NONCAUSAL LINEAR PERIODICALLY TIME-VARYING SCALING FOR DISCRETE-TIME SYSTEMS

Tomomichi Hagiwara * **Yasuhiro Ohara** *

* *Kyoto University, Japan*

Abstract: This paper is concerned with robust stability analysis of discrete-time systems. We first consider linear periodically time-varying (LPTV) nominal systems, for which we apply the discrete-time lifting to have their equivalent linear time-invariant (LTI) representations. Applying the conventional but general scaling approach to the LTI representations leads to the notion of noncausal LPTV scaling when the scaling is interpreted in the original time axis without lifting. Regarding this discrete-time noncausal LPTV scaling, we confirm its effectiveness over causal LPTV scaling and (causal) LTI scaling theoretically as well as with a numerical example. We next consider LTI nominal systems, for which we again apply noncausal LPTV scaling by regarding the LTI systems as a special case of LPTV systems and thus applying the discrete-time lifting in the same way as in the LPTV nominal systems. We then study the relationship of such an approach with the conventional LTI scaling applied directly to the LTI nominal systems without lifting treatment. In particular, we show that even static noncausal LPTV scaling yields dynamic (frequency-dependent) LTI scaling if it is interpreted in the context of lifting-free treatment, and an advantage of noncausal LPTV scaling for LTI nominal systems is investigated from this viewpoint. A numerical example is also provided that supports the advantage.

Keywords: Robust control, quadratic separator, scaling, lifting.

1. INTRODUCTION

In the study of sampled-data systems, the continuous-time lifting technique (Yamamoto, 1994) plays a significant role. A lot of important studies on robust stability of sampled-data systems rely heavily on this technique, which enables us to introduce the transfer operator and frequency response operator of sampled-data systems defined in an operator theoretic framework. Based on such a treatment, a general necessary and sufficient condition of a separator-type was given for robust stability of sampled-data systems (Hagiwara and Tsuruguchi, 2004). By introducing a technique that we call fast-lifting, this condition was further generalized, and as a result, a novel technique called causal/noncausal linear periodically time-varying (LPTV) scaling was introduced in (Hagiwara and Mori, 2006; Hagiwara, 2006). It has been demonstrated that causal/noncausal LPTV scaling is quite effective in reducing the conservativeness in the robust stability analysis of sampled-data systems (Hagiwara and Mori, 2006; Hagiwara, 2006), and as a special case of sampled-data systems, the use of causal/noncausal LPTV scaling on continuous-time systems has also been discussed. An interesting result about the use of noncausal LPTV scaling on continuous-time systems is that even static noncausal scaling has an ability of inducing some frequency-dependent scaling when it is interpreted in the context of conventional LTI scaling. Thus, it is an intriguing question if noncausal LPTV scaling could become a promising tool even in the robust stability analysis of continuous-time systems. However, due to the use of an operator theoretic framework, it seems a rather difficult question to tackle, and it seems easier to study a similar problem in the discrete-time context. This is the motivation of the present study.

The contents of this paper are as follows. We first state the robust stability analysis problem of discrete-time systems in Section 2, and give a fundamental method for dealing with such a problem via the discrete-time lifting technique. Based on this method, we introduce in Section 3 the notion of noncausal LPTV scaling in the discrete-time context, and show that it is effective for robust stability analysis of discrete-time systems with LPTV nominal systems. In Section 4, we consider the case with LTI nominal systems, and give some results that suggest the effectiveness of applying the discrete-time lifting even for robust stability analysis of discrete-time LTI feedback systems. A numerical example is also studied, which demonstrates the above effectiveness. All proofs are omitted due to limited space.

2. ROBUST STABILITY PROBLEM AND PRELIMINARIES

2.1 Robust Stability Problem

Let us consider the discrete-time closed-loop system shown in Fig. 1 consisting of the nominal system G and the uncertainty Δ. G has q inputs and p outputs and is an internally stable, finite-dimensional (FD) linear periodically time-varying (LPTV) system with period N (i.e., an N-periodic system), where N is a positive integer. Δ belongs to some given set $\boldsymbol{\Delta}$ satisfying the assumption:

A1 Every $\Delta \in \boldsymbol{\Delta}$ is FD, N-periodic, and internally stable, and $\boldsymbol{\Delta}$ is a connected set such that $0 \in \boldsymbol{\Delta}$.

