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A FINITE DIMENSIONAL MECHANICAL SYSTEM WITH A CASCADE OF NON SMOOTH CONSTITUTIVE TERMS

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ABSTRACT *We describe the model corresponding to a one degree-of-freedom mechanical system fixed to a support via a cascade of non smooth constitutive laws: The basic nonlinearity of the constitutive terms consists of dry-friction elements. We study dynamical behavior of the system. The model of the studied mechanical system corresponds to the motion of an elastoplastic chain driving one mass on a fixed support.*

1. Introduction

In this work we are going on the study of varied classes of dynamical behaviors of wide classes of nonlinear oscillators including non smooth terms of Saint-Venant (or dry friction) type. Previous works are devoted to the mechanical, numerical and mathematical study of one-degree-of-freedom or multi-degree-of-freedom oscillators that may include also delay terms or history terms under deterministic external solicitations or under stochastic excitations: See references [Bas00, SLB99, BSL00, LBB05, BS02, BS00, Ber03, LBH03, BSL04a, BSL04b, BL05, LBH05, BL07a, BL07b].

In this paper, we describe the model corresponding to a one degree-of-freedom mechanical system fixed to a support via a cascade of non smooth constitutive laws that consist of dry-friction elements. We study dynamical behavior of the system. Let us notice that quasi-static behavior could also be investigated via the same method and quasi-static model derived from the present one. The model of the studied mechanical system corresponds to the motion of an elastoplastic chain driving one mass on a fixed support. We show that mathematical expression of the system is

$$\dot{X} + M\partial\Phi(X) \ni f(t,X), \quad X(0) = X_0, \qquad (1.1)$$

where for a real $T > 0$, and convenient integer N, $X : [0,T] \mapsto \mathbb{R}^N$ is a function, $f : [0,T] \times \mathbb{R}^N$ is a Lipschitz continuous function from $[0,T]\times\mathbb{R}^N$ to $\mathbb{R}^N$ that contains external deterministic solicitation, Φ is a convex function from $\mathbb{R}^N$ to $]-\infty,+\infty]$, $\partial\Phi(X)$ is its sub-differential at X defining a maximal monotone operator (see [Bre73]), $M \in \mathcal{M}_N(\mathbb{R})$ is a symmetric positive definite matrix.

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The studied system is different from previously considered systems either classical ones (see [BSL00, LBB05]) or gephyroidal model ([BL07b]).

The mathematical model and its numerical treatment is close to the case of the gephyroidal model ([BL07b]) because en euclidean non classical metrics is also used. Nevertheless the geometry of the cascade depends on an arbitrary number of dry friction elements contrary to the gephyroidal basic model.

The paper is organized as follows. In Section 2, we describe the studied class of models. In Section 3, we give mathematical expression of the model. In Section 4, we provide numerical scheme for the model and give mathematical properties. Finally we sum up main results of this work as a conclusion.

2. Description of the model

The frame of maximal monotone operators is convenient for the study of wide classes of elastoplastic oscillators. Let us consider here a one-degree-of-freedom oscillator that consists of one mass m oscillating on an horizontal plan, fixed to a support via a cascade of $n+1$ springs with stiffness $k_i > 0$ ($i = 0, \dots, n$) and n Saint-Venant elements (dry friction elements) with threshold $\alpha_i > 0$ ($i = 1, \dots, n$) as described in Figure 1 in Appendix B. Let x denote the horizontal displacement of mass m submitted to external forcing F.

This model does not correspond to any association of elementary sub-models involving either one spring and one Saint-Venant element settled in series or one spring and one Saint-Venant element settled in parallel. It does not correspond to gephyroid model [BL07b]. Nevertheless the model can be expressed via a differential inclusion of type (1.1).

For $i \in \{0, \dots, n\}$ let us denote (see Figure 2 in Appendix B)

- u_i the displacement of the spring number i vs its reference position,
- f_i the internal force of the spring number i associated to displacement u_i .

For $i \in \{1, \dots, n\}$ let us denote (see Figure 2 in Appendix B)

- v_i the displacement of the Saint-Venant element number i vs its reference position,
- g_i the internal force of the Saint-Venant element number i associated to displacement v_i.

