A CONSPIRACY OF OSCILLATORS

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Abstract

We discuss nonlinear mechanical systems containing several oscillators whose frequecies are all much higher than frequencies associated with the remaining degrees of freedom. In this situation a near constant of the motion, an adiabatic invariant, exists which is the sum of all the oscillator actions. The phenomenon is illustrated, and calculations of the small change of the adiabatic invariant is outlined.

Key words

High-frequency oscillators, adiabatic invariants

1 Introduction

In a mechanical system with n degrees of freedom, where m of the degrees of freedom are associated with oscillatory modes whose characteristic time scale is much shorter than the characteristic time scale associated with the remaining $n - m$ degrees of freedom, the *sum* of actions from all the m oscillatory modes forms an adiabatic invariant (see, e.g., [1]) of the system, its value evolving only exponentially slowly.

This is a nontrivial sum because the individual oscillator action are not adiabatically invariant, its invariance broken by resonance with the other high-frequency oscillators. We discuss two examples of this phenomenon. In section 2, a model 1-dimensional gas of mutually interacting diatomic molecules whose vibrational time scale is much shorter than the time scale associated with the collisions between the molecules; and in section 3 a three-dimensional gas of charged particles moving in a magnetic field so strong that the Larmor period of the charged particles is much shorter that the time scale for collisions along the field lines. In both systems there is a thus a separation between the oscillator time scale and the time scale associated with the remaining degrees of freedom. In both systems one finds that the many-body sum of actions is adiabatically invariant, exponentially well preserved by the interactions. The greater the time scale separation, the more the high-frequency oscillators together will conspire to effectivly decouple themselves from the rest of the system, leaving the low frequency components to evolve on their own. Finally, section 4 will outline some analytical estimates of the decoupling.

2 Diatomic molecules

Consider a classical mechanical model of a one–dimensional gas consisting of n identical diatomic molecules. See figure 1.
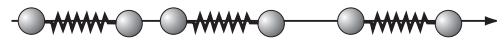

Figure 1. Several diatomic molecules move in 1 dimension and interact with long-range forces between the all the mass points.

Denoting by $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$ the canonical variables of the centers of mass of the molecules, and by $\xi = (\xi_1, \ldots, \xi_n)$, $\pi = (\pi_1, \ldots, \pi_n)$ the (Cartesian) canonical variables describing the internal vibrations, the Hamiltonian has the overall form [3]:

$$H(x, p, \xi, \pi) = H_{\text{tr}}(p) + H_{\text{vib}}(\xi, \pi) + V_{\text{int}}(x, \xi) \ , \quad (1)$$

with

$$H_{\text{tr}} = \sum_{i=1}^{n} \left(\frac{p_i^2}{2m} + U(r_i) \right) \ , \ H_{\text{vib}} = \sum_{i=1}^{n} \left(\frac{\pi_i^2}{2\mu} + \frac{\mu \omega^2 \xi_i^2}{2} \right)$$

and

$$V_{\text{int}} = \sum_{i=1}^{n-1} V(r_i, \xi_i, \xi_{i+1}) \ ,$$

where m and μ denote respectively the total mass and the reduced mass of the molecules, while ω denotes

the frequency of vibration for a single molecule, and $r_i = x_{i+1} - x_i - l$, with l the proper length of the molecule; the separation between U and V is established by requiring $V(r, 0, 0) = 0$. Both U and V are assumed to be smooth functions and to vanish for $r \to \infty$. The Hamiltonian H_{tr} describes a gas of perfectly rigid molecules, with pairwise interaction described by the potential U; as is natural, $U(r)$ will be assumed to diverge for $r \to 0$. The gas is assumed to be diluted, so that V_{int} is small compared to H_{tr} and H_{vib}.