As a special case of the above assumption, we also prepare the following alternative assumption.

A1' Every $\Delta \in \boldsymbol{\Delta}$ is FD, LTI, and internally stable, and $\boldsymbol{\Delta}$ is a connected set such that $0 \in \boldsymbol{\Delta}$.

G is also allowed to be LTI; when Δ is N-periodic, we view an LTI system G as a special case of N-periodic systems. An LTI Δ is treated similarly. Hence we assume that G and Δ are described by

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k + D_k u_k \quad (1)$$

$$\xi_{k+1} = A_{\Delta k}\xi_k + B_{\Delta k} y_k, -u_k = C_{\Delta k}\xi_k + D_{\Delta k} y_k \quad (2)$$

respectively, where the coefficient matrices are N-periodic. We denote by Σ_Δ the closed-loop system shown in Fig. 1, and assume that Σ_Δ is well-posed. We define the family $\boldsymbol{\Sigma}(\boldsymbol{\Delta}) := \{\Sigma_\Delta \mid \Delta \in \boldsymbol{\Delta}\}$.

2.2 Discrete-Time Lifting of N-Periodic Systems

The N-periodic system G can be associated with its LTI representation via the discrete-time lifting technique (Bittanti and Colaneri, 2000); by defining $\widehat{x}_\nu := x_{\nu N}, \widehat{u}_\nu := [u^T_{\nu N}, u^T_{\nu N+1}, \cdots, u^T_{\nu N+N-1}]^T$ and $\widehat{y}_\nu := [y^T_{\nu N}, y^T_{\nu N+1}, \cdots, y^T_{\nu N+N-1}]^T$, we have an alternative representation of (1) given by

$$\widehat{G} : \ \widehat{x}_{\nu+1} = \widehat{A}\widehat{x}_\nu + \widehat{B}\widehat{u}_\nu, \quad \widehat{y}_\nu = \widehat{C}\widehat{x}_\nu + \widehat{D}\widehat{u}_\nu \quad (3)$$

with appropriately defined constant matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$ that are independent of ν. G is internally stable if and only if $\widehat{A}$ is Schur stable. Similarly, the LTI representation of Δ leads to

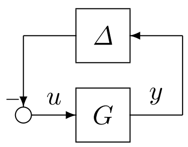

Fig. 1. Discrete-time system Σ_Δ with uncertainty Δ.

$$\widehat{\Delta} : \ \widehat{\xi}_{\nu+1} = \widehat{A}_\Delta\widehat{\xi}_\nu + \widehat{B}_\Delta\widehat{y}_\nu, -\widehat{u}_\nu = \widehat{C}_\Delta\widehat{\xi}_\nu + \widehat{D}_\Delta\widehat{y}_\nu \quad (4)$$

The feedback connection of the LTI systems $\widehat{G}$ and $\widehat{\Delta}$ is well-posed, and it is internally stable if and only if the system Σ_Δ is.

In summary, the discrete-time lifting technique allows us to convert the stability problem of LPTV systems into that of LTI systems both in the open-loop and closed-loop settings.

2.3 Stability Analysis via Separator

Let us introduce the discrete-time transfer matrices of $\widehat{G}$ and $\widehat{\Delta}$, respectively, given by

$$\widehat{G}(z) = \widehat{C}(zI - \widehat{A})^{-1}\widehat{B} + \widehat{D} \quad (5)$$

$$\widehat{\Delta}(z) = \widehat{C}_\Delta(zI - \widehat{A}_\Delta)^{-1}\widehat{B}_\Delta + \widehat{D}_\Delta \quad (6)$$

which we call the N-lifted transfer matrices of the N-periodic systems G and Δ, respectively [1]. Then, we can derive the following theorem regarding the robust stability analysis of the family $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ (see also (Iwasaki and Hara, 1998)).