Each constitutive element possesses its constitutive law that can be written as:

$$\forall i \in \{0, \dots, n\}, \quad f_i = -k_i u_i, \tag{2.1}$$

and

$$\forall i \in \{1, \dots, n\}, \quad g_i \in \alpha_i \sigma(\dot{v}_i) \tag{2.2}$$

where σ denotes the graph of the sign function defined by $\sigma(z) = \{-1\}$ if $z < 0$, $\sigma(z) = \{1\}$ if $z > 0$, $\sigma(z) = [-1, 1]$ if $z = 0$.

Taking into account that any of the springs 1 to n and any of the Saint-Venant element is only linked to the spring numbered 0 and to the mass m, we can write fundamental relation for the mass m in the form ($\dot{} = d/dt$)

$$m\ddot{x} = f_0 + F. \tag{2.3}$$

Geometrical relations have to be included in the form

$$\forall i \in \{1, \dots, n\}, \quad x = v_i + \sum_{j=0}^{i-1} u_j, \tag{2.4}$$

and

$$u_n = v_n, \tag{2.5}$$

up to constants corresponding to reference positions of each constitutive element. Equilibrium of each node of the considered system leads to

$$\forall i \in \{0, \dots, n-1\}, \quad f_i = f_{i+1} + g_{i+1}, \tag{2.6}$$

and finally

$$f_0 = \sum_{i=1}^{n} g_i + f_n. \tag{2.7}$$

Indeed, equation (2.7) is useless since it can be obtained by summation of equations (2.6).

3. Mathematical expression of the class of models

From previous Section, one can see that the model is expressed via equations (2.1), (2.2), (2.3), (2.4), (2.5), (2.6).

Using (2.3), (2.1) (for $i = 0$) and (2.4) (for $i = 1$) one has

$$m\ddot{x} = -k_0 u_0 + F = -k_0(x - v_1) + F. \tag{3.1}$$

Using successively (2.6), (2.2) then (2.1) we can obtain :

$$\forall i \in \{1,\dots,n\}, \quad k_i u_i - k_{i-1}u_{i-1} \in -\alpha_i \sigma(\dot{v}_i). \tag{3.2}$$

System of equations (2.4), (2.5) can be inverted so that

$$\begin{cases} u_0 = x - v_1, \\ \forall i \in \{1,\dots,n-1\}, \quad u_i = v_i - v_{i+1}, \\ u_n = v_n. \end{cases} \tag{3.3}$$

Finally we obtain the following constitutive model in the form

$$\begin{cases} -k_0 x + (k_0+k_1)v_1 - k_1 v_2 \in \alpha_1 \sigma(\dot{v}_1), \\ \forall i \in \{2,\dots,n-1\}, \\ -k_{i-1}v_{i-1} + (k_{i-1}+k_i)v_i - k_i v_{i+1} \in -\alpha_i \sigma(\dot{v}_i), \\ -k_{n-1}v_{n-1} + (k_{n-1}+k_n)u_n \in -\alpha_n \sigma(\dot{v}_n). \end{cases} \tag{3.4}$$

Let us introduce the inverse graph of σ, denoted β and defined by

$$\beta(x) = \begin{cases} \emptyset & \text{if } x \in (-\infty,-1)\cup(1,+\infty), \\ \{0\} & \text{if } x \in (-1,1), \\ \mathbb{R}_- & \text{if } x = -1, \\ \mathbb{R}_+ & \text{if } x = 1. \end{cases} \tag{3.5}$$

Equation (3.4) can be expressed as

$$\begin{cases} \dot{v}_1 + \beta\left(\dfrac{-k_0 x + (k_0+k_1)v_1 - k_1 v_2}{\alpha_1}\right) \ni 0, \\ \forall i \in \{2,\dots,n-1\}, \\ \dot{v}_i + \beta\left(\dfrac{-k_{i-1}v_{i-1} + (k_{i-1}+k_i)v_i - k_i v_{i+1}}{\alpha_i}\right) \ni 0, \\ \dot{v}_n + \beta\left(\dfrac{-k_{n-1}v_{n-1} + (k_{n-1}+k_n)u_n}{\alpha_n}\right) \ni 0. \end{cases} \tag{3.6}$$

Now, using a convenient change of variables, the mathematical model of our problem can be formulated in the form (1.1). Let us introduce the tridiagonal $n\times n$ matrix K defined by (A.1), in Appendix A. It can be easily proved that K is a symmetric positive definite matrix of $\mathcal{M}_n(\mathbb{R})$.

