SOME NUMERICAL RESULTS ONENERGY TRANSFER BETWEEN MECHANICAL OSCILLATORS

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Abstract: The present paper deals with energy transfer in a dissipative mechanical system. Various results, theoretical and numerical, based on recent works are given. Numerical results are given by considering a linear interaction between the considered oscillators. Specifically, we show an energy transfer from linear to nonlinear oscillator (energy pumping) as well as from nonlinear to linear oscillator.

1. Introduction

In this paper we extend previous results obtained in [1, 2] on energy transfer in a 2-DOF system composed of two coupled and damped oscillators.

We investigated analytically and numerically the occurrence of energy pumping (which consists in passive irreversible transfer of energy from a linear system to a nonlinear attachment [3–15]) as well as the occurrence of energy transfer from nonlinear to linear oscillator [6, 7].

Previous works on transfer of energy concerned mainly energy transfer from linear to nonlinear oscillator or with linear absorber with important mass. However, the case of an energy transfer from nonlinear to linear oscillator has been passed over. Besides, some works in this area try to solve the problem of minimal energy level necessary to reach the stable orbit responsible for energy pumping considering homogeneous initial conditions for the displacements for both oscillators and for the initial velocity of the nonlinear oscillator. An impulse is applied as initial velocity(the energy of the system at $t = 0^+$) for the linear oscillator. So depending on the level of energy applied, energy pumping occurs (see [4, 5]).

In this work, we showed that depending on the initial conditions, there is an energy transfer from linear to nonlinear or from nonlinear to linear oscillator.

Here we are interested in the oscillations of the energy of each oscillator calculated on the orbits of the perturbed system. Assuming that these oscillators are coupled by springs, expansions of the energies, in a small parameter ε, are obtained. We consider only the resonant case, since if the non-resonant condition holds, then there is no energy transfer.

Numerical evidences confirm our theoretical results.

2. Mathematical Model

All results presented here are based in the definition of energy transfer presented in references [1, 2]:

Let H_i, i =1, 2 and R_j, j = 1,…,5 be functions adequately smooth defined on open sets of $\Re^2$ and $\Re^5$ respectively. It is assumed that each open set contains the origin and each R_j is T-periodic in the variable t.

Consider the following system

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H_1}{\partial p_1}(q_1, p_1) + \varepsilon R_1(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{p}_1 = -\dfrac{\partial H_1}{\partial q_1}(q_1, p_1) + \varepsilon R_2(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{q}_1 = \dfrac{\partial H_2}{\partial p_2}(q_2, p_2) + \varepsilon R_3(q_1, p_1, q_2, p_2, t, \varepsilon), \\ \dot{q}_1 = -\dfrac{\partial H_2}{\partial q_2}(q_2, p_2) + \varepsilon R_4(q_1, p_1, q_2, p_2, t, \varepsilon), \end{cases} \tag{1}$$

and defining

$$\begin{aligned} E_1(t,\varepsilon,a,b,c,d) &= H_1(q_1(t,\varepsilon), p_1(t,\varepsilon)), \\ E_2(t,\varepsilon,a,b,c,d) &= H_2(q_2(t,\varepsilon), p_2(t,\varepsilon)), \end{aligned} \tag{2}$$

where ($q_1(t, \varepsilon)$, $p_1(t, \varepsilon)$, $q_2(t, \varepsilon)$, $p_2(t, \varepsilon)$) is the solution of Eq.(1) such that ($q_1(0,\varepsilon)$, $p_1(0,\varepsilon)$, $q_2(0,\varepsilon)$, $p_2(0,\varepsilon)$,) = (a, b, c, d) , it is said that there is a transfer of energy, from the oscillator 1 to oscillator 2, in the point (a, b, c, d) of the phase space of the system given by Eq.(1), if there is $T_0 = T_0(a, b, c, d) \geq 0$ such that for all finite time interval [T_1, T], $T_1 \geq T_0$, there exists $\varepsilon_0 = \varepsilon_0(a, b, c, d, T) > 0$ such that

$$\begin{aligned} E_1(t,\varepsilon,a,b,c,d) &< E_1(0,\varepsilon,a,b,c,d), \\ E_2(t,\varepsilon,a,b,c,d) &> E_2(0,\varepsilon,a,b,c,d), \end{aligned} \tag{3}$$

for all $t \in [T_1,T]$ and $\varepsilon \in (0, \varepsilon_0)$.

