AN OVERVIEW OF RECENT DEVELOPMENTS IN COMPUTATIONAL METHODS FOR PERIODIC SYSTEMS

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Abstract: At the First IFAC Workshop on Periodic Systems (Como, Italy, 2001), a state of the art survey of computational methods for periodic systems has been presented (Varga and Van Dooren, 2001). This contribution continues this survey by presenting the main achievements in this field since 2001. Besides many foreseen developments mentioned in 2001 as open problems, important new developments took place as general algorithms for analysis of periodic descriptor systems, solution of periodic Riccati equations, or computational methods for continuous-time periodic systems.

Keywords: Periodic systems, discrete-time systems, time-varying systems, numerical methods, computer aided control system design.

1. INTRODUCTION

The theory of linear discrete-time periodic systems has received a lot of attention in the last 30 years (Bittanti and Colaneri, 1996; Bittanti and Colaneri, 2000) and many applications of this theory have been reported. Practically all results for constant discrete-time systems have been extended to *standard* periodic systems of the form

$$\begin{aligned} x(k+1) &= A_k x(k) + B_k u(k) \\ y(k) &= C_k x(k) + D_k u(k) \end{aligned} \tag{1}$$

where $A_k \in \mathrm{R}^{n_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{n_{k+1}\times m}$, $C_k \in \mathrm{R}^{p\times n_k}$, $D_k \in \mathrm{R}^{p\times m}$ are periodic matrices with period $N \geq 1$.

We also noticed an increased interest to address periodic control problems for continuous-time periodic systems of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{2}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, and $D(t) \in \mathbb{R}^{p\times m}$ are periodic matrices of period T. For a recent survey of main theoretical developments see (Colaneri, 2005) and papers cited therein.

Several algorithms for standard discrete-time periodic systems have been extended to the more general periodic *descriptor* systems of the form

$$\begin{aligned} E_k x(k+1) &= A_k x(k) + B_k u(k) \\ y(k) &= C_k x(k) + D_k u(k) \end{aligned} \tag{3}$$

where $E_k \in \mathrm{R}^{\mu_{k+1}\times n_{k+1}}$, $A_k \in \mathrm{R}^{\mu_{k+1}\times n_k}$, $B_k \in \mathrm{R}^{\mu_{k+1}\times m}$, $C_k \in \mathrm{R}^{p\times n_k}$, $D_k \in \mathrm{R}^{p\times m}$ are N-periodic matrices. Periodic descriptor systems have been considered in many papers (Conte *et al.*, 1990; Sreedhar and Van Dooren, 1997; Sreedhar *et al.*, 1999; Sreedhar and Van Dooren, 1999; Coll *et al.*, 2004; Chu *et al.*, 2005; Varga, 2007*b*).

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A comprehensive account of the situation until 2001 in developing algorithms for periodic systems is given in the survey (Varga and Van Dooren, 2001). In this paper we report on new developments in this field since 2001. Besides many foreseen developments mentioned in 2001 as still open problems, important new developments took also place in algorithms for analysis of periodic descriptor systems, solution of Riccati equations, and computational methods for continuous-time periodic systems. A notable development emerging from the sustained algorithmic progress is a comprehensive Periodic Systems toolbox for MATLAB (Varga, 2005*c*).

2. LIFTED REPRESENTATIONS

Lifted representations of discrete-time periodic systems play an important role in studying periodic systems (Bittanti and Colaneri, 1996; Bittanti and Colaneri, 2000). To define some basic concepts used in this paper, which correspond to those for standard systems (e.g., transfer-function, poles, zeros, etc.), we will use the lifting introduced in (Grasselli and Longhi, 1991) to build an equivalent time-invariant descriptor system with the input, state and output vectors defined over time intervals of length N, rather than 1. For a given sampling time k, the corresponding mN-dimensional input vector, pN-dimensional output vector and $(\sum_{k=1}^{N} n_k)$-dimensional state vector are

$$\begin{aligned} u_k^S(h) &= [u^T(k+hN)\cdots u^T(k+hN+N-1)]^T, \\ y_k^S(h) &= [y^T(k+hN)\cdots y^T(k+hN+N-1)]^T, \\ x_k^S(h) &= [x^T(k+hN)\cdots x^T(k+hN+N-1)]^T. \end{aligned}$$