Let us define V, Z and W vectors of $\mathbb{R}^n$ by

$$V = \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{pmatrix}, \quad Z = \begin{pmatrix} k_0 x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.7}$$

and

$$W = KV - Z. \tag{3.8}$$

Equation (3.6) can be expressed as

$$\forall i \in \{1,\dots,n\}, \quad \dot{v}_i + \beta\left(\frac{w_i}{\alpha_i}\right) \ni 0, \tag{3.9}$$

or in the equivalent form

$$\dot{V} + \partial\psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(W) \ni 0. \tag{3.10}$$

where $\psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}$ denotes the convex function indicatrix of the convex domain $[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]\subset\mathbb{R}^n$. Clearly from (3.8) we have

$$\dot{V} = K^{-1}(\dot{W} + \dot{Z}). \tag{3.11}$$

From equations (3.1), (3.6), and (3.11), we obtain the system of differential inclusions

$$\begin{cases} m\ddot{x} + k_0 x - k_1 v_1 = F, \\ \dot{W} + K\partial\psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(W) \ni -\dot{Z}. \end{cases} \tag{3.12}$$

Let us denote $[U]_1$ the first component of any vector $U \in \mathbb{R}^m$, for m integer. Let us set $y = \dot{x}$ and $u = (1,1,\dots,1)^T \in \mathbb{R}^n$ and $v = (1,0,\dots,0)^T \in \mathbb{R}^n$. The problem defined by equations (3.1) and (3.6) can be developed in

$$\begin{cases} \dot{x} = y, \\ \dot{y} = (F - k_0 x + k_0[K^{-1}W]_1 + k_0^2 x[K^{-1}u]_1)/m, \\ \dot{W} + K\partial\psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(W) \ni -k_0 y v. \end{cases} \tag{3.13}$$

Let us introduce the $(n+2)\times(n+2)$ symmetric definite positive matrix M defined by

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & K & \\ 0 & 0 & & & \end{pmatrix} \tag{3.14}$$

Finally let us introduce vector X in $\mathbb{R}^{n+2}$ defined by

$$\begin{cases} X(t) = (x(t), y(t), W(t))^T, \\ X(0) = (x(0), y(0), W(0))^T, \end{cases} \tag{3.15}$$

and

$$\mathcal{F}(t, X(t)) = \begin{pmatrix} y \\ C \\ -k_0 y \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.16}$$

where

$$C = (F - k_0 x + k_0[K^{-1}W]_1 + k_0^2 x[K^{-1}u]_1)/m. \tag{3.17}$$

The problem can be written in the announced form:

$$\begin{cases} \dot{X} + M\partial\psi_{\mathcal{C}}(X) \ni \mathcal{F}(t, X(t)), \\ X(0) = X_0. \end{cases} \tag{3.18}$$

where $\psi_{\mathcal{C}}$ denotes the convex function indicatrix of the convex domain $\mathcal{C} = \mathbb{R} \times \mathbb{R} \times [-\alpha_1, \alpha_1] \times \cdots \times [-\alpha_n, \alpha_n] \subset \mathbb{R}^{n+2}$ and $N = n + 2$.

4. Numerical scheme for the general model

Based on previous works [BS02, BS00, Bas00], one can prove that the problem (3.18) possesses an unique solution $X \in W^{1,\infty}(0, T; \mathbb{R}^{n+2})$, if $F \in H^1(0, T)$. Due to the expression of the problem in the frame of maximal monotone operators, a numerical scheme can be built. Let $h > 0$ be time step, and to simplify $t_q = qh$ for any integer $q \geq 0$. One can write:

$$\begin{cases} \dfrac{X^{q+1} - X^q}{h} + M\partial\psi_{\mathcal{C}}(X^{q+1}) \ni \mathcal{F}(t_q, X^q), \\ X^0 = X_0. \end{cases} \tag{4.1}$$

From previous theoretical works[BS02, BS00, Bas00], we can prove that this Euler implicit type numerical scheme is convergent with optimal order 1, i.e. $O(h)$.