The governing equations of the system considered are given by (from references [1, 2])

$$\begin{cases} \ddot{x} + \omega_1^2 x + \varepsilon\left(c_0\dot{x} + \dfrac{\partial V}{\partial x}(x,y) + A\sin(\omega t)\right) = 0, \\ \ddot{y} + y^3 + \varepsilon\left(c_0\dot{y} + \dfrac{\partial V}{\partial x}(x,y)\right) = 0, \end{cases} \tag{4}$$

where c_0 is the coefficient of the viscous damping, x is the displacement of the body 1 from its equilibrium position and y is the displacement of the body 2. Moreover, V is the potential energy associated to the coupling spring, **A** and **ω** are the amplitude and frequency of the external excitation respectively. It is assumed that $V(0,0) = 0$.

Here, we will consider an energy transfer of coupled linear and nonlinear oscillators for the case of linear interaction.

3. Overview on Theoretical Results

Using the following change of variables $q_1 = x,\ p_1 = \dot{x},\ q_2 = y,\ p_2 = \dot{y}$, Eq.(4) is wrote in the stable variable (q_1, p_1, q_2, p_2) as being

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{p}_1 = -\omega_1^2 q_1 - \varepsilon\left(c_0 p_1 + \dfrac{\partial V}{\partial q_1} + A\sin(\omega t)\right), \\ \dot{q}_2 = p_2, \\ \dot{p}_2 = -q_2^3 - \varepsilon\left(c_0 p_2 + \dfrac{\partial V}{\partial q_2}\right) \end{cases} \tag{5}$$

Assuming also

$$V(q_1, q_2) = \frac{(q_1 - q_2)^2}{2} \tag{6}$$

and using the new change of variables

$$\begin{aligned} q_1 &= \sqrt{\frac{2I}{\omega_1}}\sin\theta, \quad & p_1 &= \sqrt{2\omega_1 I}\cos\theta, \\ q_2 &= J\, cn\varphi, & p_2 &= -J^2 sn\varphi\, dn\varphi, \end{aligned} \tag{7}$$

in Eq.(4), where cn[t,k], sn[t,k], and dn[t,k], are the classical elliptic Jacobian functions (see reference [1]), with argument $k = 1/\sqrt{2}$ we will obtain the following system

$$\begin{cases} \dot{I} = \varepsilon\left(-2c_0 I\cos^2\theta - \sqrt{\dfrac{2I}{\omega_1}}\cos\theta(q_1 - q_2) - A\sqrt{\dfrac{2I}{\omega_1}}\cos\theta\sin\omega t\right) \\ \dot{\theta} = \omega_1 + \varepsilon\left(c_0\sin\theta\cos\theta + \dfrac{\sin\theta}{\sqrt{2\omega_1 I}}(q_1 - q_2) + \dfrac{A}{\sqrt{2\omega_1 I}}\sin\theta\sin\omega t\right) \\ \dot{J} = \varepsilon\left(-c_0 J cn'^2\varphi - J^{-1}cn'\varphi(q_2 - q_1)\right) \\ \dot{\varphi} = J + \varepsilon\left(c_0 cn\varphi\, cn'\varphi + J^{-2}cn\varphi(q_2 - q_1)\right) \end{cases} \tag{8}$$

From Eqs.(2) and (7), where H_1 represents the unperturbed energies of the system (5), follows that,

$$E_1 = I, \quad E_2 = \frac{J^4}{4} \tag{9}$$

Taking I(0) = e_1, θ(0)= α, J(0) = e_2 and φ(0)= β as being the initial conditions, we gotten from the classical Theorem of Differentiability of the Flow of Autonomous Systems that given T > 0 there is $\varepsilon_0 > 0$ such that

$$\begin{aligned} I &= I_0 + \varepsilon I_1 + O(\varepsilon^2) \\ \theta &= \theta_0 + \varepsilon\theta_1 + O(\varepsilon^2) \\ J &= J_0 + \varepsilon J_1 + O(\varepsilon^2) \\ \varphi &= \varphi_0 + \varepsilon\varphi_1 + O(\varepsilon^2) \end{aligned} \tag{10}$$