The corresponding time-invariant descriptor system has the form

$$\begin{aligned} L_k x_k^S(h+1) &= F_k x_k^S(h) + G_k u_k^S(h) \\ y_k^S(h) &= H_k x_k^S(h) + J_k u_k^S(h) \end{aligned} \qquad (4)$$

where

$$F_k - zL_k = \begin{bmatrix} A_k & -E_k & O & \cdots & O \\ O & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots\ -E_{k+N-3} & O \\ O & & \ddots & A_{k+N-2} & -E_{k+N-2} \\ -zE_{k+N-1} & O & \cdots & O & A_{k+N-1} \end{bmatrix}$$

and

$$\begin{aligned} G_k &= \mathrm{diag}(B_k, B_{k+1}, ..., B_{k+N-1}), \\ H_k &= \mathrm{diag}(C_k, C_{k+1}, ..., C_{k+N-1}), \\ J_k &= \mathrm{diag}(D_k, D_{k+1}, ..., D_{k+N-1}). \end{aligned}$$

Assuming the pencil $F_k - zL_k$ is regular (i.e., square and $\det(F_k - zL_k) \not\equiv 0$), the *transfer-function matrix* (TFM) of the lifted system at time k is

$$W_k(z) := H_k(zL_k - F_k)^{-1}G_k + J_k. \qquad (5)$$

For minimal periodic systems (1) or (3) (i.e., completely reachable and completely observable), the poles are defined as the zeros of the pencil $F_k - zL_k$, while the system zeros are the zeros of the associated system pencil

$$\mathcal{S}_k(z) := \begin{bmatrix} F_k - zL_k & G_k \\ H_k & J_k \end{bmatrix} \qquad (6)$$

3. ALGORITHMS FOR PERIODIC SYSTEMS

We consider two main categories of algorithms for periodic systems. The so-called "fast" algorithms are *structure exploiting* methods which work on lifted representations. These methods are highly efficient, because completely avoid explicitly forming of lifted representations like (4). A typical algorithm in this category is the algorithm to compute zeros of periodic systems by exploiting the structure of the lifted system matrix (6) but without forming it explicitly. Similar "fast" algorithms can be employed to solve periodic Riccati equations. For many "fast" algorithms a certain kind of numerical stability can be proven.

In the second category there are so-called *structure preserving* methods, which work directly on the system matrices and preserve the cyclic structure of matrices of the lifted representation (4). Most algorithms in this category are very recent developments. Several methods in this category are *structurally backward stable*. For them, it can be proven that the computed results are exact for slightly perturbed original system data.

Three main requirements, *generality*, *numerical stability* and *efficiency*, have been formulated in (Varga and Van Dooren, 2001) for a satisfactory algorithm for periodic systems. The requirement of *generality* covers, among other aspects, the handling of problems with time-varying dimensions. Many computational methods have been proposed for systems with constant dimensions. While extensions to time-varying dimensions are often straightforward, these extensions usually rely on more involved computational ingredients which have been only recently developed. To promote *numerical stability*, some of the key ingredients are: using exclusively orthogonal transformations, avoiding completely forming of products of non-orthogonal matrices, and fully exploiting structure and possibly fully preserving structure by employing condensed forms. The requirement for *efficiency* implies avoiding excessive storage usage (i.e., avoiding lifting) and guaranteeing a computational complexity of at most $O(Nn^3)$, where n is the maximal state/input/output vector dimension. Specific features of algorithms for continuous-time periodic systems are addressed separately in Section 7.

4. COMPUTATIONAL INGREDIENTS

The traditional computational ingredients described in (Varga and Van Dooren, 2001), like the algorithms to compute periodic Hessenberg, Schur and QZ decompositions (Bojanczyk *et al.*, 1992; Hench and Laub, 1994), have been recently complemented by enhanced numerically stable algorithms for reordering of periodic Schur forms (Granat and Kågström, 2006) and QZ decompositions (Granat *et al.*, 2006). An elegant presentation of the periodic QR-algorithm related techniques is part of a recent book (Kressner, 2005).