A similar model was considered in 1903 by Jeans [6], in the investigation of the time scale of the approach to equilibrium between translational and vibrational degrees of freedom in diatomic gases. The basic idea of Jeans (following Boltzmann) was that energy exchanges among the translational degrees of freedom, as well as among the vibrational degrees of freedom, are easy, so that, the two sub–systems described by H_{tr} and by H_{vib} separately reach their thermodynamical equilibrium, with temperatures T_{tr} and T_{vib} not necessarily equal; whereas collisional energy exchanges between H_{tr} and H_{vib}, in case of large ω, are exponentially small, so that an effective "freezing out" of the vibrational energy for quite large times ("billions of years", in the words by Jeans!) is expected to occur, if ω is sufficiently large. [1] The conclusion by Jeans was that, in principle, the experimentally observed 'freezing' phenomena of various kind could be explained classically, without Planck's quantization. Jeans conjectured, and supported heuristically, an exponential law for the collisional transfer of energy E_0 between translational and vibrational modes in a single binary collision to be of the form

$$E_0 \approx \mathcal{E}_0 e^{-\tau_0 \omega},$$

where $\mathcal{E}_0$ is a typical microscopic energy, while the (crucial) constant τ_0 represents some time scale associated to the translational motion. Here, unfortunately, Jeans was not really precise, though he qualitatively identifies τ_0 with the typical duration of the collision process. The conclusion is that in typical conditions E_0 is negligibly small, and long–time freezing out occurs.

Today we model molecular collisions by quantum mechanics and the 'freezing out' of vibrational degrees of freedom is seen as a manifestation of the quantized energy levels and the inability of collisional energies to excite the molecules away from their vibrational ground state.

3 Charged particles in strong magnetic field

Consider a particle of mass $m = 1$ and charge $e = 1$ moving in $\mathbf{R}^3$. A homogeneous magnetic field $\mathbf{B} = Be_z$ acts on the particle. Denote by $\mathbf{A} = \mathbf{A}(\mathbf{r})$ the

magnetic vector potential at the point $\mathbf{r}$, so that $\mathbf{B} = \mathbf{curl\,A}$. It is natural to take $\mathbf{A}$ as $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, but the vector potential can be taken as any representative from an equivalence class of potentials that differ by the gradient of some scalar field, see [8]. The Hamiltonian for the system is:

$$H = \frac{(\mathbf{p} - \mathbf{A}(\mathbf{r}))^2}{2},$$

Note that we can write down the Hamiltonian function in this vector form without specifying any particular set of phase space coordinates. The motion of the single particle is integrable, and, for the above choice of direction of the magnetic field, consists of uniform circular motion (around the 'guiding center') with angular frequency $\Omega_c \equiv B$ (the *cyclotron frequency*) in the xy-plane (perpendicular to the z-direction), and free inertial drift in the z-direction.
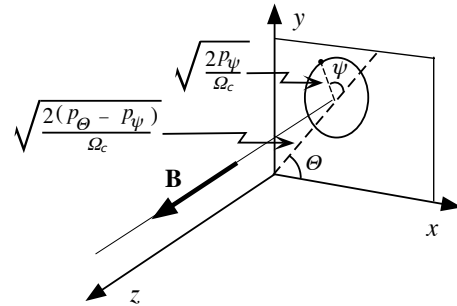
Figure 2. The coordinates

We shall employ here a particular set of coordinates for the perpendicular directions. These (canonical) coordinates $(\Theta, p_\Theta, \psi, p_\psi)$ are defined such that the angle Θ and its associated momentum p_Θ specify the polar coordinates for the guiding center, and the angle ψ and its associated momentum p_ψ specify the polar coordinates for the particle position relative to the guiding center and the direction given by Θ. See figure 2.

In terms of these *polar guiding center coordinates*, the single particle Hamiltonian is simply

$$H = \Omega_c p_\psi + \frac{p_z^2}{2},$$

which manifestly is integrable. If we perturb H to a slow spatial or temporal variation of, say, the magnetic field, the quantity p_ψ will be adiabatically invariant [1], and the motion of the single particle will such that the value of this phase space function is approximately conserved.