Substituting the initial conditions $I_0 = e_1$, $\theta_0 = \alpha$, $J_0 = e_2$ and $\varphi_0 = \beta$ and Eq.(10) into Eq.(8) we will obtain that

$$\begin{aligned} I(t,e_1,\alpha,e_2,\beta,\varepsilon) &= e_1 + \varepsilon I_1(t,e_1,\alpha,e_2,\beta,\varepsilon) + O(\varepsilon^2) \\ J(t,e_1,\alpha,e_2,\beta,\varepsilon) &= e_2 + \varepsilon J_1(t,e_1,\alpha,e_2,\beta,\varepsilon) + O(\varepsilon^2) \end{aligned} \tag{11}$$

where

$$\begin{aligned} I_1 &= -\frac{c_0 e_1}{2} t + \frac{2K}{\pi}\sqrt{\frac{2e_1}{\omega_1}} \sum_{m=0}^{\infty} a_m \cos\left(\alpha - \frac{\omega_1\beta}{e_2}\right)\Lambda_1 - \\ &\quad - \frac{2AK}{\pi e_2}\sqrt{\frac{2e_1}{\omega_1}}\Lambda_2 + b_1(t,\alpha,\beta) \\ J_1 &= -\frac{c_0 e_2 k}{2} t + \frac{1}{e_2^2}\sqrt{\frac{2e_1}{\omega_1}} os\left(\alpha - \frac{\omega_1\beta}{e_2}\right) \times \\ &\quad \times \sum_{m=0}^{\infty} (2m_0+1) a_m \Lambda_3 + b_2(t,\alpha,\beta) \end{aligned} \tag{12}$$

and

$$\Lambda_1 = \int_{\frac{\beta\pi}{2K}}^{\frac{e_2\pi}{2K}\left(t+\frac{\beta}{e_2}\right)} \frac{1}{2}\cos\left(\left(-2\frac{K\omega_1}{e_2\pi} + 2m + 1\right)u\right)du$$

$$\Lambda_2 = \int_{\frac{\beta\pi}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)} \cos\left(\frac{2K\omega_1}{e_2\pi}u - \frac{\omega_1\beta}{e_2} + \alpha\right)\sin\left(\frac{2K\omega}{e_2\pi}u - \frac{\omega\beta}{e_2}\right)du$$

$$\Lambda_3 = \int_{\frac{\beta\pi}{2K}}^{\frac{\pi}{2K}(e_2 t+\beta)} \frac{1}{2}\cos\left(\left(2m + 1 - \frac{2K\omega_1}{e_2\pi}\right)u\right)du \tag{13}$$

Here b_1 and b_2 are limited functions.

Under the internal resonance (that is, $\omega_1 = \omega$), if the condition

$$e_2 = \frac{2K}{(2m_0+1)\pi}, \tag{14}$$

is satisfied for some $m_0 \in \aleph$, where K is given by

$$K = \int_0^1 \left(1-t^2\right)^{-1/2}\left(1 - \frac{1}{2}t^2\right)^{-1/2} dt, \tag{15}$$

Then we will get

$$\begin{aligned} E_1 &= e_1 + \varepsilon\left(\frac{-c_0 e_1}{2} + a_{m_0} e_2\sqrt{\frac{e_1}{2\omega}}\cos\left(\alpha - \frac{\omega\beta}{e_2}\right)\right) + \\ &\quad + \varepsilon\left(A\sqrt{\frac{e_1}{2\omega}}\sin\alpha + O(1)\right)t + O\left(\varepsilon^2\right), \end{aligned} \tag{16a}$$

$$\begin{aligned} E_2 &= \frac{e_2^4}{4} + e_2^3\varepsilon\left(-\cos\left(\alpha - \frac{\omega\beta}{e_2}\right)\frac{(2m_0+1)\pi a m_0}{4Ke_2}\sqrt{\frac{2e_1}{\omega}}\right) - \\ &\quad - \varepsilon\left(\frac{-c_0 e_2 k}{2} + O(1)\right)t + O\left(\varepsilon^2\right), \end{aligned} \tag{16b}$$

where k and a_{mo} depend on the Fourier expansion of the periodic function cn(t).

From now, we will assume that **A =0**. Then, the problem of energy transfer reduces itself, to the analysis of the signal of the coefficients of t in the Eqs.(16).

