A study of control systems from the geometry of ring of scalars

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Abstract

In this paper we relate the problem of finding all feedback classes of Brunovsky and locally Brunovsky linear systems defined on a commutative ring with combinatorial problems of visiting respectively all Ferrers and colored Ferrers diagrams of a fixed size.

On the other hand, in the case of rings of real continuous functions defined on a compact topological space we point out the topological properties related to the problem. We study the dimension 1 case and comment the 2-dimensional case.

AMS classification: 93B10; 93B25; 93B55.
PACS: 02.30.Yy; 02.10.Uw; 02.40.Re
Keywords: Feedback classification; Projective modules; Enumeration.

1 Introduction

Let R be a commutative ring with identity element $1 \neq 0$. A sequential linear control system is a discrete time ($\mathcal{T} = \mathbb{Z}$) dynamical system following a linear rule (or right hand side) on the form

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array}$$

where $x(-) : \mathbb{Z} \to X$ is the sequence of internal states, $y(-) : \mathbb{Z} \to Y$ is the sequence of outputs of the system, and $u(-) : \mathbb{Z} \to U$ is the sequence of external controls (usually fixed or designed by the controler).

Sets X (of internal states), Y (of outputs), and U (of controls or external inputs) are R-modules in our context while maps $A : X \to X$, $B : U \to X$, and $C : X \to Y$ are R-linear maps. We describe the above sequential linear control system (linear system) by using the diagram

$$\Sigma : \boxed{\begin{array}{cccccc} U & & & & & Y \\ & \searrow^{B} & & & {}^{C}\nearrow & \\ & & X & \rightarrow^{A} & X & \end{array}}$$

The classical case is when $R = \mathbb{K}$ is either the field of real numbers or the field of complex numbers and $X \cong \mathbb{K}^n$, $Y \cong \mathbb{K}^p$, and $U \cong \mathbb{K}^m$ are finite dimensional based $\mathbb{K}$-vector spaces. In this case, a linear system is given by a triple of matrices.

$$\Sigma : \boxed{\begin{array}{cccccc} \mathbb{K}^m & & & & & \mathbb{K}^p \\ & \searrow^{B} & & & {}^{C}\nearrow & \\ & & \mathbb{K}^n & \rightarrow^{A} & \mathbb{K}^n & \end{array}}$$

There are many subjects in control theory. Some of them may be sumarized as "related to reachability" and involves only the left half of above picture; that is, output space plays no rôle. One of that subjects is feedback equivalence. Hence in the sequel outputs will not be considered and linear systems are, in this context, reduced to

$$\Sigma : \boxed{\begin{array}{cccc} U & & & \\ & \searrow^{B} & & \\ & & X & \rightarrow^{A} \; X \end{array}}$$

For general reading on linear systems over commutative rings the reader is referred to [1].

Feedback equivalence of linear systems

The algebraic equivalence of linear systems deals with the study of linear systems up to isomorphisms in the input and state modules . But it is more interesting (from the control theoretic side) to allow linear feedback actions on linear systems; that is to say we are allowed to design controls as linear functions of the current state $u = Fx$. This "closed loop" is at the very heart of control theory. To be concise:

Linear systems:

$$\Sigma : \boxed{\begin{array}{cccc} U & & & \\ & \searrow^{B} & & \\ & & X & \rightarrow^{A} \; X \end{array}}$$

and

$$\Sigma' : \boxed{\begin{array}{cccc} U' & & & \\ & \searrow^{B'} & & \\ & & X' & \rightarrow^{A'} \; X' \end{array}}$$

are said to be *Feedback Equivalent* if we can bring one of them into the another by a finite composition of the following *Basic Feedback Actions*:

1. Isomorphisms $Q : U \to U'$ in the input module which transforms
$$(A, B) \to (A, BQ)$$
2. Isomorphisms $P : X \to X'$ in the state module which transforms
$$(A, B) \to (PAP^{-1}, PB)$$
3. Feedback actions $F : X \to U$ which transforms
$$(A, B) \to (A + BF, B)$$

Consequently a general feedback action (P, Q, F) brings linear system $\Sigma = (A, B)$ to system:
$$(A, B) \to (P(A + BF)P^{-1}, PBQ)$$

If two linear systems $\Sigma = (A, B)$ and $\Sigma' = (A', B')$ are feedback equivalent via (P, Q, F) then linear maps
$$\varphi_i^{\Sigma} = \begin{pmatrix} B & AB & \cdots & A^{i-1}B \end{pmatrix} : U^{\oplus i} \longrightarrow X$$
and
$$\varphi_i^{\Sigma'} = \begin{pmatrix} B' & A'B' & \cdots & (A')^{i-1}B' \end{pmatrix} : (U')^{\oplus i} \longrightarrow X'$$
are equivalent linear maps (in the sense of there exists an isomorphism $\alpha : (U')^{\oplus i} \to U^{\oplus i}$ such that $\varphi_i^{\Sigma'} = P \cdot \varphi_i^{\Sigma} \cdot \alpha$). This is because one has an equivalence of maps φ_i^{Σ} and $\varphi_i^{\Sigma'}$ for any basic feedback action:

1. Isomorphism Q in input modules brings (A, B) to (A, BQ) yields the equivalence
$$\varphi_i^{\Sigma'} = \varphi_i^{\Sigma} \begin{pmatrix} Q & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & Q & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & Q \end{pmatrix}$$
2. Isomorphism P in state modules brings (A, B) to (PAP^{-1}, PB) yields the equivalence
$$\varphi_i^{\Sigma'} = P\varphi_i^{\Sigma}$$

3. Feedback action $F: X \to U$ of states onto inputs brings (A, B) to $(A+BF, B)$ and consequently one has the equivalence

$$\varphi_i^{\Sigma'} = \varphi_i^{\Sigma} \begin{pmatrix} \mathbf{1} & FB & F(A+BF)B & \cdots & F(A+BF)^{i-1}B \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & F(A+BF)B \\ \vdots & & \ddots & \ddots & FB \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Therefore linear maps φ_i^{Σ} and $\varphi_i^{\Sigma'}$ are equivalent and *a fortiori* cokernels (and images) are isomorphic. This gives us our main invariants. To be concise, we have proven the following result:

Lemma 1.1. *(cf. [4] Lemma 3) Consider the linear system*

$$\Sigma : \boxed{\begin{array}{ccc} U & & \\ & \searrow^{B} & \\ & X & \to^{A} \quad X \end{array}}$$

The R-modules $N_i^{\Sigma} = Im(\varphi_i^{\Sigma})$ and $M_i^{\Sigma} = Coker(\varphi_i^{\Sigma}) = X/Im(\varphi_i^{\Sigma})$ are feedback invariants, up to isomorphism associated to linear system Σ.

Some properties of the invariant R-modules are directly obtained:

Lemma 1.2. *With the above notations the following properties hold:*

1. *There exists an index $s \geq 0$ such that:*

$$0 = N_0^{\Sigma} \subsetneq N_1^{\Sigma} \subsetneq \cdots \subsetneq N_{s-1}^{\Sigma} \subsetneq N_s^{\Sigma} = N_{s+1}^{\Sigma} = \cdots$$

2. *Quotient module $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ is the kernel of the natural surjective map $M_i^{\Sigma} \to M_{i+1}^{\Sigma}$*

3. *Quotient modules $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ are also feedback invariants associated to system Σ.*

Proof.- To prove (1) first note that a Cayley-Hamilton Theorem applies on endomorphism $A : X \to X$ of finitely generated projective R-module X (see [11] Theorem IV.17) and A satisfies a monic polynomial $\chi(z) \in R[z]$. Thus there exists an index $s = \deg(\chi)$ such that A^s is a linear combination of $\mathbf{1}, A, ..., A^{s-1}$. Consequently $\Im(A^{s+1}B) \subseteq N_i^{\Sigma}$.

Now to conclude the proof of (1) note that if $N_i^{\Sigma} = N_{i+1}^{\Sigma}$ then it follows that $A(N_i^{\Sigma}) \subseteq N_i^{\Sigma}$ and consequently $A^j(N_i^{\Sigma}) \subseteq N_i^{\Sigma}$ for all j. Hence it follows that $(N_i^{\Sigma}) = N_{i+j}^{\Sigma}$ for all j.

(2) The natural quotient map $M_i^\Sigma \to M_{i+1}^\Sigma$ sending $x + N_i^\Sigma \mapsto x + N_{i+1}^\Sigma$ is obviously well defined and onto. Its kernel is the module

$$\{x + N_i^\Sigma : x \in N_{i+1}^\Sigma\} = N_{i+1}^\Sigma / N_i^\Sigma$$

To prove (3) consider two feedback equivalent linear systems Σ and Σ' and the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & N_{i+1}^\Sigma / N_i^\Sigma & \to & M_i^\Sigma & \to & M_{i+1}^\Sigma & \to & 0 \\
 & & \downarrow & \circlearrowleft & \updownarrow & \circlearrowleft & \updownarrow & & \\
0 & \to & N_{i+1}^{\Sigma'} / N_i^{\Sigma'} & \to & M_i^{\Sigma'} & \to & M_{i+1}^{\Sigma'} & \to & 0
\end{array}$$

where the isomorphisms are those obtained in Lemma 1.1. The first linear map is the quotient of the restriction of isomorphism $P : X \to X$ in the feedback action. Short Five Lemma implies that this linear map is also an isomorphism □

A natural question is: When does the above set of invariants characterize the feedback class of a linear system?

