# SHARPENED ESTIMATES FOR THE NUMBER OF SLIPPED CYCLES IN CONTROL SYSTEMS WITH PERIODIC DIFFERENTIABLE NONLINEARITIES

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**Abstract**

For multidimensional systems of indirect control with periodic differentiable nonlinearities the problem of cycle-slipping is investigated. By means of generalized Lyapunov periodic functions and Yakubovich-Kalman-Popov lemma, new frequency-algebraic estimates for a number of slipped cycles are obtained.

**Key words**

Phase system, cycle-slipping, Yakubovich-Kalman-Popov lemma, Lyapunov-type periodic function.

## 1 Introduction

In this paper the systems of indirect control with periodic differentiable nonlinearities and denumerable sets of equilibria are considered. Such control systems are often called phase systems.

Phase systems are widespread in mechanics, radio and electric engineering, communication. That is why the asymptotic behavior of phase systems is of great interest, and many published works are devoted to problems which arise here.

In most works the problem of gradient-like behavior is studied with the aim to obtain conditions which guarantee that every solution converges to a certain equilibrium. For details and bibliography one can turn to the papers [Leonov, 2006] and [Perkin,Smirnova, and Shepeljavyi, 2012].

This paper is devoted to the problem of cycle-slipping in gradient-like phase control systems. The problem was introduced in the book [Stoker, 1950], for mathematical pendulum which underwent the strength of the resistance of the medium proportional to the square of its angular speed. The problem was then extended to multidimensional phase systems in paper [Yershova, and Leonov, 1983].

The essence of cycle-slipping is as follows. Suppose the phase system is gradient-like. Let it have a $\Delta$ - periodic scalar input and let $\sigma(t)$ be its angular scalar output. They say that the output function has slipped $k \in \mathbf{N} \bigcup \{0\}$ cycles if there exists such a moment $\widehat{t} \geq 0$ that

$$|\sigma(\widehat{t}) - \sigma(0)| = k\Delta, \qquad (1)$$

but

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta \qquad (2)$$

for all $t \geq 0$. The problem which arises here is to establish certain estimates for a number of slipped cycles.

In the paper [Yershova, and Leonov, 1983] such estimates were obtained by means of Lyapunov-type functions and Yakubovich-Kalman-Popov lemma. The estimates had the form of frequency inequalities with varying parameters which ought to satisfy algebraic restrictions.

Ideas and theorems of [Yershova, and Leonov, 1983] were exploited then in a series of published works in order to investigate the cycle-slipping of various types of phase systems.

In [Yang, and Huang, 2007] the results of [Yershova, and Leonov, 1983] were joined with the LMI method. In [ Leonov, Reitmann, and Smirnova, 1992] and [Smirnova, Utina, and Shepeljavyi, 2006] the theorems proved in [Yershova, and Leonov, 1983] were extended to distributed phase systems and discrete phase systems correspondingly. At the same time in [ Leonov, Reitmann, and Smirnova, 1992] differential properties of periodic nonlinearities were taken into consideration and new frequency inequalities with more varying parameters were introduced.

In this paper frequency inequalities intended for differentiable nonlinearities are used. At the same time generalized periodic Lyapunov functions introduced in [Perkin,Smirnova, and Shepeljavyi, 2012] are exploited here. These functions give the opportunity to establish for varying parameters new algebraic restrictions which are less limitative than the restrictions formulated in [ Leonov, Reitmann, and Smirnova, 1992].

## 2 Frequency-Algebraic Estimates for Cycle-Slipping of Multidimensional Phase System

Consider an autonomous control system

$$\left.\begin{aligned} \frac{dz}{dt} &= Az + b\varphi(\sigma) \quad (z \in \mathbf{R}^m, \sigma \in \mathbf{R}), \\ \frac{d\sigma}{dt} &= c^* z + \rho\varphi(\sigma). \end{aligned}\right\} \tag{3}$$

Here $A$ — $m \times m$ - real matrix, $b$ and $c$ are real $m$ - vectors, $\rho$ is a number, $\varphi(\sigma)$ is a $\Delta$ - periodic function and symbol (*) is used for Hermitian conjugation.

We suppose that $A$ is a Hurwitz matrix, the pairs $(A, b)$ and $(A, c)$ are controllable and observable respectively. We suppose also that $\varphi(\sigma)$ has simple zeros on $[0, \Delta)$ and is continuously differentiable. So

$$\alpha_1 \le \frac{d\varphi}{d\sigma} \le \alpha_2 \quad (\sigma \in \mathbf{R}) \tag{4}$$

with $\alpha_1\alpha_2 < 0$. Let us define the function

$$\Phi(\sigma) = \sqrt{(1 - \alpha_1^{-1}\varphi'(\sigma))(1 - \alpha_2^{-1}\varphi'(\sigma))}.$$

Let us for definiteness suppose that

$$\int_0^\Delta \varphi(\sigma)d\sigma \le 0. \tag{5}$$

First we shall present a number of auxiliary Lyapunov-type assertions. We shall need the functions

$$r_j(k, æ, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{æk}}{\int_0^\Delta |\varphi(\sigma)|d\sigma} \quad (j = 1, 2),$$

$$r_{0j}(k, æ, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{æk}}{\int_0^\Delta \Phi(\sigma)|\varphi(\sigma)|d\sigma} \quad (j = 1, 2).$$

**Lemma 1.** Suppose there exist such numbers $k \in \mathbf{N}$, $a \in [0, 1]$, $æ \neq 0$, positive $\varepsilon$, $\delta$, $\tau$ and such continuously differentiable functions $\sigma(t)$ and $W(t)$ that the following conditions are fulfilled:

1)$W(t) \ge 0$ for $t \ge 0$;