Theorem 1. Suppose that $\boldsymbol{\Delta}$ satisfies **A1** (or **A1**') and Σ_Δ is well-posed for every $\Delta \in \boldsymbol{\Delta}$. Then, $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ is robustly stable if and only if there exists $\widehat{\Theta}(z) = \widehat{\Theta}(z)^*$ for all $z \in \partial\mathbf{D}$ such that

$$\begin{bmatrix} I & \widehat{G}(z)^* \end{bmatrix} \widehat{\Theta}(z) \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \leq 0 \quad (7)$$

$$\begin{bmatrix} -\widehat{\Delta}(z)^* & I \end{bmatrix} \widehat{\Theta}(z) \begin{bmatrix} -\widehat{\Delta}(z) \\ I \end{bmatrix} > 0 \quad (\forall \Delta \in \boldsymbol{\Delta}) \quad (8)$$

where $\partial\mathbf{D}$ denotes the unit circle $\{z : |z| = 1\}$.

The matrix $\widehat{\Theta}(z)$ contained in this theorem is called a (dynamic) separator (Iwasaki and Hara, 1998). We say that $\widehat{\Theta}(z)$ is a static separator if it is in fact independent of z.

3. NONCAUSAL LPTV SCALING FOR LPTV NOMINAL SYSTEMS

This paper aims at investigating the use of the discrete-time lifting technique in the robust stability analysis of LPTV (and LTI) systems. In this section, we first introduce a natural notion that accompanies such a treatment, which we call noncausal LPTV scaling. We also introduce causal LPTV scaling as a special case, and study their mutual relationship as well as their relationship to the conventional LTI scaling.

This section first introduces the notions of causal/noncausal LPTV scaling and then demonstrate their effectiveness when the nominal system G is LPTV. The same technique can be applied to the case when the nominal system G is LTI, and such a case will be studied independently in Section 4.

[1] We can view G as a μN-periodic system, where μ is a positive integer. We then obtain another lifted representation and an associated transfer matrix, which will be called the μN-lifted transfer matrix of the N-periodic system G.

3.1 Causal/Noncausal LPTV Scaling

Let us assume for the moment that G and Δ are both square systems (i.e., $p = q$), and let us consider a typical separator of the form

$$\widehat{\Theta}(z) = \begin{bmatrix} -\gamma^2 \widehat{W}(z)^* \widehat{W}(z) & 0 \\ 0 & \widehat{W}(z)^* \widehat{W}(z) \end{bmatrix} \quad (9)$$

where $\widehat{W}(z)$ is invertible for every $z \in \partial\mathbf{D}$ and γ is a positive scalar. Then, (7) and (8) are equivalent to the following conditions, respectively.

$$\|\widehat{W}(z)\widehat{G}(z)\widehat{W}(z)^{-1}\| \le \gamma \quad (10)$$

$$\|\widehat{W}(z)\widehat{\Delta}(z)\widehat{W}(z)^{-1}\| < 1/\gamma \quad (\forall \Delta \in \boldsymbol{\Delta}) \quad (11)$$

That is, taking the separator (9) corresponds to applying the small-gain condition scaled with $\widehat{W}(z)$. Suppose for simplicity the case when the scaling factor is independent of z and is in fact a constant matrix $\widehat{W}$. If we interpret the corresponding scaling in the time domain (the time domain with respect to k in (1) and (2) rather than that with respect to ν in (3) and (4)), it generally leads to periodically time-varying scaling of the systems G and Δ with some noncausal operation with respect to time k. In view of this observation, we say that the separator (9) generally induces noncausal scaling on LPTV systems G and Δ.

Here, it would be worth noting that every $\widehat{W}^*\widehat{W}$ can also be represented as $\widehat{W_0}^*\widehat{W_0}$ with some $\widehat{W_0}$ with some special block lower triangular structure. Hence, we can replace $\widehat{W}$ in (10) and (11) with $\widehat{W_0}$. Since $\widehat{W_0}^{-1}$ has the same structure as $\widehat{W_0}$ and thus both $\widehat{W_0}$ and $\widehat{W_0}^{-1}$ conform to the causality constraint with respect to the time axis k, introducing the term "noncausal scaling" might sound misleading. Thus, we might better use the term like "lifted scaling" instead of "noncausal scaling." Nonetheless, we use the term "noncausal scaling" in this paper partly because we would like to make clear the connection of the arguments of this paper to those in (Hagiwara and Mori, 2006; Hagiwara, 2006) in the sampled-data/continuous-time setting, by which this paper is strongly motivated. We introduce a few special classes of separators $\widehat{\Theta}(z)$ in the following, from which the use of the term "noncausal scaling" will be supported and validated further (see Remark 5).