Answer is that those invariants are sufficient in the case of reachable linear systems over a finite dimensional vector space (that is, the Classical Brunovsky's Theorem in [2]). For reader's convenience we review the notion of reachable linear system in our context:

Definition 1.3. Linear system

$$\Sigma : \boxed{\begin{array}{ccccc} U & & & & \\ & \searrow^{B} & & & \\ & & X & \to^{A} & X \end{array}}$$

is called reachable if there exists an index s (this s is the same s above which stabilize the chain of N_i^Σ) such that the following equivalent conditions hold:

1. $N_s^\Sigma = X$ (not only isomorphic, but equal)

2. $M_s^\Sigma = 0$

In the general case of commutative rings, the invariants we introduced are not sufficient to state the feedback class of a linear system even for the dimension $m = n = 1$ case: Think in systems $\Sigma = ((2), (0))$ and $\Sigma' = ((2), (1))$ defined over $U = U' = X = X' = \mathbb{Z}$.

In fact it is proven in [6] that the class of commutative rings where those invariants do characterize the feedback classes of reachable linear systems is exactly the class of fields.

2 Brunovsky systems and their Ferrer's Diagrams

The linear system is a reachable linear system

$$\Sigma : \boxed{\begin{array}{cccc} R^m & & & \\ & \searrow^B & & \\ & & R^n & \to^A & R^n \end{array}}$$

is called of Brunovsky type if it is equivalent to a Brunovsky canonical form (see [6], [7]). In the case of $R = \mathbb{K}$ being a field, a Brunovsky linear system is just a reachable linear system. The same is true for the case of linear systems such that all its invariants are free defined over commutative rings R such that finitely generated projective R-modules are free (see [7]).

The key is that, in the case of reachable linear systems over a field or, in the more general framework of projective-free rings, if all the R-modules $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ are free then they are really a complete set of invariants verifying that

$$X = N_1^{\Sigma} \oplus N_2^{\Sigma}/N_1^{\Sigma} \oplus \cdots \oplus N_s^{\Sigma}/N_{s-1}^{\Sigma}$$

Let us denote by ξ_i^{Σ} the dimension of the free R-module $N_i^{\Sigma}/N_{i-1}^{\Sigma}$ in the Brunovsky case. It follows that

$$\sum_{i\geq 1}^{s} \xi_i^{\Sigma} = n = \dim X$$

On the other hand, the set

$$\xi_1^{\Sigma}, \xi_2^{\Sigma}, ..., \xi_s^{\Sigma}$$

characterizes the feedback class of system Σ.

Moreover, because of one has the splitting surjective map

$$\overline{A} : \begin{array}{ccc} N_i^{\Sigma}/N_{i-1}^{\Sigma} & \to & N_{i+1}^{\Sigma}/N_i^{\Sigma} \\ x + N_{i-1}^{\Sigma} & \to & Ax + N_i^{\Sigma} \end{array}$$

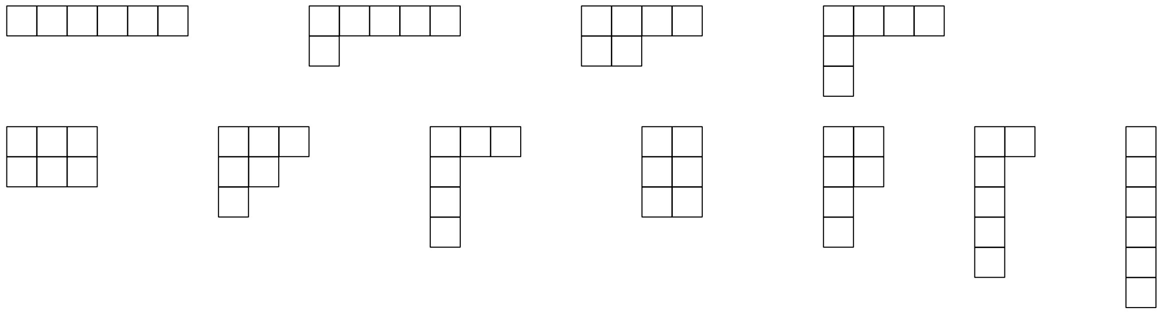
it follows that the sequence of ξ_i is decreasing; that is, $\xi_i^{\Sigma} \geq \xi_{i+1}^{\Sigma}$ for all i.

Thus, once we have fixed a projective-free ring R and the dimensions m and n, all feedback classes of linear systems of the form

$$\Sigma : \boxed{\begin{array}{cccc} R^m & & & \\ & \searrow^B & & \\ & & R^n & \to^A & R^n \end{array}}$$

are in one-to-one correspondence with the set of partitions of integer n in decreasing sequences, or, equivalently or by all the Ferrers diagrams of integer n.

For example, if we set $n = 6$ we have the following Ferrers diagrams visited following the reverse lexicographical order (see [9]):



Consequently, the description of all types of Brunovsky linear systems over a commutative ring R does not depend on the ring R but on the dimension n of free state module X. In fact, there are exactly $p(n)$ Brunovsky linear systems over a free module $X \cong R^n$, where $p(n)$ is the number of partitions of integer n into decreasing integers.