An algorithm to compute periodic Kronecker-like forms has been recently developed (Varga, 2004*c*). For given N-periodic matrix pairs (S_k, T_k) with $S_k \in \mathbb{R}^{\mu_k \times \nu_k}$ and $T_k \in \mathbb{R}^{\mu_k \times \nu_{k+1}}$, this algorithm determines orthogonal N-periodic transformation matrices Q_k and Z_k such that

$$Q_k S_k Z_k = \begin{bmatrix} B_k^r & A_k^r & * & * & * \\ O & O & A_k^\infty & * & * \\ O & O & O & A_k^f & * \\ O & O & O & O & A_k^l \\ O & O & O & O & C_k^l \end{bmatrix},$$

$$Q_k T_k Z_{k+1} = \begin{bmatrix} O & E_k^r & * & * & * \\ O & O & E_k^\infty & * & * \\ O & O & O & E_k^f & * \\ O & O & O & O & E_k^l \\ O & O & O & O & O \end{bmatrix},$$

where:

(a) E_k^r is invertible and the periodic pair $\left((E_k^r)^{-1}A_k^r, (E_k^r)^{-1}B_k^r\right)$ is completely reachable;
(b) E_k^l is invertible and the periodic pair $\left(C_k^l, (E_k^l)^{-1}A_k^l\right)$ is completely observable;
(c) A_k^∞ is invertible and the product $(A_k^\infty)^{-1}E_k^\infty \dots (A_{k+N-1}^\infty)^{-1}E_{k+N-1}^\infty$ is nilpotent;
(d) E_k^f is non-singular.

Note that $Q_k S_k Z_k$ and $Q_k T_k Z_{k+1}$ have the same row partition which however generally depends on k. For a fixed column partitioning of $Q_k S_k Z_k$, the corresponding column partitioning of $Q_k T_k Z_{k+1}$ is uniquely determined by the conditions (a)-(d) above. The periodic pair (A_k^∞, E_k^∞) specifies the structure at infinity of the periodic pair (S_k, T_k), while the pair (A_k^f, E_k^f) specifies its finite structure. Similarly, the periodic triples (A_k^r, E_k^r, B_k^r) and (A_k^l, E_k^l, C_k^l) specify the right and left Kronecker structures of the pair (S_k, T_k), respectively.

The algorithms proposed in (Varga, 2004*c*) as well as particularizations of them have many straightforward applications, as for example, computation of periodic systems poles and zeros, minimal realizations, solution of periodic Riccati equations in the most general setting, etc.

5. ALGORITHMS FOR STANDARD DISCRETE-TIME PERIODIC SYSTEMS

5.1 System conversions and analysis

Several open computational problems for periodic systems mentioned in (Varga and Van Dooren, 2001) have been solved in the meantime. To employ frequency-domain methods for the analysis of periodic systems, an algorithm to evaluate the associated $Np \times Nm$ TFM $W_k(z)$ (5) has been developed (Varga, 2003*a*). Conversely, the minimal realization of a given proper lifted TFM $W_k(z)$ can be reliably computed by an algorithm proposed in (Varga, 2004*d*). To compute frequency responses, the evaluation of $W_k(e^{j\theta})$ is necessary for a range of values of θ. Methods to compute the frequency response efficiently by exploiting the sparse structure of the matrices of the lifted representation (4) have been proposed in (Varga, 2006*b*). Here, the periodic Hessenberg and Schur forms (Bojanczyk *et al.*, 1992) play important roles to improve the computational efficiency.

Efficient and numerically stable algorithms have been developed for the computation of the periodic Kalman reachability and observability canonical forms (Varga, 2004*b*). These algorithms can be used to compute minimal realizations of periodic systems by eliminating the non-reachable and non-observable parts of the system. Recently the reduction technique of (Varga, 2004*b*) has been extended to address the computation of minimal dynamic covers for periodic systems (Varga, 2007*a*).

For a minimal (completely reachable and completely observable) periodic system the zeros of the associated TFM $W_k(z)$ (5) can be computed using a numerically stable "fast" algorithm which computes the zeros of $S_k(z)$ in (6) using structure exploiting orthogonal reduction (Varga and Van Dooren, 2002). A structure preserving backward stable approach to compute zeros is based on the algorithm to compute the periodic Kronecker-like forms (Varga, 2004*c*) applied to the periodic pair (S_k, T_k) defined by matrices

$$S_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad T_k = \begin{bmatrix} I_{n_{k+1}} & 0 \\ 0 & 0 \end{bmatrix}.$$

The finite zeros can be determined as the generalized eigenvalues of the quotient-product

$$\left(E_{k+N-1}^f\right)^{-1} A_{k+N-1}^f \cdots \left(E_k^f\right)^{-1} A_k^f,$$

while the infinite zeros structure is determined by the nilpotent matrix quotient-product

$$(A_k^\infty)^{-1}E_k^\infty \dots (A_{k+N-1}^\infty)^{-1}E_{k+N-1}^\infty.$$

For these computations extensions of the periodic QZ decomposition techniques of (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) to time-varying dimensions is necessary. Works in this direction are in progress (Kressner, 2007).