This subsection aims at introducing the definitions of causal LPTV scaling and noncausal LPTV scaling in general situations. We begin with the following definition on causal LPTV scaling.

Definition 2. We say that the separator $\widehat{\Theta}(z)$ induces causal LPTV scaling (or equivalently, $\widehat{\Theta}(z)$ is a causal LPTV separator) in an N-periodic feedback system if it can be represented as

$$\widehat{\Theta}(z) = \begin{bmatrix} \widehat{V}_1(z) & \widehat{V}_2(z) \end{bmatrix}^* \widehat{\Lambda} \begin{bmatrix} \widehat{V}_1(z) & \widehat{V}_2(z) \end{bmatrix} \quad (12)$$

where $\widehat{V}_1(z)$ and $\widehat{V}_2(z)$ are the N-lifted transfer matrices of a causal N-periodic system V_1 with q input and a causal N-periodic system V_2 with p input, respectively, and $\widehat{\Lambda} = \widehat{\Lambda}^*$ is a constant matrix of the form $\widehat{\Lambda} = \mathrm{diag}[\Lambda_1, \cdots, \Lambda_N]$ with the size of Λ_i being the same for all $i = 1, \cdots, N$.

The arguments of this paper hold *mutatis mutandis* even if $\widehat{\Lambda}$ in (12) is replaced by the N-lifted transfer matrix $\widehat{\Lambda}(z)$ of a causal N-periodic system such that $\widehat{\Lambda}(z)^* = \widehat{\Lambda}(z)$. Since the essential parts of the arguments remain the same, however, we simply deal with the case of $\widehat{\Lambda}(z) = \widehat{\Lambda}$. Definition 2 applies also to the case with $N = 1$, i.e., an LTI feedback system, when (12) reduces to

$$\Theta(z) = V(z)^* \Lambda V(z), \ V(z) := [V_1(z), \ V_2(z)] \quad (13)$$

with the transfer matrix $V(z)$ of an LTI system V with $p + q$ inputs and a constant matrix $\Lambda = \Lambda^*$. This would be worth calling a causal LTI separator, which is nothing but the conventional separator in the analysis of LTI feedback systems, and is general enough in the sense that every matrix $\Theta = \Theta^*$ can be represented as

$$\Theta = V^* \Lambda V, \ \Lambda = \Lambda^* \quad (14)$$

We believe that our Definition 2 gives a quite natural extension of causal LTI separators to the LPTV setting. We next introduce the following definition of noncausal LPTV scaling.

Definition 3. We say that the separator $\widehat{\Theta}(z)$ induces noncausal LPTV scaling (or equivalently, $\widehat{\Theta}(z)$ is a noncausal LPTV separator) in an N-periodic feedback system if it can be represented as

$$\widehat{\Theta}(z) = \widetilde{\Gamma}^* \widehat{V}(z)^* \Gamma \widehat{V}(z) \widetilde{\Gamma} \quad (15)$$

where $\Gamma = \Gamma^*$ and $\widetilde{\Gamma}$ are constant matrices and $\widehat{V}(z)$ is the transfer matrix of a causal LTI system *defined on the lifted time axis* ν.

The arguments of this paper hold *mutatis mutandis* even if Γ in (15) is replaced by $\Gamma(z) = \Gamma(z)^*$.

Remark 4. In the above definition, the matrix $\widetilde{\Gamma}$ in (15) could in fact be removed since we could redefine $\widehat{V}(z)\widetilde{\Gamma}$ as $\widehat{V}(z)$. However, we leave $\widetilde{\Gamma}$ as it is for some reasons; the details are omitted due to limited space.