Defining

$$\Psi_1 = \frac{-c_0 e_1}{2} + a_{m_0} e_2 \sqrt{\frac{e_1}{2\omega}} \cos\left(\alpha - \frac{\omega\beta}{e_2}\right)$$

$$\Psi_2 = \frac{-c_0 e_2 k}{2} - \cos\left(\alpha - \frac{\omega\beta}{e_2}\right) \frac{(2m_0+1)\pi a_{m_0}}{4Ke_2} \sqrt{\frac{2e_1}{\omega}} \quad (17)$$

So, there are three cases to be considered:

a) If $\Psi_1 < 0$ and $\Psi_2 > 0$ then energy pumping occurs.

b) If $\Psi_1 > 0$ and $\Psi_2 < 0$ then the linear oscillator suffers an increase of energy and the non-linear loses energy.

c) If $\Psi_1 < 0$ and $\Psi_2 < 0$ then both oscillators lose energy.

Observe that from Eq.(14), the conditions (a), (b) and (c) are equivalent to, respectively

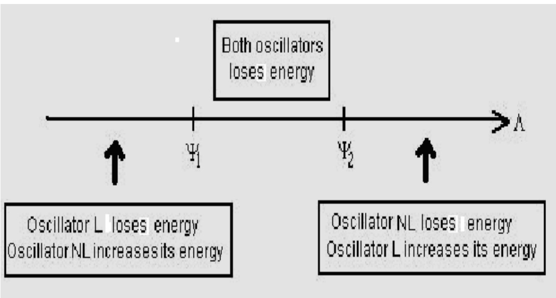

Figure 1: Conditions for energy transfer between oscillators (L for linear and NL for nonlinear oscillator)

where

$$\Psi_1 = \frac{4c_0 k K^3 \sqrt{2\omega_1}}{(2m_0+1)^3 \pi^3 a_{m_0} \sqrt{e_1}}, \quad (18)$$

$$\Psi_2 = \frac{\pi c_0 (2m_0+1)\sqrt{2e_1\omega_1}}{4Ka_{m_0}}, \quad (19)$$

$$\Lambda = \cos\left(\alpha - \frac{\omega\beta}{e_2}\right) \quad (20)$$

Next, we will discuss some numerical results, those confirm the theoretical results presented in this section.

4. Numerical Results

For the numerical simulations we have used the C language and the Matlab environment. A fourth order Runge-Kutta with fixed time step was used to the numerical integrations.

The following parameters are common for all simulations: $\omega = 1.01$, $c_0 = 1.0$, $k^2 = 0.5$, $K = 1.854075$, $m_0 = 3.0$, $\varepsilon = 0.025$ and $a_{mo} = 0.95$.

Particular cases satisfying conditions (a), (b) and (c) in previous section were obtained for the system subjected to different initial conditions.

For the plots in Fig.2 we choose the initial conditions $e_1 = 1.0$, $e_2 = 0.01$, $\alpha = \pi$ and $\beta = \pi/4$. Note that both oscillators perform damped free oscillations and no energy transfer occurs. This case corresponds to $\Psi_1 < 0$ and $\Psi_2 < 0$ or, equivalently, $\Psi_1 < \Lambda < \Psi_2$ (see Fig.1).

Analysis of the behavior of energies (Figs. 2a, 3a and 4a) can be done in a similar fashion that in reference [4].

In Fig.2a, we observe that the energies I and J (see Eq.(8) and Eq.(9)) decay nearly exponentially to zero indicating absence of resonance capture. The second plot in that figure corresponds to the transient response of Eq.(5).

In Fig.3, we have $\Lambda > \Psi_2$, accordingly to the notation in Fig.1, that is, the linear oscillator increases its energy, while the nonlinear attachment loses energy.

Here we have used the initial conditions $e_1 = e_2 = 1.0$, $\alpha = \beta = \pi/4$.

Note that in Fig. 3a as the time progresses the energy of the linear oscillator surpasses the energy of the nonlinear oscillator.

Finally, in Fig.4 the irreversible energy transfer (that is, energy pumping) from the linear to the nonlinear oscillator takes place. The numerical time decays of energies are depicted in Fig.4a for initial conditions $e_1 = 1.0$, $e_2 = 0.002$, $\alpha = \pi$ and $\beta = \pi/4$.

Fig.4b and c depict the transient response of the linear oscillator considering coupled and uncoupled system (Eq.5), and the motion of the linear oscillator together the nonlinear attachment, respectively.