2)

$$\begin{aligned} &\tfrac{dW(t)}{dt} + æ\varphi(\sigma(t))\tfrac{d\sigma(t)}{dt} + \varepsilon\left(\tfrac{d\sigma(t)}{dt}\right)^2 + \delta\varphi^2(\sigma(t)) \\ &+\tau\Phi^2(\sigma(t))\left(\tfrac{d\sigma(t)}{dt}\right)^2 \le 0, \forall t \ge 0; \end{aligned} \tag{6}$$

3) matrices $T_j(W(0)) \quad (j = 1, 2)$, where $T_j(x) =$

$$\left\| \begin{matrix} \varepsilon & , & \dfrac{aær_j(k, æ, x)}{2} & , & 0 \\ \dfrac{aær_j(k, æ, x)}{2} & , & \delta & , & \dfrac{a_0ær_{0j}(k, æ, x)}{2} \\ 0 & , & \dfrac{a_0ær_{0j}(k, æ, x)}{2} & , & \tau \end{matrix} \right\|$$

with $a_0 = 1 - a$, are positive definite. Then

$$|\sigma(t) - \sigma(0)| < k\Delta, \quad t \ge 0. \tag{7}$$

Lemma 1 is proved in [Perkin,Smirnova, Shepeljavyi, and Utina,2009]. We shall prove here a modification of Lemma 1.

Consider the functions

$$r_{1j}(k, æ, \varepsilon, \tau, x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j \frac{x}{æk}}{\int_0^\Delta |\varphi(\sigma)|\sqrt{\varepsilon + \tau\Phi^2(\sigma)}d\sigma}$$

for $j = 1, 2$.

**Lemma 2.** Suppose there exist such numbers $k \in \mathbf{N}$, $æ \neq 0$, $\varepsilon$, $\delta$, $\tau > 0$ and such functions $\sigma(t) \in C^1(0, \infty)$ and $W(t) \in C^1(0, \infty)$ that the conditions 1) and 2) of Lemma 1 are fulfilled and

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, W(0)))^2 \quad (j = 1, 2). \tag{8}$$

Then the assertion (7) is true.

*Proof.* Let $\varepsilon_0$ be so small that

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, W(0)+\varepsilon_0)^2 \quad (j = 1, 2). \quad (9)$$

Define the functions

$$Y_j(\sigma) = \varphi(\sigma) - r_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma) \quad (j = 1, 2),$$

where $r_{1j} = r_{1j}(k, æ, \varepsilon, \tau, W(0) + \varepsilon_0)$, $P(\varepsilon, \tau, \sigma) = \sqrt{\varepsilon + \tau\Phi^2(\sigma)}$, and consider two Lyapunov-type functions

$$V_j(t) = W(t) + æ \int_{\sigma(0)}^{\sigma(t)} Y_j(\sigma)d\sigma \quad (j = 1, 2).$$

Their derivatives are as follows

$$\frac{dV_j(t)}{dt} = \frac{dW(t)}{dt} + æY_j(\sigma(t))\dot{\sigma}(t). \quad (10)$$

From condition 2) of Lemma 1 we conclude that

$$\begin{aligned} \frac{dV_j}{dt} &\le -\varepsilon\dot{\sigma}^2 - \delta\varphi^2(\sigma) - \tau(\Phi(\sigma)\dot{\sigma})^2 - \\ &-ær_{1j}P(\varepsilon, \tau, \sigma)\,|\varphi(\sigma)|\,\dot{\sigma} \quad (j = 1, 2), \end{aligned} \quad (11)$$

or

$$\begin{aligned} \frac{dV_j}{dt} &\le -(P(\varepsilon, \tau, \sigma)\dot{\sigma})^2 - \delta|\varphi(\sigma)|^2 \\ &-ær_{1j}|\varphi(\sigma)|P(\varepsilon, \tau, \sigma)\dot{\sigma}. \end{aligned} \quad (12)$$

It follows from (9) that

$$\frac{dV_j(t)}{dt} \le 0 \quad (j = 1, 2), \quad (13)$$

and consequently for all $t > 0$

$$V_j(t) \le V_j(0) = W(0) \quad (j = 1, 2). \quad (14)$$

Suppose now that for a certain $\bar{t}$ we have

$$\sigma(\bar{t}) = \sigma(0) + k\Delta. \quad (15)$$

Then

$$V_1(\bar{t}) = W(\bar{t}) + æk\int_0^{\Delta} Y_1(\sigma)d\sigma = W(\bar{t}) + W(0) + \varepsilon_0 \quad (16)$$

So $V_1(\bar{t}) > W(0)$ which contradicts (14).

If we suppose that

$$\sigma(\bar{t}) = \sigma(0) - k\Delta \quad (17)$$

we can use $V_2(t)$ and establish that

$$\begin{aligned} V_2(\bar{t}) &= W(\bar{t}) - æk\int_0^{\Delta} Y_2(\sigma)d\sigma = \\ &= W(\bar{t}) + W(0) + \varepsilon_0 > W(0). \end{aligned} \quad (18)$$

Lemma 2 is proved.

In order to use the two lemmas for estimation of solutions of system (3) we exploit certain approaches from monographs [Yakubovich,Leonov, and Gelig, 2004] and [ Leonov, Reitmann, and Smirnova, 1992].