Remark 5. Regarding $\widehat{W_0}$ that we discussed just before Definition 2, we can see that the corresponding separator is indeed classified among static *noncausal* LPTV separators, in general, even if $\widehat{W_0}$ has a block lower triangular form. The noncausality can play a significant role in reducing the conservativeness in the robust stability analysis (see Sections 3.2 and 4.3). As we will see later, the effect of the noncausality can also be interpreted equivalently as introducing frequency-dependent scaling in the context of the conventional lifting-free treatment of LTI systems. This is an interesting phenomenon and will be studied in Subsection 4.2.

Definition 3 also applies to the case with $N = 1$, but (15) again reduces to the causal LTI separator (13) when $N = 1$. Thus, there exists no (strictly) "noncausal LTI separator," and hence it would be justified to refer to (13) simply as an LTI separator rather than a causal LTI separator. Furthermore, we can see that possible noncausality of separators is a feature that is specific to the very treatment in this paper in which the discrete-time lifting is employed. Note, however, that it does not necessarily mean that noncausality of separators cannot be exploited in the analysis of LTI feedback systems. This is because we can always regard LTI systems as a special case of N-periodic systems, and thus we can apply noncausal LPTV separators by taking such a viewpoint. Such an approach is deferred to Section 4.

3.2 Numerical Example with Causal/Noncausal LPTV Scaling for LPTV Nominal G

We confine ourselves to the case with LPTV nominal G and demonstrate the effectiveness of causal/noncausal LPTV separators through a numerical example; we study a numerical example with the nominal system G being a stable 2-periodic LPTV system given by

$$A_k = \begin{bmatrix} 0 & I \\ 0.1 & a_k^T \end{bmatrix}, \ (k = 1, 2) \tag{16}$$

$$a_1^T = [0.1\ 1\ \ {-1}\ 1.9],\ a_2^T = [-0.1\ 0.01\ \ -0.5\ 0.2] \tag{17}$$

$$B_1 = B_2 = e_5, \quad C_1 = \begin{bmatrix} 0\ 0.6\ 0.6\ 1.5\ 0 \end{bmatrix}, \tag{18}$$

$$C_2 = \begin{bmatrix} 0\ 0.3\ 0.3\ 0.3\ 0 \end{bmatrix}, \quad D_1 = D_2 = 0 \tag{19}$$

Here we assume that Δ is static, and consider the two situations: (i) Δ is a time-invariant scalar, which we denote by δ; (ii) Δ is also 2-periodic, and thus takes δ_1 and δ_2 alternately. We intend to analyze via LPTV scaling (a lower bound of) the robust stability radius regarding each of these uncertainties.

Let us consider the first situation, in which case the lifting of Δ with $N = 2$ leads to $\widehat{\Delta} = \delta I$ with I being the 2×2 identity matrix. For simplicity, we confine ourselves to the separator of the form

$$\widehat{\Theta} = \begin{bmatrix} -\gamma^2 \widehat{W}^T \widehat{W} & 0 \\ 0 & \widehat{W}^T \widehat{W} \end{bmatrix}, \ \widehat{W}^T \widehat{W} > 0, \ \gamma > 0 \tag{20}$$

that induces the so-called D-scaling. When $\widehat{W}$ is unstructured, this corresponds to using static noncausal LPTV separators. Here, the minimization of $\|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty$ leads to $\gamma_{\min} = 2.4455 =: \gamma_{\min}^{\text{noncausal}}$. Thus, we obtain $1/\gamma_{\min}^{\text{noncausal}} = 0.4089$ as a lower bound with static noncausal LPTV scaling. For reference, the l_2-induced norm of G is given by $\gamma_0 = 5.3114$, and thus the conventional small-gain approach leads to a lower bound of the robust stability radius given by $1/\gamma_0 = 0.1883$, which is quite conservative.

Next, let us consider the second situation. In this case, it is reasonable to confine $\widehat{W}$ in (20) to the form $\widehat{W} = \text{diag}[\widehat{w}_1,\ \widehat{w}_2] > 0$ so that (8) reduces to the $\widehat{W}$-independent condition $\max(|\delta_1|, |\delta_2|) < 1/\gamma$. Taking the above restricted form of $\widehat{W}$ in fact corresponds to employing static causal LPTV separators. The minimization of $\|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty$ under the restricted $\widehat{W}$ leads to $\gamma_{\min} = 3.4897 =: \gamma_{\min}^{\text{causal}}$, and thus we obtain $1/\gamma_{\min}^{\text{causal}} = 0.2866$ as a lower bound of the robust stability radius against static 2-periodic Δ. Obviously, the small-gain approach can also be applied in this second situation and leads to the same lower bound as in the first situation, $1/\gamma_0 = 0.1883$, but this is again quite conservative.