The study of the number of partitions goes back to L. Euler, who introduced in the XVIII century the main tools and seminal results. However, $p(n)$ is easily obtained for small values of n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

For large values of n we may estimate $p(n)$ by analytic methods (see [9] for details).

To conclude, note that we didn't say a word on the dimension of U in the above discussion. Because N_1^{Σ} is an homomorphic image of U and (in the particular case of Brunovsky systems) it is also a direct summand of U; it follows that the first row of Ferrers diagram of Σ (the dimension of N_1^{Σ}) needs to be lower than the dimension of U. Consequently if m is fixed then the number of feedback classes of Brunovsky systems is in fact $p_m(n)$, the number of decreasing partitions of n into pieces lower than m. Note that the extremal case $m = 1$ yields $p_1(n) = 1$ representing the Canonical Controller Form.

3 Locally Brunovsky systems and coloured Ferrer's Diagrams

Let R be a commutative ring, U and X finitely generated projective R-modules and

$$\Sigma : \boxed{\begin{array}{ccc} U & & \\ & \searrow^{B} & \\ & & X \xrightarrow{A} X \end{array}}$$

a linear system. A ring homomorphism $f : R \to S$ defines an additive functor $\mathbf{Proj}(R) \to \mathbf{Proj}(S)$ from finitely generated projective R-modules to finitely generated projective S-modules sending P to $P \otimes_R S$ (see [15] for example). Thus a new linear system

$$f^*(\Sigma) : \boxed{\begin{array}{lll} U \otimes_R S & & \\ & \searrow^{B\otimes 1} & \\ & & X \otimes_R S \quad \to^{A\otimes 1} \quad X \otimes_R S \end{array}}$$

arises as extension of scalars from R to S via f.

If U and X are based free modules (hence A and B are given by its matrices in the fixed bases) and $S = R/I$ where I is a ideal of R and f is the standard quotient map $r \mapsto \overline{r}$, then system $f^*(\Sigma)$ is just given by residual matrices $\overline{A}$ and $\overline{B}$ modulo I.

If K is a compact topological space and $R = \mathcal{C}(K)$ is the ring of real continuous functions defined on K then the map $P \mapsto \mathfrak{m}_P = \{f \in \mathcal{C}(K) : f(P) = 0\}$ is an homeomophism from K to the topological space Max(R) (the maximal spectrum of R together with the Zariski topology). Hence the ring homomorphism $R \to R/\mathfrak{m}$ is the evaluation of functions at P and provides the puntual study of systems.

On the other hand, if we consider the ring homomorphism $R \to R_\mathfrak{p}$ we have the localization of systems and the local study. The interested reader is referred to [5] for detais.

Definition 3.1. Let

$$\Sigma : \boxed{\begin{array}{lll} U & & \\ & \searrow^{B} & \\ & & X \quad \to^{A} \quad X \end{array}}$$

be a linear system and $\mathcal{P}$ a property of linear systems (i.e. being reachable, Brunovsky, controllable, pole-assignable,...). We say that Σ is locally-$\mathcal{P}$ (over R) if and only if all localizations

$$\Sigma_\mathfrak{p} : \boxed{\begin{array}{lll} U \otimes R_\mathfrak{p} & & \\ & \searrow^{B\otimes 1} & \\ & & X \otimes R_\mathfrak{p} \quad \to^{A\otimes 1} \quad X \otimes R_\mathfrak{p} \end{array}}$$

are $\mathcal{P}$ over $R_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ of R.

A property $\mathcal{P}$ of linear systems is local if one has

$$\mathcal{P} \Leftrightarrow \text{Locally-}\mathcal{P}$$

Note that because of U and X are finitely generated projective R-modules it follows that their localizations at $\mathfrak{p}$: $U \otimes R_\mathfrak{p}$ and $X \otimes R_\mathfrak{p}$ are free of finite dimension.

One the other hand, note that reachability is a local property of linear systems because it is stated in terms of surjectivity of a linear map. Some other properties are not local: Controllability, Pole-Assignability.

Next we deal with the locally Brunovsky property: A locally-Brunovsky linear system verifies that the localizations $\Sigma_{\mathfrak{p}}$ are Brunovsky. Hence the invariants $M_i^{\Sigma_{\mathfrak{p}}}$ are free. Because tensor product is right exact functor it follows that $M_i^{\Sigma_{\mathfrak{p}}} \cong M_i^{\Sigma} \otimes R_{\mathfrak{p}}$ and the result:

Lemma 3.2. *A locally Brunovsky linear system Σ has locally free (i.e. projective) invariants M_i^{Σ}.*

The converse is also true and a locally Brunovsky linear system Σ is a reachable linear system such that R-modules M_i^{Σ} are projective for all i.