Let us introduce the function $\xi(t) = \frac{d}{dt}\varphi(\sigma(t))$ and the $(m+1)$-vector-function

$$y(t) = \begin{Vmatrix} z(t) \\ \varphi(\sigma(t)) \end{Vmatrix}.$$

We introduce also $(m + 1)$ -vectors

$$L = \begin{Vmatrix} O \\ 1 \end{Vmatrix},\; D = \begin{Vmatrix} c \\ \rho \end{Vmatrix}$$

and the $(m + 1) \times (m + 1)$-matrix

$$Q = \begin{Vmatrix} A & b \\ O & 0 \end{Vmatrix}.$$

Functions $y(t)$ and $\xi(t)$ satisfy the system

$$\left.\begin{aligned} \frac{dy(t)}{dt} &= Qy(t) + L\xi(t), \\ \frac{d\sigma(t)}{dt} &= D^*y(t) \end{aligned}\right\}. \quad (19)$$

The controlability of $(A, b)$ implies the controlability of $(Q, L)$ [Yakubovich,Leonov, and Gelig, 2004].

Consider the following quadratic form of $y \in \mathbf{R}^{m+1}$, $\xi \in \mathbf{R}$:

$$\begin{aligned} G(y, \xi) &= 2y^*H(Qy + L\xi) + \varepsilon y^*DD^*y + æy^*LD^*y \\ &-\tau(D^*y - \alpha_1^{-1}\xi)(\alpha_2^{-1}\xi - D^*y) + \delta y^*LL^*y. \end{aligned}$$

Here $H = H^*$ is an real $(m+1)\times(m+1)$-matrix and $\varepsilon$, æ, $\tau$ and $\delta$ are parameters.

Consider the function

$$K(p) = -\rho + c^*(A - pE_m)^{-1}b \qquad (p \in \mathbf{C}),$$

where $E_m$ is a unit $m \times m$-matrix, which is the transfer function of linear part of (3) from the input $\varphi$ to the output $(-\dot{\sigma}(t))$. It is demonstrated in [Yakubovich,Leonov, and Gelig, 2004] by means of Yakubovich-Kalman-Popov lemma that if the frequency inequality

$$Re\{æK(i\omega) - \tau(K(i\omega) + \alpha_1^{-1}i\omega)^*(K(i\omega) + \alpha_2^{-1}i\omega)\} \\ -\varepsilon|K(i\omega)|^2 - \delta \geq 0 \quad (i^2 = -1) \tag{20}$$

is true for $\omega \geq 0$ then there exists such real matrix $H = H^*$ that

$$G(y, \xi) \leq 0, \quad \forall y \in \mathbf{R}^{m+1}, \quad \xi \in \mathbf{R}. \tag{21}$$

Let $(z(t), \sigma(t))$ be a solution of (3) with the initial data $(z(0), \sigma(0))$. The corresponding solution $y(t)$ of (19) has the initial data
$y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$.
Note that $y(t)$ is bounded (since $A$ is Hurwitz matrix and $\varphi(\sigma)$ is a bounded function).

**Theorem 1.** Suppose there exist such $k \in \mathbf{N}$, $a \in [0, 1]$, positive $\varepsilon$, $\delta$, $\tau$ and $æ \neq 0$ that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency inequality (20) is true;

2) matrices $T_j\,(y^*(0)Hy(0) - I)$ $(j = 1, 2)$, where $I = inf_{t \in R_+} y^*(t)Hy(t)$, are positive definite for a certain matrix $H = H^*$, satisfying (21).

Then for the solution of (3) with initial data $(z(0), \sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < \Delta k \tag{22}$$

is true for all $t > 0$ .

Theorem 1 is proved in in [Perkin, Perieva, Smirnova, and Shepelyavyi, 2012]. The proof is based on Lemma 1. Lemma 2 gives the opportunity to prove a certain modification of Theorem 1.

**Theorem 2.** Suppose for certain parameters $\varepsilon$, $\delta$, $\tau > 0$, $æ \neq 0$ and $k \in \mathbf{N}$ the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency inequality (20) is true;

2)

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, y(0)^*Hy(0) - I))^2 \quad (j = 1, 2) \tag{23}$$

where $I$ and $H$ are defined in the text of Theorem 1.

Then for the solution of (3) with the initial data $(z(0), \sigma(0))$ the estimate (22) is true for all $t \geq 0$ .

*Proof.* The proof is based on Lemma 2. Let $\{z(t), \sigma(t)\}$ be the solution of system (3) with initial data $(z(0), \sigma(0))$ . Let

$$W(t) = y^*(t)Hy(t) - I$$

where $y(t)$ is the solution of (19) with $y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$. We have $W(t) \geq 0$ for $t \geq 0$ and

$$\frac{dW(t)}{dt} = 2y^*(t)H\left(Qy(t) + L\xi(t)\right). \tag{24}$$

Condition 1) of the theorem guarantees that (21) is true, whence as $D^*y = \dot{\sigma}$ and $L^*y = \varphi(\sigma(t))$ it follows that

$$\frac{dW(t)}{dt} \leq -æ\varphi(\sigma(t))\dot{\sigma}(t) - \varepsilon\dot{\sigma}^2(t) - \delta\varphi^2(\sigma(t)) - \\ -\tau\dot{\sigma}^2(t)\left(1 - \alpha_1^{-1}\varphi'(\sigma(t))\right)\left(1 - \alpha_2^{-1}\varphi'(\sigma(t))\right). \tag{25}$$

So condition 2) of Lemma 1 is fulfilled. Estimates (8) coincide with condition 2) of the theorem. It follows from Lemma 2 that estimate (22) holds for all $t \geq 0$. Theorem 2 is proved.

## 3 Sharpened Estimates for Cycle-Slipping

In this section we continue to establish various frequency-algebraic estimates for cycle-slipping of system (3). We are going to handle the same tools as in the previous section. We shall use system (19), the quadratic form $G(y, \xi)$, the frequency inequality (20). Our aim is to simplify algebraic conditions 2) of Theorems 1 and 2.