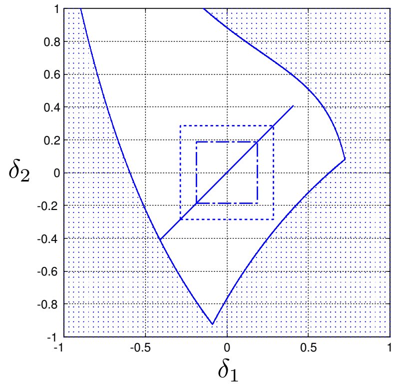

Fig. 2. Robust stability analysis for an LPTV nominal system.

The above results together with the stability region in the (δ_1, δ_2) plane computed via fine gridding are shown in Fig. 2; the unshaded region corresponds to the stability region and the solid line corresponds to the lower bound of the robust stability radius obtained by static noncausal LPTV scaling assuming that Δ is time-invariant, while the dash square corresponds to the lower bound obtained by static causal LPTV scaling assuming that Δ is 2-periodic, and the dash-dot square corresponds to the lower bound obtained by applying the conventional small-gain approach.

4. NONCAUSAL LPTV SCALING APPLIED TO LTI NOMINAL SYSTEMS

In this section, we consider the special case in which G is in fact LTI. In this case, we can drop the subscript k from all the matrices in (1), and G has the transfer matrix $G(\zeta) := C(\zeta I - A)^{-1}B + D$. Note that the forward shift in time k is denoted by ζ to distinguish from the symbol z.

If Δ is also LTI, then we can also introduce the associated transfer matrix $\Delta(\zeta)$, and it is obvious that we can arrive at the following theorem, which is just parallel to Theorem 1.

Theorem 6. Suppose that G is LTI, $\boldsymbol{\Delta}$ satisfies **A1**' and Σ_Δ is well-posed for every $\Delta \in \boldsymbol{\Delta}$. Then, $\boldsymbol{\Sigma}(\boldsymbol{\Delta})$ is robustly stable if and only if there exists $\Theta(\zeta) = \Theta(\zeta)^*$ for all $\zeta \in \partial\mathbf{D}$ such that

$$\begin{bmatrix} I & G(\zeta)^* \end{bmatrix} \Theta(\zeta) \begin{bmatrix} I \\ G(\zeta) \end{bmatrix} \le 0 \tag{21}$$

$$\begin{bmatrix} -\Delta(\zeta)^* & I \end{bmatrix} \Theta(\zeta) \begin{bmatrix} -\Delta(\zeta) \\ I \end{bmatrix} > 0 \quad (\forall \Delta \in \boldsymbol{\Delta}) \tag{22}$$

Now, regarding the robust stability analysis problem for an LTI system G under the Assumption **A1**', we have two options. One is the conventional method, in which we apply the above Theorem 6, while the other is to apply Theorem 1 using the N-lifted transfer matrices associated with G and Δ viewed as N-periodic systems. The former option corresponds to the use of the conventional LTI separators while the latter corresponds to using causal/noncausal LPTV separators. It is an interesting topic to study the relationship between these two options, or in other words, to clarify which is more effective for robust stability analysis. This section is devoted to such a study.

4.1 Construction of LPTV Separators from Lifting-Free LTI Separators

Regarding the relationship between the two options for robust stability analysis with an LTI nominal system stated just before, we have the following fundamental result.

Theorem 7. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies the Assumption **A1**'. If there exists an LTI separator $\Theta(\zeta)$ satisfying (21) and (22), then there also exists a causal LPTV separator $\widehat{\Theta}(z)$ satisfying (7) and (8). In particular, if $\Theta(\zeta)$ is in fact a static LTI separator, there also exists a static causal LPTV separator $\widehat{\Theta}(z)$ satisfying (7) and (8).

