac{æ_k a_{0k} \nu_{0k}}{2} & \tau_k \dfrac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}} \end{matrix} \right\|, \tag{86}$$

where $a_{0k} = 1 - a_k$ and α_{0k} are defined in the text of theorem 5, are positive definite. Then the conclusion of Theorem 5 is true.

The proof of Theorem 6 is alike the proof of Theorem 5. It is based on the Lyapunov-type sequence

$$V(n) = W(n) + \sum_{k=1}^{l} æ_k \left(a_k \int\limits_{\sigma_k(0)}^{\sigma_k(n)} F_k(\sigma)d\sigma + a_{0k} \int\limits_{\sigma_k(0)}^{\sigma_k(n)} \Psi_k(\sigma)d\sigma \right), \tag{87}$$

where the sequence $W(n)$ is defined in the text of theorem 5 and functions F_k and Ψ_k $(k=1,\dots,l)$ are borrowed from the text of theorem 2. The estimates for $a_k \int\limits_{\sigma_k(0)}^{\sigma_k(n)} F_k(\sigma)d\sigma$ and $a_{0k} \int\limits_{\sigma_k(0)}^{\sigma_k(n)} \Psi_k(\sigma)d\sigma$ are taken from [Leonov and Smirnova, 2000] and [Smirnova and Shepeljavyi, 2007] respectively. The full text of this proof can be found in [Perkin, Smirnova and Shepeljavyi, 2009].

6 Conclusion

The paper is devoted to the problem of gradient-like behavior for lumped, distributed and discrete phase systems. The problem is investigated by two methods traditionally used in absolute stability theory. They are Lyapunov direct method for multidimensional systems and the method of a priori integral estimates for distributed systems. In the paper new types of periodic Lyapunov-type functions and Popov-type functionals are exploited. As a result new frequency-algebraic stability criteria are obtained. The new criteria give the opportunity to improve the estimates for the regions of gradient-like behavior in the space of parameters of the systems. They are applied to stability investigation of second order phase-locked loops with time delay and to the problem of self-synchronization of two rotors on a vibrator with one degree of freedom.

References

Bakaev, Ju.N., and Guzh, A.A. (1965) Optimal reception of frequency modulated signals under Doppler effect conditions. *Radiotekhnika i Electronika*, **10**(1), pp. 175–196.

Belyustina, L.N., Kinyapina, M.S., and Fishman, L.Z. (1990) Dynamics of phase synchronization systems with delay. In: *Teoreticheskaya Electrotehnika*, L'vov.

Blekhman, I.I. (1981) *Synchronization in Nature and Technology*. Nauka, Moscow (English translation by ASME Press, USA, N.Y., 1988).

Blekhman, I.I. (2000) *Vibrational Mechanics*. World Scientific, Singapure.

. Blekhman, I.I. (2012) Oscillatory strobodynamics — a new area in nonlinear oscillations theory, nonlinear dynamics and cybernetical physics. *Cybernetics and Physics*.**1**(1), pp. 5-10.

Pena-Ramirez, J., Fey, R., and Nijmeijer, H. (2012) In-phase and anti-phase synchronization of oscillators with Huygens' coupling. *Cybernetics and Physics*.**1**(1), pp. 58-66.

Brockett, R.W. (1982) On the asymptotic properties of solution of differential equations with multiple equilibra. *J. Diff. Equations*, **44**(4), pp. 249–262.

Gelig, A.Kh. (1966) On the stability of a nonlinear system with infinitely many degrees of freedom. *Prikl. Matem. i Mekh.*, **30**(4), pp. 789–795.

Yakubovich, V.A., Leonov, G.A., and Gelig, A.Kh. (2004) *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, World Scietific, Syngapore–New Jersey–London–Hong Kong.

Leonov, G.A., Ponomarenko, D.V., and Smirnova V.B., (1996) *Frequency-Domain Metods for Nonlinear Analysis. Theory and Applications*, World Scietific, Syngapore–New Jersey–London–Hong Kong.

Karpichev, A.N., Koryakin, Ju.A., Leonov, G.A., and Shepeljavyi, A.I. (1990) Frequency-domain criteria for stability and instability of multidimensional discrete phase synchronization systems. *Voprosy kibernetiki i vychislitel'noi tehniki. Diskretnye sistemy.* Kiev, **87**.

Leonov, G., and Smirnova, V. (2000) *Mathametical Problems in the Theory of Phase Synchronization*. Nauka, St. Petrsburg.

Perkin, A., Smirnova, V., and Shepeljavyi, A. (2009) Frequency-domain criteria for gradient-like behavior of phase control systems with vector nolinearities. In: *Proc. 3rd IEEE Multi-Conf. on Systems and Control (MSC 2009).* St. Petersburg, Russia, pp. 142–146.

Popov, V.M. (1973) *Hyperstability of Control Systems*. Springer, Berlin.

Smirnova, V.B., and Shepeljavyi, A.I. (2007) A modified frequency-domain criterion for gradient-like behavior of discrete phase systems. In: *Proc. of Third IFAC Workshop "Periodic Control Systems"*. St. Petersburg, Russia. http://lib.physcon.ru/getfile.html?item1427.

Sperling, L., Merten F., and Duckstein H. (1997) Rotation and vibration in Beispielen zur Methode der directen Bewegungsteilung. *Technische Mechanik*, Bd. 17, H. 3, pp. 231-243.

Duan, Z., Wang, J., and Huang, L. (2007) Criteria for dichotomy and gradient-like behavior of a class of nonlinear systems with multiple equilibria. *Automatica*, **43**, pp. 1583–1589.