5. Conclusion

The main results of this paper are

- in the mechanical point of view, the description of a system that can not be described by classical assemblies of springs and Saint-Venant elements or gephyroid types models,
- in the mathematical point of view, the description of the model of the system in the frame of maximal monotone operators leading to an unique solution of the problem approximated by a non event driven numerical scheme with optimal convergence order 1.

References

[Bas00] Jérôme Bastien. *Étude théorique et numérique d'inclusions différentielles maximales monotones. Applications à des modèles élastoplastiques.* PhD thesis, Université Lyon I, 2000. Numéro d'ordre : 96-2000.

[Ber03] Frédéric Bernardin. Multivalued stochastic differential equations: convergence of a numerical scheme. *Set-Valued Anal.*, 11(4):393–415, 2003.

[BL05] Jérôme Bastien and Claude-Henri Lamarque. Maximal monotone model with history term. *Non linear Analysis*, 63(5-7):e199–e207, 2005.

[BL07a] Jérôme Bastien and Claude-Henri Lamarque. Non smooth dynamics of mechanical systems with history term. *Nonlinear Dynam.*, 47(1-3):115–128, 2007.

[BL07b] Jérôme Bastien and Claude Henri Lamarque. Persoz's gephyroidal model described by a maximal monotone differential inclusion. to appear in *Archive of Applied Mechanics*, DOI 10.1007/s00419-007-0171-8, 2007.

[Bre73] Haïm Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[BS00] Jérôme Bastien and Michelle Schatzman. Schéma numérique pour des inclusions différentielles avec terme maximal monotone. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(7):611–615, 2000.

[BS02] Jérôme Bastien and Michelle Schatzman. Numerical precision for differential inclusions with uniqueness. *M2AN Math. Model. Numer. Anal.*, 36(3):427–460, 2002.

[BSL00] Jérôme Bastien, Michelle Schatzman, and Claude-Henri Lamarque. Study of some rheological models with a finite number of degrees of freedom. *Eur. J. Mech. A Solids*, 19(2):277–307, 2000.

[BSL04a] Frédéric Bernardin, Michelle Schatzman, and Claude-Henri Lamarque. Second-order multivalued stochastic differential equations on Riemannian manifolds. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 460(2051):3095–3121, 2004.

[BSL04b] Frédéric Bernardin, Michelle Schatzman, and Claude-Henri Lamarque. A Stochastic differential equation from friction mechanics. *C. R. Math. Acad. Sci. Paris*, 338(11):837–842, 2004.

[LBB05] Claude-Henri Lamarque, Frédéric Bernardin, and Jérôme Bastien. Study of a rheological model with a friction term and a cubic term: deterministic and stochastic cases. *Eur. J. Mech. A Solids*, 24(4):572–592, 2005.

[LBH03] Claude-Henri Lamarque, Jérôme Bastien, and Matthieu Holland. Study of a maximal monotone model with a delay term. *SIAM J. Numer. Anal.*, 41(4):1286–1300 (électronique), 2003.

[LBH05] Claude-Henri Lamarque, Jérôme Bastien, and Matthieu Holland. Maximal monotone model with delay term of convolution. *Math. Probl. Eng.*, (4):437–453, 2005.

[SLB99] Michelle Schatzman, Claude-Henri Lamarque, and Jérôme Bastien. An ill-posed mechanical problem with friction. *Eur. J. Mech. A Solids*, 18(3):415–420, 1999.