Roughly speaking, the proof of the above theorem says that if there exists an LTI separator that "resolves" the original lifting-free robust stability analysis problem, then the "lifted version" of that separator is also causal and "resolves" the lifted restatement of the same problem. This in particular implies that we never lose anything in recasting the lifting-free problem into the lifted counterpart as far as the solvability issues of these problems are concerned. This fact may not be surprising, but this guarantee does support and suggest to study possible advantages of treating the lifted counterpart instead of the original lifting-free problem. This is particularly because, in the lifted counterpart, we can also consider *noncausal* LPTV separators, and this is indeed the case even when we confine ourselves to the simple class of static separators (recall that we have seen the effectiveness of static noncausal LPTV separators in Section 3.2 for LPTV nominal systems). In other words, it can be interpreted that the lifted counterpart of the problem allows us to extend the class of possible (tractable) separators even when we confine ourselves to the class of, e.g., static separators. This could make the inequality (7) about G easier to hold, while, more importantly, among the extend class are a lot of separators that satisfy the inequality (8) only for a smaller class of Δ. For example, we can easily construct a noncausal LPTV separator for which (8) holds for any norm-bounded LTI Δ but not necessarily for a norm-bounded LPTV Δ (see Section 3.2). This implies that noncausal LPTV separators could possibly reduce the conservativeness of the robust stability analysis in the practical situation in which we cannot sweep over all possible separators $\Theta(\zeta)$ but have to restrict the class of $\Theta(\zeta)$ to a tractable one, e.g., a class of static separators.

The following subsection studies possible advantages of noncausal LPTV separators also from a different viewpoint (i.e., a frequency-domain viewpoint).

4.2 Implication of Noncausal LPTV Separators in the Lifting-Free Treatment

We can show another relation between the two options stated before, which is given by the following theorem.

Theorem 8. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies **A1**'. If there exists a separator $\widehat{\Theta}(z)$ satisfying (7) and (8), then there also exists a separator $\Theta(\zeta)$ satisfying (21) and (22). One such $\Theta(\zeta)$ is given by

$$\Theta(\zeta) := \mathrm{diag}[T_q(\zeta),\ T_p(\zeta)]^* \widehat{\Theta}(\zeta^N)\mathrm{diag}[T_q(\zeta),\ T_p(\zeta)] \quad (23)$$

with $T_m(\zeta) := [I_m,\ \zeta I_m,\ \cdots,\ \zeta^{N-1} I_m]^T$.

This theorem suggests that even a static noncausal LPTV separator $\widehat{\Theta}(z) = \widehat{\Theta}$ in the lifted treatment leads to $\Theta(\zeta)$ corresponding to the lifting-free treatment that is frequency-dependent (i.e., depends on $\zeta \in \partial\mathbf{D}$). To put it another way, the lifted treatment possibly has an ability to convert the problem of searching for frequency-dependent separators satisfying (21) and (22) in the lifting-free treatment into a simpler problem of finding a static separator satisfying (7) and (8). Compared with static Θ, it is obvious that ζ-dependent Θ has more freedom, and thus we could expect the above ability to reduce the conservativeness in the robust stability analysis. The following subsection is devoted to examining if this is indeed the case through a numerical example. Before proceeding, however, we state the following result, which implies that the above expectation is denied for static *causal* LPTV separators.

Theorem 9. Suppose that G is LTI, and $\boldsymbol{\Delta}$ satisfies **A1**'. There exists a static causal LPTV separator $\widehat{\Theta}$ satisfying (7) and (8) if and only if there exists a static LTI separator Θ satisfying (21) and (22).

4.3 Numerical Example with Causal/Noncausal LPTV Scaling for LTI Nominal G

We study a numerical example with the nominal system G being a stable LTI system given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.2 & 0.5 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (24)$$

$$C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{25}$$

Here we assume that Δ is static and has a diagonal structure, and in particular consider the two situations: (i) $\Delta = \delta I$ with a time-invariant scalar δ; (ii) $\Delta = \delta I$ with an N-periodically time-varying scalar δ that takes δ_i $(i = 1, \cdots, N)$ circularly. In the first situation, we intend to examine an advantage of introducing the lifted treatment together with noncausal LPTV scaling to the robust stability analysis of LTI feedback systems. A relevant problem is dealt with in the second situation.