Let $z(t), \sigma(t)$ be a solution of (3) with the initial data $(z(0), \sigma(0))$. Let $\sigma_0$ be such a zero of $\varphi(\sigma)$ that

$$\sigma_0 - \Delta < \sigma(0) \leq \sigma_0.$$

In this section we replace functions $r_j, r_{0j}, r_{1j}$ $(j = 1, 2)$ by the functions

$$\nu_1(k, æ, x) = \frac{\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int\limits_0^{\Delta} \varphi(\sigma)d\sigma - æ^{-1}x}{\int\limits_{\sigma(0)}^{\sigma_0} |\varphi(\sigma)|d\sigma + k\int\limits_0^{\Delta} |\varphi(\sigma)|d\sigma},$$

$$\nu_2(k, æ, x) = \frac{\int\limits_{\sigma_0 - \Delta}^{\sigma(0)} \varphi(\sigma)d\sigma + k\int\limits_0^{\Delta} \varphi(\sigma)d\sigma + æ^{-1}x}{\int\limits_{\sigma_0 - \Delta}^{\sigma(0)} |\varphi(\sigma)|d\sigma + k\int\limits_0^{\Delta} |\varphi(\sigma)|d\sigma},$$

$$\nu_{01}(k, æ, x) = \frac{\int\limits_{\sigma(0)}^{\sigma_0} \varphi(\sigma)d\sigma + k\int\limits_0^{\Delta} \varphi(\sigma)d\sigma - æ^{-1}x}{\int\limits_{\sigma(0)}^{\sigma_0} \Phi(\sigma)|\varphi(\sigma)|d\sigma + k\int\limits_0^{\Delta} \Phi(\sigma)|\varphi(\sigma)|d\sigma},$$

$$\nu_{02}(k,æ,x)=\frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)}\varphi(\sigma)d\sigma+k\int\limits_0^{\Delta}\varphi(\sigma)d\sigma+æ^{-1}x}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)}\Phi(\sigma)|\varphi(\sigma)|d\sigma+k\int\limits_0^{\Delta}\Phi(\sigma)|\varphi(\sigma)|d\sigma},$$

$$\nu_{11}(d,x)=\frac{\int\limits_{\sigma(0)}^{\sigma_0}\varphi(\sigma)d\sigma+k\int\limits_0^{\Delta}\varphi(\sigma)d\sigma-æ^{-1}x}{\int\limits_{\sigma(0)}^{\sigma_0}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma+k\int\limits_0^{\Delta}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma},$$

$$\nu_{12}(d,x)=\frac{\int\limits_{\sigma_0-\Delta}^{\sigma(0)}\varphi(\sigma)d\sigma+k\int\limits_0^{\Delta}\varphi(\sigma)d\sigma+æ^{-1}x}{\int\limits_{\sigma_0-\Delta}^{\sigma(0)}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma+k\int\limits_0^{\Delta}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma}$$

with $d=\{k,æ,\varepsilon,\tau\}$, $P(\varepsilon,\tau,\sigma)=\sqrt{\varepsilon+\tau\Phi^2(\sigma)}$.

**Lemma 3.** Let $\sigma(0)\in(\sigma_0-\Delta,\sigma_0)$, where $\varphi(\sigma_0)=0$. Suppose there exist such numbers $k\in\mathbf{N}$, $a\in[0,1]$, $\varepsilon$, $\delta$, $\tau>0$, $æ\neq 0$ and such functions $\sigma(t),W(t)\in C^1[0,\infty)$ that the following conditions are fulfilled:

1) $W(t)\geq 0$ if $\varphi(\sigma(t))=0$;

2)

$$\frac{dW(t)}{dt}+æ\varphi(\sigma(t))\frac{d\sigma(t)}{dt}+\varepsilon\left(\frac{d\sigma(t)}{dt}\right)^2+ +\delta\varphi^2(\sigma(t))+\tau\Phi^2(\sigma(t))\left(\frac{d\sigma(t)}{dt}\right)^2\leq 0,\forall t\geq 0; \quad (26)$$

3) matrices $U_j(W(0))\quad(j=1,2)$, where $U_j(x)=$

$$\left\|\begin{matrix}\varepsilon, & \frac{aæ\nu_j(k,æ,x)}{2}, & 0\\ \frac{aæ\nu_j(k,æ,x)}{2}, & \delta, & \frac{a_0æ\nu_{0j}(k,æ,x)}{2}\\ 0, & \frac{a_0æ\nu_{0j}(k,æ,x)}{2}, & \tau\end{matrix}\right\|$$

with $a+a_0=1$, are positive definite.

Then for all $t\geq 0$ the estimate

$$|\sigma(t)-\sigma(0)|<(k+1)\Delta \quad (27)$$

is true.

*Proof.* Let $\varepsilon_0>0$ be a small number, such that matrices $U_j(W(0)+\varepsilon_0)\quad(j=1,2)$ are positive definite. Let us introduce functions

$$F_j(\sigma)=\varphi(\sigma)-\nu_j|\varphi(\sigma)|,$$

$$\Psi_j(\sigma)=\varphi(\sigma)-\nu_{0j}|\varphi(\sigma)|\Phi(\sigma),$$

with

$$\nu_j=\nu_j(k,æ,W(0)+\varepsilon_0);\nu_{0j}=\nu_{0j}(k,æ,W(0)+\varepsilon_0).$$

Consider Lyapunov-type functions

$$V_j(t)=W(t)+æ\left(a\int\limits_{\sigma(0)}^{\sigma(t)}F_j(\sigma)d\sigma+a_0\int\limits_{\sigma(0)}^{\sigma(t)}\Psi_j(\sigma)d\sigma\right).$$

Then

$$\frac{dV_j}{dt}=\frac{dW}{dt}+æ\left(aF_j(\sigma(t))+a_0\Psi_j(\sigma(t))\right)\dot\sigma(t). \quad (28)$$