Appendix A. Definition of K

$$K = \begin{pmatrix} k_0+k_1 & -k_1 & 0 & 0 & \dots & 0 & 0 & 0 & \\ -k_1 & k_1+k_2 & -k_2 & 0 & \dots & \dots & 0 & 0 & \\ 0 & -k_2 & k_2+k_3 & -k_3 & 0 & \dots & \dots & 0 & \\ & \vdots & \vdots & \dots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & -k_{n-2} & k_{n-2}+k_{n-1} & -k_{n-1} & \\ 0 & 0 & \dots & \dots & \dots & 0 & -k_{n-1} & k_{n-1}+k_n & \end{pmatrix} \tag{A.1}$$

Appendix B. Figures

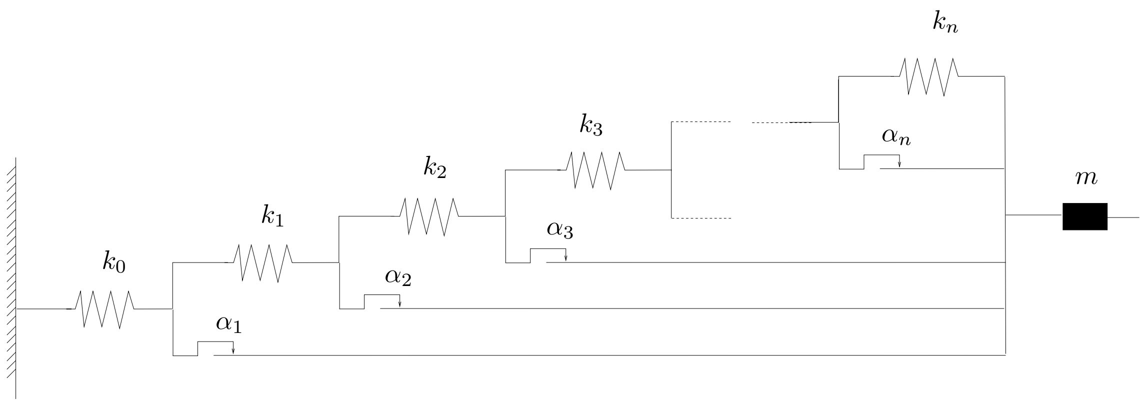

FIGURE 1. One-degree-of-freedom system with a cascade of Saint-Venant elements.

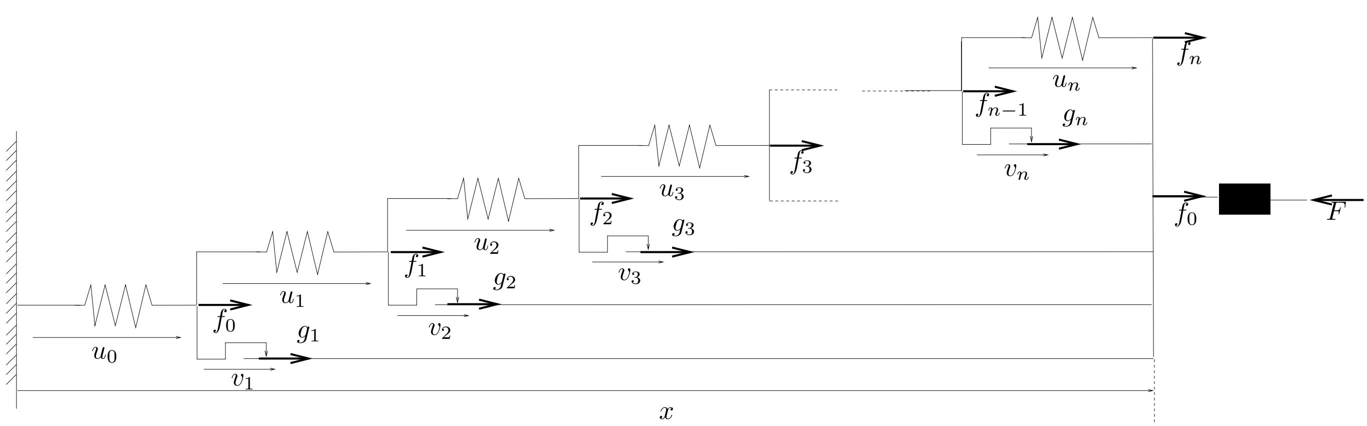

FIGURE 2. One-degree-of-freedom system with a cascade of Saint-Venant elements with displacements x, u_i and v_i and forces f_i and g_i.