As an example, consider the *periodic reverse discrete-time algebraic Riccati equation* (PRDARE)

$$X_k = Q_k + A_k^T X_{k+1} A_k - (A_k^T X_{k+1} B_k + S_k) \times (R_k + B_k^T X_{k+1} B_k)^{-1} (A_k^T X_{k+1} B_k + S_k)^T \quad (7)$$

where $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $B_k \in \mathbb{R}^{n_{k+1}\times m_k}$, $Q_k \in \mathbb{R}^{n_k\times n_k}$, $R_k \in \mathbb{R}^{m_k\times m_k}$ and $S_k \in \mathbb{R}^{n_k\times m_k}$ are N-periodic matrices ($N \geq 1$). All Q_k and R_k are assumed symmetric matrices. We are interested to compute the unique symmetric stabilizing N-periodic solution X_k of equation (7). This solution allows, for example, to determine a stabilizing periodic state-feedback which solves a linear-quadratic optimization problem.

The solution of the PRDARE (7) for constant dimensions has been considered in (Bojanczyk *et al.*, 1992; Hench and Laub, 1994; Benner *et al.*, 2002), but PRDAREs with time-varying dimensions have been considered only recently in (Chu *et al.*, 2004). A standard assumption in all these algorithms is the invertibility of R_k. Recently, general algorithms to solve PRDARE for time-varying dimensions and possibly singular R_k have been developed (Varga, 2005*a*; Varga, 2007*c*). The new algorithms can be seen as extensions of both the periodic QZ decomposition based approach (Bojanczyk *et al.*, 1992; Hench and Laub, 1994) as well as of "fast" methods (Benner *et al.*, 2002; Chu *et al.*, 2004).

6. ALGORITHMS FOR DISCRETE-TIME PERIODIC DESCRIPTOR SYSTEMS

The main computational tool for the analysis of periodic descriptor systems is the reduction of periodic pairs (S_k, T_k) to periodic Kronecker-like forms (Varga, 2004*c*). With this tool, various analysis tasks can be performed using suitable choices of the periodic matrices S_k and T_k.

The *solvability* of the periodic descriptor system (3) is equivalent to check the regularity of the pencil $F_k - zL_k$ (Sreedhar and Van Dooren, 1999). To do this, we can perform the structurally stable reduction of the periodic pair (A_k, E_k) to a periodic Kronecker-like form using the algorithm of (Varga, 2004*c*). If this pair has no left or right Kronecker structure, then the periodic descriptor system (3) is *regular*, and thus solvable.

For the computation of system zeros, a "fast" algorithm has been proposed in (Varga and Van Dooren, 2003) and a structure preserving strongly stable algorithm has been proposed in (Varga, 2003*b*). These algorithms are also useful to compute the poles (as the zeros of a particular system without inputs and outputs). A structure preserving backward stable approach to compute zeros is based on applying the algorithm to compute the periodic Kronecker-like forms to the periodic pair (S_k, T_k) defined by the matrices

$$S_k = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad T_k = \begin{bmatrix} E_k & 0 \\ 0 & 0 \end{bmatrix}$$

For the analysis of reachability or observability, similar computations as described in (Varga, 2007*b*) can be performed for

$$S_k = [\, A_k \; B_k \,], \quad T_k = [\, E_k \; 0 \,] \quad (8)$$

or

$$S_k = \begin{bmatrix} A_k \\ C_k \end{bmatrix}, \quad T_k = \begin{bmatrix} E_k \\ 0 \end{bmatrix}, \quad (9)$$

to compute the input or output decoupling zeros, respectively. Their presence is equivalent to the existence of non-reachable part and/or non-observable part. Stabilizability and detectability can be assessed by checking the stability of input or output decoupling zeros, respectively.

To solve the minimal realization problem, two approaches are proposed in (Varga, 2007*b*). The first approach combines the forward/backward separation with specialized minimal realization procedures for the forward and backward parts. In the second approach, the non-reachable and non-observable parts are successively eliminated by performing special reductions of periodic matrix pairs as defined in (8) or (9), respectively.