Of course if commutative ring R verifies that finitely generated projective R-modules are free (for example if R is a field, if R is local, or the ring of rational integers $\mathbb{Z}$, or a polynomial ring $k[t_1, ..., t_n]$ where k is a field, ...) then Brunovsky = Locally Brunovsky. But, in general, we have Brunovsky $\Rightarrow$ Locally Brunovsky and not the converse (see bellow).

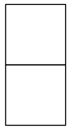
Note that if K is a compact topological space and we consider the ring $R = \mathcal{C}(K)$ of real valued continuous functions defined on K then finitely generated projective R-modules are in one to one correspondence with vector bundles over K (Swan's Theorem [14]). Thus finitely generated R-modules (and a fortiori locally Brunovsky linear systems) are characterized by the topological properties of K.

Consider $K = [0, 1]$ be the unit closed interval in $\mathbb{R}$ or, in general, K be a compact contractible space. Because of every vector bundle is trivial it follows that finitely generated $\mathcal{C}(K)$-modules are free and therefore every locally Brunovsky linear system over $\mathcal{C}(K)$ is of Brunovsky type.

On the other hand, the converse does not hold if the compact space admits nontrivial vector bundles: Put $K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\} = \mathbb{S}^1$ the unit sphere in $\mathbb{R}^2$, and the linear system (cf. [3])

$$\Sigma : \boxed{\begin{array}{ccc} \mathcal{C}(\mathbb{S}^1)^2 & & \\ & \searrow^{B=\begin{pmatrix} x-1 & y \\ -y & x+1 \end{pmatrix}} & \\ & \mathcal{C}(\mathbb{S}^1)^2 \xrightarrow{A=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} & \mathcal{C}(\mathbb{S}^1)^2 \end{array}}$$

This linear system is locally of Brunovsky type. Localization $\Sigma_{\mathfrak{p}}$ of Σ at any point (maximal ideal $\mathfrak{p}$ of $\mathcal{C}(\mathbb{S}^1)$) gives us a linear system with Ferrers diagram $\xi_1^{\Sigma_{\mathfrak{p}}} = \xi_2^{\Sigma_{\mathfrak{p}}} = 1$; that is:



But however Σ is not Brunovsky because M_1^Σ is a non-free rank 1 projective $\mathcal{C}(\mathbb{S}^1)$-module.

How can we study the feedback classes of locally Brunovsky linear systems? Next we introduce our main result which is based in the splitting properties of projective modules.

Theorem 3.3. *Let R be a commutative ring and let U, X be two finitely generated R-modules. Consider the linear system Σ given by:*

$$\Sigma : \boxed{\begin{array}{ccc} U & & \\ & \searrow^{B} & \\ & & X \quad \to^{A} \quad X \end{array}}$$

If M_1^Σ is projective then:

1. $X \cong N_1^\Sigma \oplus M_1^\Sigma$ *and* $U \cong N_1^\Sigma \oplus ker(B)$
2. *System Σ is feedback equivalent to linear system:*

$$\Sigma' : \boxed{\begin{array}{ccccc} N_1^\Sigma \oplus ker(B) & & & & \\ & \searrow \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & & \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a_{21} & a_{22} \end{pmatrix} & \\ & & N_1^\Sigma \oplus M_1^\Sigma & \longrightarrow & N_1^\Sigma \oplus M_1^\Sigma \end{array}}$$

Proof.- Part (1) follows from the exact splitting natural sequence (M_1^Σ is projective):

$$0 \quad \to \quad N_1^\Sigma \quad \to_{inclusion} \quad X \quad \to_{quotient} \quad M_1^\Sigma \quad \to \quad 0$$

On the other hand, the second isomorphism follows directly of the exact splitting sequence

$$0 \quad \to \quad \ker(B) \quad \to_{inclusion} \quad U \quad \to_B \quad N_1^\Sigma \quad \to \quad 0$$

To prove (2) we need only to consider that, according to decompositions in (1) we have that B is given by the matrix of linear maps

$$B = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where **1** is the identity linear map and **0** is the zero linear map. Thus the feedback action

$$F = \begin{pmatrix} -a_{11} & -a_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

brings Σ to Σ' □

Definition 3.4. With the above notations we define $\delta(\Sigma) = (a_{22}, a_{21})$ as the linear system:

$$\delta(\Sigma) : \boxed{\begin{array}{cccc} N_1^{\Sigma} & & & \\ & \searrow^{a_{21}} & & \\ & & M_1^{\Sigma} \xrightarrow{a_{22}} & M_1^{\Sigma} \end{array}}$$

Lemma 3.5. *Suppose that linear system*

$$\Sigma : \boxed{\begin{array}{cccc} U & & & \\ & \searrow^{B} & & \\ & & X \to^{A} & X \end{array}}$$

is locally of Brunovsky type; that is, Σ is reachable and all invariant modules M_i^{Σ} are projective. Then the following properties hold:

1. $M_1^{\delta(\Sigma)} \cong M_2^{\Sigma}$*, consequently one can obtain* $\delta^2(\Sigma) = \delta(\delta(\Sigma))$
2. *In general,* $M_i^{\delta(\Sigma)} \cong M_{i+1}^{\Sigma}$
3. *We have a finite sequence of decreasing systems $\delta^i(\Sigma)$ in the sense of their state spaces are isomorphic to M_i^{Σ}.*
4. $N_{i+1}^{\delta(\Sigma)}/N_i^{\delta(\Sigma)} \cong N_i^{\delta^2(\Sigma)}/N_{i-1}^{\delta^2(\Sigma)}$
5. $U \cong N_1^{\Sigma} \oplus \mathit{ker}(B)$
6. $X \cong \bigoplus_{i\geq 1} N_i^{\Sigma}/N_{i-1}^{\Sigma}$

Proof.- To prove (1) and (2) just write down and compare matrices of linear maps $\varphi_i^{\delta(\Sigma)}$ and φ_{i+1}^{Σ}. Remain items follows directly by standard calculations on the direct sum decomposition of involved modules □

Consequently, for a reachable linear system Σ with all invariant R-modules M_i^Σ being projective, the feedback class characterizes, up to isomorphism the finitely generated R-modules

$$\begin{array}{ccc} \ker(B) & & \\ N_1^\Sigma & = & \mathrm{Im}(B) \\ N_2^\Sigma/N_1^\Sigma & = & \mathrm{Im}(B,AB)/\mathrm{Im}(B) \\ \vdots & \vdots & \vdots \\ N_{i+1}^\Sigma/N_i^\Sigma & = & \mathrm{Im}(B,AB,...,A^iB)/\mathrm{Im}(B,AB,...,A^{i-1}B) \\ \vdots & \vdots & \vdots \end{array},$$

The converse is also true; that is, above data characterizes the feedback class of Σ:

Lemma 3.6. *If $\delta(\Sigma)$ and $\delta(\Sigma')$ are feedback equivalent and $\ker(B)$ and $\ker(B')$ are isomorphic then Σ and Σ' are feedback equivalent.*

Proof.- Suppose that $\delta(\Sigma) = (a_{22}, a_{21})$ and $\delta(\Sigma') = (a'_{22}, a'_{21})$ are feedback equivalent by a basic feedback action of type state-space isomorphism. Then

$$a'_{22} = Pa_{22}P^{-1}a'_{21} = Pa_{21}$$

Consequently

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a'_{21} & a'_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & P^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Now suppose that $\delta(\Sigma) = (a_{22}, a_{21})$ and $\delta(\Sigma') = (a'_{22}, a'_{21})$ are feedback equivalent by a basic feedback action of type input-space isomorphism. Then

$$a'_{22} = a_{22}a'_{21} = a_{21}Q$$

Consequently

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a'_{21} & a'_{22} \end{pmatrix} = \begin{pmatrix} Q^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} Q^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Now suppose that $\delta(\Sigma) = (a_{22}, a_{21})$ and $\delta(\Sigma') = (a'_{22}, a'_{21})$ are feedback equivalent by a basic feedback action of type feddback. Then

$$a'_{22} = a_{22} + a_{21}F a'_{21} = a_{21}$$

Consequently

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a'_{21} & a'_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & -F \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mathbf{1} & F \\ \mathbf{0} & \mathbf{1} \end{pmatrix} +$$

$$+ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} F a_{21} & F a_{21} F - F a_{22} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & F \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

□

Theorem 3.7. *The classification problem (in the case of projective invariants) is actually equivalent to the problem of characterization of all possible decompositions of finitely generated R-modules U and X on the form*

$$\begin{array}{rcl} U & = & P_0 \oplus P_1 \\ X & = & P_1 \oplus P_2 \oplus \cdots \oplus P_s \end{array}$$

where P_0 represents a solution for $\ker(B)$ and P_i represents a solution $N_i^{\Sigma}/N_{i-1}^{\Sigma}$. Thus the only restriction to solve the system of equations is that P_{i+1} must be a direct summand of P_i for all i.

Consequently to give the complete classification of locally Brunovsky linear systems is needed to know exactly the monoid $(\mathbf{Proj}(R), \oplus)$ of isomorphism classes of finitely generated R-modules with the direct sum as internal operation.

The full description of the monoid $(\mathbf{Proj}(R), \oplus)$ is a great task. Of course if finitely generated projectives are free then $(\mathbf{Proj}(R), \oplus)$ is isomorphic to $(\mathbb{N} \cup \{0\}, +)$ but in general this is not the case.

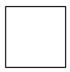
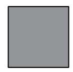
If $R = \mathcal{C}(K)$ is the ring of continuous functions defined on a compact topological space K then $(\mathbf{Proj}(R), \oplus) \equiv (\mathbf{Vect}(K), \oplus)$ depend, of course, on the topology of K. For instance if $K = \mathbb{S}^1$ is the real unit circumference then $(\mathbf{Proj}(R = \mathcal{C}(\mathbb{S}^1)), \oplus)$ is the commutative monoid generated by the symbols R (representing trivial vector bundle) and P (representing the Möbius Strip) modulo the relation $P \oplus P = R \oplus R = R^2$ (see [13]).