For *two* interacting particles moving in the homogeneous magnetic field, one can separate the (now 6 de-

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gree of freedom) problem

$$H = \frac{(\mathbf{p}_1 - \mathbf{A}(\mathbf{r}_1))^2}{2} + \frac{(\mathbf{p}_2 - \mathbf{A}(\mathbf{r}_2))^2}{2} + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \,,$$

by introducing, as in the celestial mechanics two-body problem, relative and center-of-mass frames $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, and $\mathbf{R} \equiv \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$. In terms of these vectors,

$$H = H^{CM} + H^{rel} \equiv \frac{(\mathbf{P} - \mathbf{A})^2}{2} + \frac{(\mathbf{p} - \mathbf{a})^2}{2} + \frac{1}{|\mathbf{r}|} \,,$$

where $\mathbf{A} = \mathbf{A}(\mathbf{r}_1) + \mathbf{A}(\mathbf{r}_2)$ and $\mathbf{a} = \frac{1}{2}(\mathbf{A}(\mathbf{r}_1) - \mathbf{A}(\mathbf{r}_2))$. The center-of-mass dynamics decouples completely, reducing the problem to one of single particle motion. Thus, the nontrivial dynamics takes place only in the relative coordinates. The relative Hamiltonian H^{rel} describes (as in celestial mechanics) the collision of a single particle with another particle held fixed at the origin of the coordinate frame. Using the polar guiding-center coordinates $(\Theta, p_\Theta, \psi, p_\psi)$ defined above, H^{rel} takes the coordinate form

$$H^{rel} = \Omega_c p_\psi + \frac{p_z^2}{2m} + \frac{1}{\sqrt{\rho^2 + z^2}} \,,$$

where

$$\rho^2 = \frac{4p_\Theta}{\Omega_c} + \frac{8}{\Omega_c} \sqrt{p_\psi(p_\Theta - p_\psi)} \, \cos\psi$$

specifies the xy-radial position of the guiding center of the moving particle.

We now introduce the criterion for *strong magnetization*. We say the magnetic field is *strong*, when the initial values of the z-velocity, and the value of Ω_c respectively, are such that the cyclotron period $2\pi/\Omega_c$ is much shorter than the duration of one collision (this duration is taken to be the distance of closests approach divided by the z-velocity). The criterion simply states that the moving particle completes many cyclotron orbits during its collision with the charge fixed at the origin, or equivalently, that the interaction with the field of the fixed particle is a slow perturbation. The *total* perpendicular kinetic energy

$$\begin{aligned} E_\perp &= \Omega_c p_\psi = E_\perp^{cm} + E_\perp^{rel} \\ &= E_\perp(\text{particle 1}) + E_\perp(\text{particle 2}) \,, \end{aligned}$$

will thus change only by an exponentially small amount in a binary collision.

Note that $E_\perp(\text{particle 1})$ and $E_\perp(\text{particle 2})$ each will undergo substantial change in the course of a collision, since each individual particle moves in a rapidly fluctuating electric field from the other particle. But their *sum* will only change by a small amount, if the magnetic field is strong. As in the diatomic molecules, the near constancy of the sum is *not* the result of a near constancy of each of its parts.

3.1 Many Particles

For $N \gg 1$ particles there is little to be gained in separating out the center-of-mass motion, since this will rid the problem of only 3 degrees of freedom. Nevertheless the key result from the 2-particle problem is still valid, as the following argument demonstrates. The Hamiltonian, can, in the polar guiding center coordinates be written as

$$H = \sum_{j=1}^{N} \left[\Omega_c p_{\psi_j} + \frac{p_{z_j}^2}{2} + \sum_{i>j} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right]$$

where $\mathbf{r}_i - \mathbf{r}_j$ are assumed to be expressed as a function of $\Theta_i, \Theta_j, p_{\Theta_i}, p_{\Theta_j}, \psi_i, \psi_j$, etc. To uncover the many-particle invariant we introduce a new set of angle variables, $\{\varphi_j\}$, given by

$$\begin{aligned} \varphi_1 &= \psi_1 \,, \\ \varphi_j &= \psi_j - \psi_1 \,, \quad j = 2, \cdots, N \,. \end{aligned}$$