Condition 2) of Lemma 3 implies that

$$\frac{dV_j}{dt}\leq-\varepsilon\dot\sigma^2-\tau\left(\Phi(\sigma)\dot\sigma\right)^2-æa\nu_j|\varphi(\sigma)|\dot\sigma- -æa_0\nu_{0j}|\varphi(\sigma)|\Phi(\sigma)\dot\sigma-\delta\varphi^2(\sigma). \quad (29)$$

By condition 3) of the lemma we have that

$$\frac{dV_j}{dt}\leq 0\quad(j=1,2). \quad (30)$$

So

$$V_j(t)\leq V_j(0)=W(0). \quad (31)$$

Suppose that

$$\sigma(\bar t)=\sigma_0+k\Delta. \quad (32)$$

for a certain $\bar t$. Then

$$V_1(\bar t)=W(\bar t)+ +æ\left(a\int\limits_{\sigma(0)}^{\sigma_0+k\Delta}F_1(\sigma)d\sigma+a_0\int\limits_{\sigma(0)}^{\sigma_0+k\Delta}\Psi_1(\sigma)d\sigma\right)= =W(\bar t)+æ\left(a\int\limits_{\sigma(0)}^{\sigma_0}F_1(\sigma)d\sigma+a_0\int\limits_{\sigma(0)}^{\sigma_0}\Psi_1(\sigma)d\sigma+ +ak\int\limits_0^{\Delta}F_1(\sigma)d\sigma+a_0k\int\limits_0^{\Delta}\Psi_1(\sigma)d\sigma\right). \quad (33)$$

The following equalities are true

$$\int\limits_{\sigma(0)}^{\sigma_0}F_1(\sigma)d\sigma+k\int\limits_0^{\Delta}F_1(\sigma)d\sigma=\int\limits_{\sigma(0)}^{\sigma_0}\varphi(\sigma)d\sigma+ k\int\limits_0^{\Delta}\varphi(\sigma)d\sigma-\nu_1(\int\limits_{\sigma(0)}^{\sigma_0}|\varphi(\sigma)|d\sigma+k\int\limits_0^{\Delta}|\varphi(\sigma)|d\sigma)= =æ^{-1}(W(0)+\varepsilon_0). \quad (34)$$

Similarly

$$\int_{\sigma(0)}^{\sigma_0} \Psi_1(\sigma)d\sigma + k\int_0^{\Delta} \Psi_1(\sigma)d\sigma = æ^{-1}(W(0)+\varepsilon_0). \tag{35}$$

It follows from (34) and (35) that

$$V_1(\bar t) = W(\bar t) + W(0) + \varepsilon_0. \tag{36}$$

Since $\varphi(\sigma(\bar t)) = \varphi(\sigma_0 + k\Delta) = 0$ we conclude from condition 1) of the lemma that $W(\bar t) \geq 0$, and

$$V_1(\bar t) > W(0), \tag{37}$$

which contradicts with (31). So our assumption is wrong and for all $t$

$$\sigma(\bar t) < \sigma_0 + k\Delta. \tag{38}$$

Suppose now that

$$\sigma(\bar t) = \sigma_0 - (k+1)\Delta. \tag{39}$$

Then

$$\begin{aligned} &V_2(\bar t) = W(\bar t) - \\ &-æ\left(a\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} F_2(\sigma)d\sigma + a\int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma + \right.\\ &\left.+a_0\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \Psi_2(\sigma))d\sigma + a_0\int_{\sigma_0-\Delta}^{\sigma(0)} \Psi_2(\sigma))d\sigma\right). \end{aligned} \tag{40}$$

It is easy to see that

$$\begin{aligned} &\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} F_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma = \\ &= k\int_0^{\Delta} F_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} F_2(\sigma)d\sigma = \\ &= -\frac{1}{æ}(W(0)+\varepsilon_0), \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\int_{\sigma_0-(k+1)\Delta}^{\sigma_0-\Delta} \Psi_2(\sigma)d\sigma + \int_{\sigma_0-\Delta}^{\sigma(0)} \Psi_2(\sigma)d\sigma = \\ &= -\frac{1}{æ}(W(0)+\varepsilon_0). \end{aligned} \tag{42}$$

As a result we obtain that

$$V_2(\bar t) = W(\bar t) + W(0) + \varepsilon_0 > W(0), \tag{43}$$

which contradicts with (31). So for all $t \geq 0$

$$\sigma_0 - (k+1)\Delta < \sigma(t). \tag{44}$$

From (38) and (44) we conclude that for all $t \geq 0$

$$\sigma_0 - \sigma(0) - (k+1)\Delta < \sigma(t) - \sigma(0) < \sigma_0 - \sigma(0) + k\Delta, \tag{45}$$

whence

$$-(k+1)\Delta < \sigma(t) - \sigma(0) < (k+1)\Delta. \tag{46}$$

Thus Lemma 3 is proved.

**Lemma 4.** Suppose there exist such numbers $k \in \mathbf{N}$, $a \in [0,1]$, $\varepsilon$, $\delta$, $\tau > 0$, $æ \neq 0$ and such functions $\sigma(t), W(t) \in C^1[0,\infty)$ that the conditions 1) and 2) of Lemma 3 are true. Suppose also that $\sigma(0) = \sigma_0$, where $\varphi(\sigma_0) = 0$, and matrices $T_j(W(0))$ $(j = 1,2)$ with $a_0 + a = 1$ are positive definite. Then for all $t \geq 0$ the estimate

$$|\sigma(t) - \sigma(0)| < k\Delta \tag{47}$$

is true.