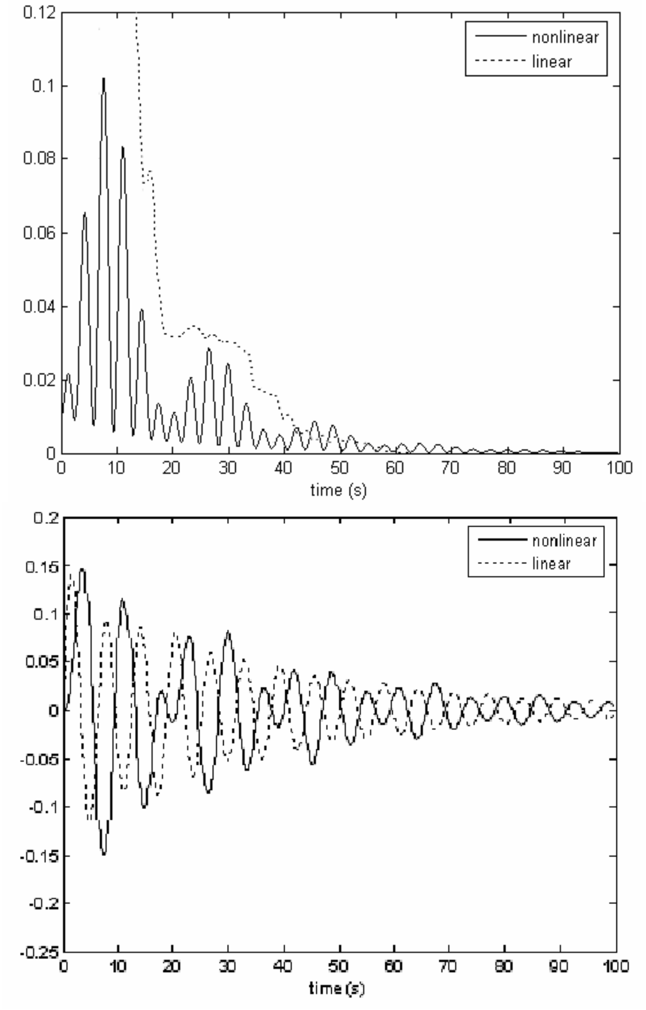

Figure 2: Absence of energy transfer.
(a) Energies; (b) transient response of the system (5).

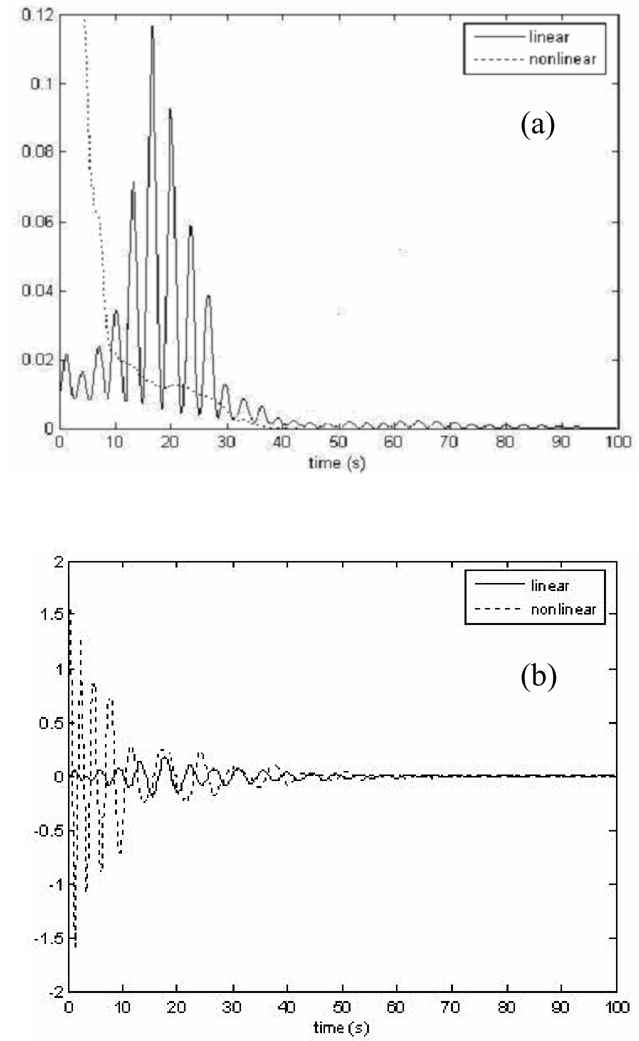

Figure 3: Energy transfer from nonlinear to linear oscillator.
(a) Energies; (b) transient response of the system (5).