An enhanced general algorithm has been recently proposed to compute the $\mathcal{L}_\infty$-norm of a periodic descriptor system with time-varying dimensions (Varga, 2006*a*). This algorithm relies on an efficient method to compute the frequency-response of descriptor systems (Varga, 2006*b*) and on specialized algorithms to compute poles and zeros of particular descriptor systems. Recently developed reliable algorithms to compute left/right inverses of periodic systems (Varga, 2004*a*) or left/right annihilators (Varga, 2004*e*) also rely on the computation of periodic Kronecker-like forms.

7. ALGORITHMS FOR CONTINUOUS-TIME PERIODIC SYSTEMS

Among the first methods for continuous-time periodic systems which can be labeled "computational", we mention just two classes of methods proposed to evaluate system norms (Zhou and Hagiwara, 2002; Lampe and Rosenwasser, 2004). In the first class of methods, frequency-lifting is employed to build equivalent infinite-dimensional system representations. Approximations obtained using various finite truncations of infinite-dimensional Toeplitz matrices are then employed to evaluate the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms using complex-valued computation based algorithms (e.g., to solve Lyapunov and Riccati equations).

The second class of methods relies on closed-form formulas to compute the $\mathcal{H}_2$-norm. Still, they involve the evaluation of several integrals and the integration of several matrix differential equations.

Recently, a new algorithmic paradigm called *multi-shot method* evolved for solving continuous-time problems (e.g., various periodic matrix differential equations). Using appropriate *exact* discretizations, the continuous-time problems are reduced to equivalent single- or multi-point discrete-time periodic problems for which reliable computational algorithms are available. By solving the discrete-time problems, so-called *periodic generators* are computed (e.g., initial or multi-point conditions), which serve to determine the continuous periodic solutions (usually by integrating the underlying ordinary matrix differential equations with known initial or multi-point conditions).

Let $\Phi_A(t,\tau)$ denote the transition matrix corresponding to a T-periodic $A(t)$ satisfying

$$\frac{\partial \Phi_A(t,\tau)}{\partial t} = A(t)\Phi_A(t,\tau), \quad \Phi_A(\tau,\tau) = I$$

The basis of several discretization techniques is the product form representation of the *monodromy* matrix

$$\Phi_A(T,0) = \Phi_A(T,T-\Delta)\cdots\Phi_A(2\Delta,\Delta)\Phi_A(\Delta,0)$$

where $\Delta = T/N$ for a suitably chosen integer period N. The matrix $F_k := \Phi_A(k\Delta,(k-1)\Delta)$ for $k = 1,2,\ldots,$ is obviously N-periodic. Thus, the eigenvalues of $\Phi_A(T,0)$ can be alternatively computed using the periodic Schur form based algorithm (Bojanczyk *et al.*, 1992). The main advantage of using product form representations is that unstable integration of differential equations over long periods can be completely avoided.

Similar discretizations of appropriate Hamiltonian matrices are employed to solve periodic Lyapunov, Sylvester or Riccati differential equations (Varga, 2005*b*). The applications of these multi-shot methods to evaluate Hankel, $\mathcal{H}_2$-, or $\mathcal{H}_\infty$-norms (Varga, 2005*b*) or to solve continuous-time output feedback linear-quadratic problems (Viganò *et al.*, 2007) are relatively straightforward. Recently, an evaluation of the multi-shot method to solve periodic Riccati equations in conjunction with special structure preserving (symplectic) integrators has shown the high effectiveness of these techniques (Johansson *et al.*, 2007).

8. CONCLUSIONS

The expected new algorithmic developments are driven by stringent needs in several application areas. The development of new algorithms for periodic descriptor systems is important in solving fault detection problems for multi-rate systems (modelled as discrete-time periodic systems). Special needs arise from fault detection or controller synthesis, where algorithms are necessary to compute various factorizations (stable coprime, normalized coprime, inner-outer) or to solve periodic model matching problems (exactly or via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization). Many new developments are also expected in algorithms for continuous-time problems, as for example, efficient computation of frequency-responses, solution of stabilization and pole assignment problems, order reduction, solution of various controller synthesis problems. Among new tools to be investigated in the future is the applicability of continuous matrix decompositions to solve various computational problems for time-varying periodic systems (Dieci and Eirola, 1999).

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