*Proof.* Note that for $\sigma(0) = \sigma_0$ we have

$$\nu_1(k, æ, x) = r_1(k, æ, x);$$

$$\nu_{01}(k, æ, x) = r_{01}(k, æ, x)$$

and

$$\nu_2(k-1, æ, x) = r_2(k, æ, x);$$

$$\nu_{02}(k-1, æ, x) = r_{02}(k, æ, x)$$

We shall use the proof of Lemma 3, though it is necessary to make certain changes in it. We shall exploit the functions $F_j(\sigma)$ and $\Psi_j(\sigma)$ $(j = 1,2)$ with

$$\nu_1 = \nu_1(k, æ, W(0) + \varepsilon_0);$$

$$\nu_{01} = \nu_{01}(k, æ, W(0) + \varepsilon_0).$$

and

$$\nu_2 = \nu_2(k-1, æ, W(0)+\varepsilon_0);$$

$$\nu_{02} = \nu_{02}(k-1, æ, W(0)+\varepsilon_0).$$

The Lyapunov-type functions $V_j(t)\quad (j = 1, 2)$ are also borrowed from Lemma 3.

For all these functions the proof of Lemma 3 can be repeated up to the estimate (38). In order to complete the proof we suppose that

$$\sigma(\bar{t}) = \sigma_0 - k\Delta. \tag{48}$$

for a certain $\bar{t} > 0$. Then

$$\begin{gathered} V_2(\bar{t}) = W(\bar{t}) - æ\left(a\int\limits_{\sigma_0-k\Delta}^{\sigma_0} F_2(\sigma)d\sigma + \right. \\ \left. + a_0\int\limits_{\sigma_0-k\Delta}^{\sigma_0}\Psi_2(\sigma))d\sigma\right) = \\ = W(\bar{t}) - æk\left(a\int\limits_0^\Delta F_2(\sigma)d\sigma + a_0\int\limits_0^\Delta \Psi_2(\sigma)d\sigma\right) = \\ = W(\bar{t}) + W(0) + \varepsilon_0 > W(0), \end{gathered} \tag{49}$$

which contradicts with (31). So for all $t \geq 0$

$$\sigma_0 - k\Delta < \sigma(t)$$

and thus Lemma 4 is proved.

**Lemma 5.** Let $\sigma(0) \in (\sigma_0 - \Delta, \sigma_0)$, where $\varphi(\sigma_0) = 0$. Suppose there exist such numbers $k \in \mathbf{N}$, $\varepsilon$, $\delta$, $\tau > 0$, $æ \neq 0$ and such functions $\sigma(t) \in C^1(0,\infty)$ and $W(t) \in C^1(0,\infty)$ that the conditions 1) and 2) of Lemma 3 are fulfilled and

$$4\delta > æ^2(\nu_{1j}(k, æ, \varepsilon, \tau, W(0))^2 \quad (j = 1, 2). \tag{50}$$

Then the estimate (27) holds for all $t \geq 0$.

*Proof.* The proof is alike that of Lemma 3. It is true that

$$4\delta > æ^2(\nu_{1j}(k, æ, \varepsilon, \tau, W(0)+\varepsilon_0))^2 \tag{51}$$

for a positive $\varepsilon_0$, small enough. We introduce functions

$$Z_j(\sigma) = \varphi(\sigma) - \nu_{1j}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma) \quad (j = 1, 2),$$

where $\nu_{1j} = \nu_{1j}(k, æ, \varepsilon, \tau, W(0)+\varepsilon_0)$ and consider Lyapunov-type functions

$$V_j(t) = W(t) + æ\int_{\sigma(0)}^{\sigma(t)} Z_j(\sigma)d\sigma.$$

Their derivatives are as follows

$$\frac{dV_j(t)}{dt} = \frac{dW(t)}{dt} + æZ_j(\sigma(t))\dot{\sigma}(t), \tag{52}$$

and in virtue of condition 2) of Lemma 3 we have

$$\begin{gathered} \frac{dV_j}{dt} \leq -\varepsilon\dot{\sigma}^2 - \delta\varphi^2(\sigma) - \tau(\Phi(\sigma)\dot{\sigma})^2 - \\ -æ\nu_{1j}P(\varepsilon,\tau,\sigma)\left|\varphi(\sigma)\right|\dot{\sigma} = \\ = -(P(\varepsilon,\tau,\sigma)\dot{\sigma})^2 - \delta|\varphi(\sigma)|^2 - \\ -æ\nu_{1j}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)\dot{\sigma}. \end{gathered} \tag{53}$$

It follows from (51) that

$$\frac{dV_j(t)}{dt} \leq 0 \quad (j = 1, 2), \tag{54}$$

and consequently for all $t \geq 0$

$$V_j(t) \leq W(0) \quad (j = 1, 2). \tag{55}$$

Suppose that

$$\sigma(\bar{t}) = \sigma_0 + k\Delta. \tag{56}$$

Then

$$\begin{gathered} V_1(\bar{t}) = W(\bar{t}) + æ\int\limits_{\sigma(0)}^{\sigma_0+k\Delta} Z_1(\sigma)d\sigma = \\ W(\bar{t}) + æ(\int\limits_{\sigma(0)}^{\sigma_0}\varphi(\sigma)d\sigma + k\int\limits_0^\Delta \varphi(\sigma)d\sigma) - \\ -æ\nu_{11}(\int\limits_{\sigma(0)}^{\sigma_0}|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma + \\ k\int\limits_0^\Delta|\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma) = W(\bar{t}) + W(0) + \varepsilon_0 \end{gathered} \tag{57}$$

So

$$V_1(\bar{t}) > W(0). \tag{58}$$

Similarly, if

$$\sigma(\bar{t}) = \sigma_0 - (k+1)\Delta \tag{59}$$

then

$$V_2(\bar{t}) > W(0). \tag{60}$$

Both (58) and (60) contradict with (55). Consequently

$$\sigma_0 - (k+1)\Delta < \sigma(t) < \sigma_0 + k\Delta \tag{61}$$

for all $t > 0$. It follows that estimate (27) holds for all $t > 0$.