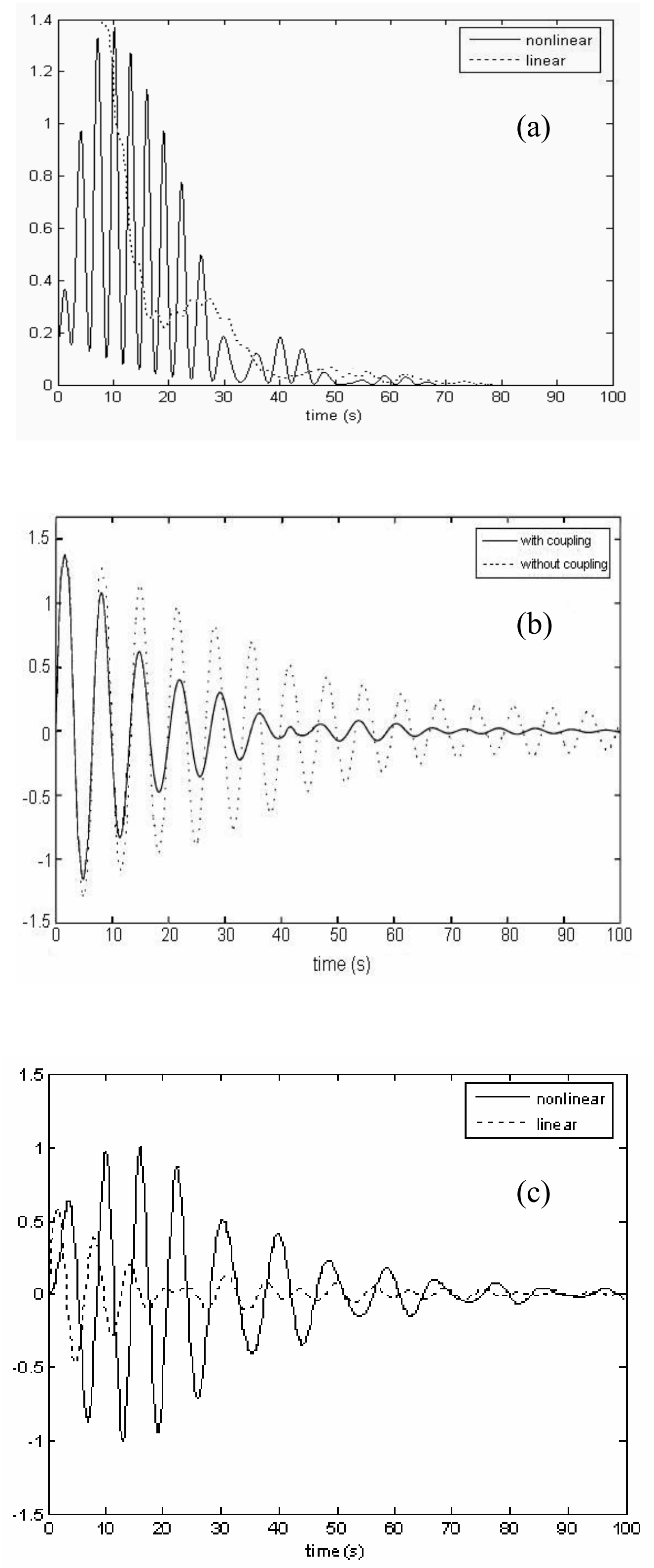

Figure 4: Energy transfer from linear to nonlinear oscillator.
(a) Energies; (b) Transient response of the system (5); (c) Energy pumping.

5. Conclusion

In this paper, we analyzed the problem of energy transfer in a dissipative mechanical system. The presented definition of energy transfer was discussed through a practice example.

The results presented in this paper take into account a more general dynamical phenomenon than energy pumping.

As a result we showed that depending on the initial conditions, there is energy transfer from linear to nonlinear oscillator and from nonlinear to linear oscillator. Moreover, there are initial conditions such that both oscillators lose energy.

Numerical simulations are in complete agreement with theoretical results.

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