The unit circle

Now we can describe a method to visit all feedback classes of locally Brunovsky linear systems when it is fixed $R = \mathcal{C}(\mathbb{S}^1)$ the ring of real continuous functions defined on the unit circle.

First note that there exist only two isomorphism classes of rank r projective R-modules: R^r, which is the free one; and $R^{r-1} \oplus P$.

Thus we may characterize the feedback class of a locally Brunovsky linear system Σ over R by a ”colored” Ferrers diagram: Because $(\mathbf{Proj}(R), \oplus)$ is the commutative monoid generated by the symbols R and P then every ”building-block” is a rank 1 projective module, and there are two classes depicted by

 and 

for classes R and P. Because we have the relation $P \oplus P = R^2$, we have the rule ”grey + grey = white + white”.

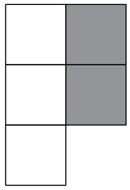
Example 3.8. Let $R = \mathcal{C}(\mathbb{S}^1)$ and Σ be the locally Brunovsky linear system given by

$$\Sigma : \quad R^3 \xrightarrow{B=\begin{pmatrix} x-1 & y & 0 \\ -y & x+1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} R^5 \xrightarrow{A=\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}} R^5$$

then $s = 3$ and the invariants are:

$$\begin{array}{rcc} N_1^\Sigma & = & R \oplus P \\ N_2^\Sigma / N_1^\Sigma & \cong & R \oplus P \\ N_3^\Sigma / N_2^\Sigma & \cong & R \end{array}$$

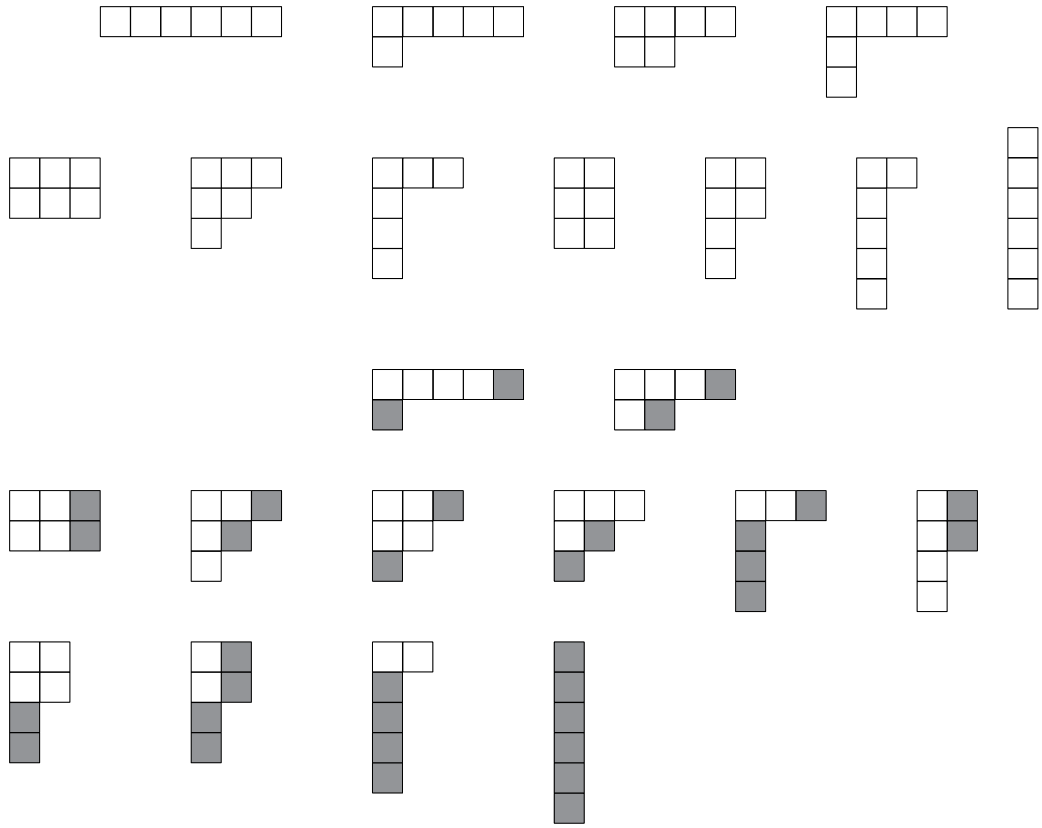
Thus its colored Ferrers diagram is



Then locally Brunovsky linear systems over the finitely generated -module X of rank n would be described by a "colored" Ferrers diagram with exactly n building blocks (white or grey) where the following restrictions apply:

- There is at most one grey block on each row (remember: "grey + grey = white + white")
- About the possible state-spaces:
  - Either $X = R^n$ and there are an even number of gray blocks in the whole colored Ferrers diagram.
  - Or $X = R^{n-1} \oplus P$ and there are an odd number of gray blocks in the whole colored Ferrers diagram.
- The i-th row is at most as long as the $i-1$-th row (decreasing ranks in the sequence of $N_i^\Sigma / N_{i-1}^\Sigma$).
- If two rows have the same number of squares (white or grey), then they are equal. (Neither R^r is a direct summand of $R^{r-1} \oplus P$ nor the converse).