Lemma 5 is proved.

**Lemma 6.** Suppose there exist such numbers $k \in \mathbf{N}$, $\varepsilon$, $\delta$, $\tau > 0$, æ $\neq 0$ and such functions $\sigma(t) \in C^1(0, \infty)$ and $W(t) \in C^1(0, \infty)$ that the conditions 1), 2) of Lemma 3 are fulfilled. Suppose also that $\sigma(0) = \sigma_0$, where $\varphi(\sigma_0) = 0$, and

$$4\delta > \text{æ}^2 (r_{1j}(k, \text{æ}, \varepsilon, \tau, W(0))^2 \quad (j = 1, 2) \tag{62}$$

Then for all $t \geq 0$ the estimate

$$|\sigma(t) - \sigma(0)| < k\Delta \tag{63}$$

is true.

*Proof.* Note that for $\sigma(0) = \sigma_0$ we have

$$\nu_{11}(k, \text{æ}, \varepsilon, \tau, x) = r_{11}(k, \text{æ}, \varepsilon, \tau, x); \tag{64}$$

$$\nu_{12}(k-1, \text{æ}, \varepsilon, \tau, x) = r_{12}(k, \text{æ}, \varepsilon, \tau, x). \tag{65}$$

So it follows from inequality (62) that

$$4\delta > \text{æ}^2 (\nu_{11}(k, \text{æ}, \varepsilon, \tau, W(0) + \varepsilon_0)^2; \tag{66}$$

$$4\delta > \text{æ}^2 (\nu_{12}(k-1, \text{æ}, \varepsilon, \tau, W(0) + \varepsilon_0)^2 \tag{67}$$

for positive $\varepsilon_0$ small enough. We shall use the proof of Lemma 5. We shall exploit functions $Z_j(\sigma)$ with

$$\nu_{11} = \nu_{11}(k, \text{æ}, \varepsilon, \tau, W(0) + \varepsilon_0), \tag{68}$$

$$\nu_{12} = \nu_{12}(k-1, \text{æ}, \varepsilon, \tau, W(0) + \varepsilon_0). \tag{69}$$

All the argument of the proof of Lemma 5 up to the estimate (58) remains valid So we shall use all its formulas and conclusions. In particular (58) contradicts with (55) and so the assumption (56) that

$$\sigma(\bar{t}) = \sigma_0 + k\Delta \tag{70}$$

for a certain $\bar{t} > 0$ is wrong. Then we suppose that

$$\sigma(\bar{t}) = \sigma_0 - k\Delta \tag{71}$$

for a certain $\bar{t} > 0$. It follows from (71) that

$$\begin{gathered} V_2(\bar{t}) = W(\bar{t}) + \text{æ} \int\limits_{\sigma_0}^{\sigma_0 - k\Delta} Z_2(\sigma) d\sigma = \\ = W(\bar{t}) + W(0) + \varepsilon_0 > W(0). \end{gathered} \tag{72}$$

Inequality (72) contradicts with (55). So the assumption (71) is wrong. Consequently

$$|\sigma(t) - \sigma(0)| < k\Delta \tag{73}$$

and Lemma 6 is proved.

Consider now a solution $(z(t), \sigma(t))$ of system (3) with initial data $(z(0), \sigma(0))$ and corresponding solution $y(t)$ of system (19) with $y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$.

**Theorem 3.** Let $\sigma(0) \in (\sigma_0 - \Delta, \sigma_0)$, where $\varphi(\sigma_0) = 0$. Suppose there exist such numbers $\varepsilon$, $\delta$, $\tau > 0$, $k \in \mathbf{N}$, $a \in [0, 1]$ and æ $\neq 0$ that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the inequality (20) is true;

2) matrices $U_j$ $(y^*(0)Hy(0))$ $(j = 1, 2)$ are positive definite for a certain matrix $H = H^*$ satisfying (21).

Then for solution of (3) with initial data $(z(0), \sigma(0))$ the estimate

$$|\sigma(t) - \sigma(0)| < (k+1)\Delta. \tag{74}$$

is true for all $t \geq 0$.

*Proof.* The proof is based on Lemma 3. Let $\{z(t), \sigma(t)\}$ with initial data $(z(0), \sigma(0))$ be a solution of (3) and $W(t) = y^*(t)Hy(t)$, where $y(t)$ is a solution of (19) with $y(0) = \left\| \begin{matrix} z(0) \\ \varphi(\sigma(0)) \end{matrix} \right\|$. Repeating the argument of the proof of Theorem 2 we can demonstrate that condition 1) of the theorem provide that condition 2) of Lemma 3 is fulfilled. Condition 2) of the theorem coincides with condition 3) of Lemma 3. It remains only to show that condition 1) of Lemma 3 is fulfilled as well. Let

$$H = \left\| \begin{matrix} H_0 & h \\ h^* & \alpha \end{matrix} \right\|,$$

where $H_0 = H_0^*$, is an $m \times m$-matrix, $h$ is an $m$-vector, $\alpha$ is a number. It follows from (21) that

$$G(y, 0) \leq 0, \quad y \in \mathbf{R}^{m+1}. \tag{75}$$

So for $y \in \mathbf{R}^{m+1}$ it is true that

$$2y^*HQy + æy^*LDy + (\varepsilon + \tau)|y^*D|^2 + \delta|y^*L|^2 \leq 0. \tag{76}$$