In the first situation, we can consider the static noncausal separator $\widehat{\Theta}$ of the D-scaling type given by (20) (i.e., without any structural constraint on $\widehat{W}$) after regarding the LTI feedback system as a special case of N-periodic LPTV feedback systems. We took $N = 1, \cdots, 5$ in such a treatment (where $N = 1$ corresponds to the conventional lifting-free treatment) and computed $\inf_{\widehat{W}} \|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty$ for each N. The results are shown in Table 1, which clearly show the advantage of applying the lifting approach [2] together with static noncausal LPTV separators [3]. From the result for $N = 5$ in Table 1, we have a lower bound of the robust stability radius with respect to time-invariant $\Delta = \delta I$ given by $1/1.6533 = 0.6049$.

In the second situation, on the other hand, it is reasonable to confine $\widehat{W}$ in (20) to a positive-definite diagonal matrix of the form

$$\widehat{W} = \mathrm{diag}[\widehat{W}_1, \cdots, \widehat{W}_N] \tag{26}$$

with 2×2 positive-definite matrices $\widehat{W}_i$ $(i = 1, \cdots, N)$, so that (8) reduces to the $\widehat{W}$-independent condition $\max(|\delta_1|, \cdots, |\delta_N|) < 1/\gamma$. Taking this form of $\widehat{W}$ in fact corresponds to employing static causal LPTV separators. The minimization of $\|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty$ under the restricted $\widehat{W}$ led to $\gamma_{\min}^{\text{causal}} = 3.6215$ (i.e., the same as the value for $N = 1$ in Table 1) regardless of N, which is indeed a consequence of Theorem 9. We thus have a lower bound of the robust stability radius with respect to N-periodic $\Delta = \delta I$ given by $1/\gamma_{\min}^{\text{causal}} = 0.2761$.

The above results are shown in Fig. 3 (for the cases of time-invariant/2-periodic $\Delta = \delta I$) under the same meaning for the solid, dash and dash-

Table 1. The results of static noncausal LPTV scaling.

N	1	2	3	4	5
$\gamma_{\min}^{\text{noncausal}}$	3.6215	1.7918	1.7146	1.6674	1.6533

[2] Note that this advantage in particular suggests a possible advantage of considering the μN-lifted transfer matrix (rather than the N-lifted transfer matrix) in the robust stability analysis involving N-periodic systems.

[3] As far as the D-scaling is concerned, however, it is not hard to prove that noncausal LPTV separators lead to no advantage over LTI separators if G and Δ are both SISO systems. This problem does not apply to (D, G)-scaling.

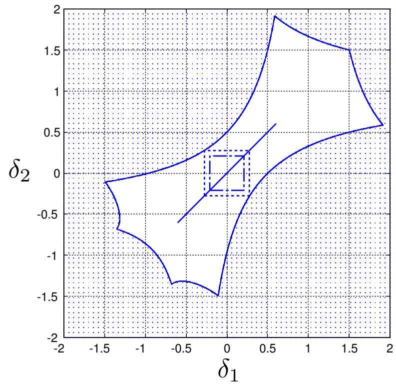

Fig. 3. Robust stability analysis for an LTI nominal system.

dot lines as well as the shaded region; as for static noncausal LPTV scaling, the figure corresponds to the case of $N = 5$, and the (unscaled) H_∞-norm of G is $\gamma_0 = 4.7803$ and thus the conventional small-gain theorem (dash-dot line) leads only to $1/\gamma_0 = 0.2092$ as an estimate of the robust stability radius. We can see that the result for 2-periodic Δ is almost exact, while the result for static Δ is still conservative. However, we have confirmed that applying static noncausal LPTV scaling of the (D, G)-scaling type (instead of the D-scaling type employed here) leads to an almost exact robust stability radius $1/\gamma_{\min,DG}^{\text{noncausal}} = 1/1.0833 = 0.9231$ (when $N = 2$) for static $\Delta = \delta I$, where fine gridding shows that stability is retained under $-0.9231 < \delta < 1.5001$. We finally remark that in the lifting-free treatment (i.e., $N = 1$), static (D, G)-scaling led to no improvement over the D-scaling studied above.

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