Consequently, as an example, we can list all feedback classes of locally Brunovsky linear systems over the free $\mathcal{C}(\mathbb{S}^1)$-module of rank 6:



In the general case, there is not described yet either a procedure to visit all locally Brunovsky classes or a way to bound the number of such classes. On the other hand note that the problem we study above we take no restriction about which is the input module U. For input modules of low rank, the feedback classes decrease dramatically; in the critical case of $U = R$ is free of rank 1, there exists only one feedback class known as the Canonical Controller Form.

Further study

The dimension 2 case is more complicated because, for instance, $\mathbf{Proj}(\mathcal{C}(\mathbb{S}^2))$ is not finitely generated as monoid (interested reader can see a full description in [13], exercise 1.1.7).

Of course one can study more topological cases (one-dimensional) from contractible spaces and circles by using algebraic methods as Milnor's Patching (see [12]). But we are interested here in point out that K-theory may give some chance:

Denote by $\Lambda^{p,q}$ the trivial linear system

$$\Lambda^{p,q}: \boxed{\begin{array}{cccc} R^p & & & \\ & \searrow^{\mathbf{0}} & & \\ & & R^q & \rightarrow^{\mathbf{0}} \quad R^q \end{array}}$$

and, for a linear system

$$\Sigma: \boxed{\begin{array}{cccc} U & & & \\ & \searrow^{B} & & \\ & & X & \rightarrow^{A} \quad X \end{array}}$$

consider the *augmented* linear system

$$\Sigma \oplus \Lambda^{p,q}: \boxed{\begin{array}{cccc} U \oplus R^p & & & \\ & \searrow \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & & \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ & & X \oplus R^q & \rightarrow \quad X \oplus R^q \end{array}}$$

representing, in the free case, the same linear control system (with some blank input and state variables)

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ 0 & = & 0 \end{array}$$

Of course, system $\Sigma \oplus \Lambda^{p,q}$ is never reachable if $q > 0$, thus it is not locally Brunovsky and hence not Brunovsky. But the point is the following:

Definition 3.9. Two linear systems Σ and Σ' are feedback-a-equivalent if and only if augmented systems $\Sigma \oplus \Lambda^{p,q}$ and $\Sigma' \oplus \Lambda^{p,q}$ are feedback equivalent for some p and some q.

Of course we have that *Feedback equivalence* $\Rightarrow$ *feedback-a-equivalence*, but the converse doesn't hold:

Example 3.10. Put $R = \mathcal{C}(\mathbb{S}^2)$ the ring of continuous real functions defined on the unit sphere immersed in $\mathbb{R}^3$ (with usual ortogonal coordinates (x, y, z)). Linear systems

$$\Sigma: \boxed{\begin{array}{cccc} R^3 & & & \\ & \searrow^{(x,y,z)} & & \\ & & X & \rightarrow^{(0)} \quad X \end{array}}$$

and

$$\Sigma': \boxed{\begin{array}{cccc} R^3 & & & \\ & \searrow^{(1,0,0)} & & \\ & & X & \rightarrow^{(0)} \quad X \end{array}}$$

are both locally Brunovsky and are not feedback equivalent because row $(1,0,0)$ may be completed to a basis of R^3 (ker(B') is free of rank 2) while row (x,y,z) cannot be completed to a basis of R^3 (Hairy-Ball Theorem).

But, on the other hand, systems

$$\Sigma \oplus \Lambda^{1,0} : \boxed{\begin{array}{lll} R^4 & & \\ & \searrow^{(x,y,z,0)} & \\ & & X \quad \rightarrow^{(0)} \quad X \end{array}}$$

and

$$\Sigma' \oplus \Lambda^{1,0} : \boxed{\begin{array}{lll} R^4 & & \\ & \searrow^{(1,0,0,0)} & \\ & & X \quad \rightarrow^{(0)} \quad X \end{array}}$$

are easily seen to be feedback equivalent.

As a consequence of the above discussion we have that if we are allowed to add new blank variables (trivial equations) then feedback equivalence turns to a new notion. In the case of locally Brunovsky linear systems, the feedback equivalence it is no more ruled by the isomorphism class of finitely generated projective R-modules rather than for isomorphism classes of finitely generated projective modules up to finitely generated free direct summands. This leads to Grothendieck $K_0(R)$ group in the case of feedback-a-equivalence in the same way we deal with $\mathbf{Proj}(R)$ in the case of feedback equivalence.

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