The corresponding momentae are:

$$\begin{aligned} p_{\varphi_1} &= \sum_{j=1}^{N} p_{\psi_j} \,, \\ p_{\varphi_j} &= p_{\psi_j} \,, \quad j = 2, \cdots, N \,. \end{aligned}$$

so that, in terms of the $(\varphi, p_\varphi, \Theta, p_\Theta, z, p_z)$ variables, the Hamilton function takes the form

$$H = \Omega_c p_{\varphi_1} + \sum_{j=1}^{N} \left[\frac{p_{z_j}^2}{2} + \sum_{i>j} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} \right] \,.$$

Under the condition of strong magnetization, the variable φ_1 is the only fast angle in the problem. Consequently, p_{φ_1} is an adiabatic invariant. For Ω_c constant, $\Omega_c p_\varphi = \sum E_{j\perp}$ is therefore nearly conserved. Thus, generalizing the 2-particle result, the *sum* of all the individual particle perpendicular actions, for a homogeneous, strong magnetic field, is an adiabatic invariant of the full N-particle dynamics.

4 How good is the conspiracy?

Since Landau and Teller's pioneering work in the 1930's, the 'breaking' or shift in an adiabatically invariant quantity when subjected to a given perturbation has been to calculate along unperturbed orbits, i.e., evaluate the oscillatory integral of the time derivative of the action.

For the case of binary collision of two charged particles in a a strong magnetic field (equivalent to the collision of one charged particle with a charge fixed at the origin), one can explicitly compute the exponentially small change in $E_\perp^{rel} = \Omega_c p_\psi$ through a collision where initially and finally the moving particle is at infinity.

The change in perpendicular kinetic energy is calculated as

$$\Delta E_\perp^{rel} = \int_{-\infty}^{+\infty} \frac{\cos(\Omega_c t)}{[\rho^2 + z(t)^2]^{3/2}} \mathrm{d}t$$

where ρ is the perpendicular distance from the guiding centre to the origin, and $z(t)$ is a solution to the differential equation

$$\dot{z}^2(t) + \frac{1}{[\rho^2 + z(t)^2]} = \dot{z}^2(-\infty)$$

corresponding to the 'infinite field' motion of a particle sliding on the field line.

By extending the integral to complex values of the time parameter and deforming the contour away from the real line as far as permitted by the branch cuts of the function in the denominator, one finds that the contour integral has the form

$$\Delta E_\perp^{rel} = h(\rho) \exp\left(-g(\rho)\Omega_c\right) \cos\delta \ ,$$

where δ is (a random) initial phase, the function h is neither exponentially small or large, and the function $g(\rho)$ is given by

$$g(\rho) = \int_1^\rho \frac{x^3/2}{\sqrt{(x-1)(\rho^2 - x^2)}} \mathrm{d}x$$
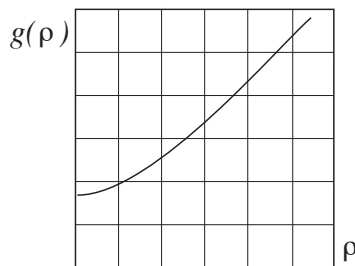

Figure 3. The function $g(\rho)$ which determines the exponential smallness of the change of the adiabatic invariant in a single collision.

The function $g(\rho)$ is monotone with $g(0) = \pi/2$. Details can be found in [5].

To estimate the evolution of the many-particle invariant, one can assume that the evolution proceeds as a sequence of binary collisions, and then use statistical mechanics to find evolution of the average perpendicular kinetic energy of an ensemble of particles. This process is discussed in [3].

5 Conclusion

The deep mechanical phenomenon of adiabatic invariance manifests itself not only as a single-particle phenomenon, but may also take the form of a non-trivial collective behavior, a conspiracy of high-frequency oscillators. The result also generalizes to the case where there are a number of well-separated time scales; the oscillators associated with each time scale will exchange energy among themselves and tend to decouple from oscillators associated with other time scales.

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