Let $\bar{y}^* = (z^*, 0)$, $z \in \mathbf{R}^m$. Then $\bar{y}^*L = 0$, $D^*\bar{y} = c^*z$, $Q\bar{y} = \left\| \begin{matrix} Az \\ 0 \end{matrix} \right\|$, $\bar{y}^*HQ\bar{y} = z^*H_0Az$. So for $z \in \mathbf{R}^m$ we have

$$2z^*H_0Az + (\varepsilon + \tau)|c^*z|^2 \leq 0. \tag{77}$$

Then since $(A, c)$ is observable and $A$ is a Hurwitz matrix it follows from (77) that $H_0$ is positive definite [Yakubovich, 1973]. For all $\tilde{\tau}$ such that $\varphi(\sigma(\tilde{\tau})) = 0$ we have

$$W(\tilde{\tau}) = y^*(\tilde{\tau})Hy(\tilde{\tau}) = z^*(\tilde{\tau})H_0z(\tilde{\tau}) \geq 0. \tag{78}$$

Theorem 3 is proved.

**Theorem 4.** Suppose there exist such numbers $\varepsilon$, $\delta$, $\tau > 0$, $k \in \mathbf{N}$, $a \in [0, 1]$ and $æ \neq 0$ that for all $\omega \geq 0$ the inequality (20) is true. Suppose also what that $\sigma(0) = \sigma_0$, where $\varphi(\sigma_0) = 0$, and matrices $T_j\ (y^*(0)Hy(0)))\ (j = 1, 2)$ with $a_0 + a = 1$ are positive definite for a certain matrix $H = H^*$, satisfying (21). Then for all $t \geq 0$ the estimate

$$|\sigma(t) - \sigma(0)| < k\Delta \tag{79}$$

is true.

The proof is based on Lemma 4. It is alike that of Theorem 3. The function $W(t)$ is the same. For particular case of $\sigma(0) = \sigma_0$ matrices $U_j$ transform to matrices $T_j$.

**Theorem 5.** Let $\sigma(0) \in (\sigma_0 - \Delta, \sigma_0)$, where $\varphi(\sigma_0) = 0$. Suppose there exist such numbers $\varepsilon$, $\delta$, $\tau > 0$, $k \in \mathbf{N}$, and $æ \neq 0$, that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the inequality (20) is true;

2)

$$4\delta > æ^2(\nu_{1j}(k, æ, \varepsilon, \tau, y(0)^*Hy(0))^2 \quad (j = 1, 2), \tag{80}$$

where $H = H^*$ satisfies (21).

Then for solution of (3) with initial data $(z(0), \sigma(0))$ the estimate (74) is true for all $t \geq 0$.

The proof of Theorem 5 is based on Lemma 5. It is alike the proof of Theorem 3. We exploit the same function $W(t)$. Then conditions 1) and 2) of the theorem provide that condition 2) of Lemma 3 and the estimate (50) are fulfilled. The fact that the condition 1) of Lemma 3 is true, is proved in Theorem 3.

**Theorem 6.** Suppose there exist such numbers $\varepsilon$, $\delta$, $\tau > 0$, $k \in \mathbf{N}$, and $æ \neq 0$, that for all $\omega \geq 0$ the inequality (20) is true. Suppose also what that $\sigma(0) = \sigma_0$, where $\varphi(\sigma_0) = 0$, and

$$4\delta > æ^2(r_{1j}(k, æ, \varepsilon, \tau, y(0)^*Hy(0))^2 \quad (j = 1, 2), \tag{81}$$

where $H = H^*$ satisfies (21). Then for solution of (3) with initial data $(z(0), \sigma(0))$ the estimate (79) is true for all $t \geq 0$.

The proof is based on Lemma 6.

## 4 Example

Let us consider a phase-locked loop with a proportionally integrating filter and a sine-shaped characteristic of phase detector. Its mathematical description is as follows:

$$\left.\begin{aligned} &\frac{dz}{dt} = -\frac{1}{T}z - (1 - s)\varphi(\sigma), \\ &\frac{d\sigma}{dt} = z - Ts\varphi(\sigma), \\ &\varphi(\sigma) = sin(\sigma) - \beta \quad (T, \beta > 0; s \in (0; 1)). \end{aligned}\right\} \tag{82}$$

We shall restrict ourselves to the solutions with $\varphi(\sigma(0)) = 0$ and apply to the system (82) Theorem 4. Let $a = 1, a_0 = 0, æ = 1, \alpha_2 = -\alpha_1 = 1$. Then the frequency inequality (20) takes the form

$$\tau\omega^2 + Re\{æK(i\omega)\} - (\varepsilon + \tau)|K(i\omega)|^2 - \delta \geq 0. \tag{83}$$

with

$$K(p) = T\frac{Tsp + 1}{Tp + 1}.$$

For any solution with $\varphi(\sigma(0)) = 0$ the algebraic condition of Theorem 4 has the form

$$2\sqrt{\varepsilon\delta} > \frac{2\pi\beta + h_{11}z^2(0)k^{-1}}{4(\beta \arcsin(\beta) + \sqrt{1 - \beta^2})}, \tag{84}$$

where $k$ is a number of slipped cycles, matrix $H = ||h_{rj}||$. (When $k$ goes to the infinity the inequalities (83) and (84) become the conditions of gradient-like behavior).

If $T$ is small enough ($T < 0.1$) the parameters $\tau, \varepsilon, \delta$ and $h_{11}$ can be chosen in such a way that the inequality

$$1 > \frac{2\pi\beta + 0.25z^2(0)(1 - s)^{-2}k^{-1}}{4(\beta \arcsin(\beta) + \sqrt{1 - \beta^2})}. \tag{85}$$

guarantees that (83) and (84) are fulfilled. Let us compare this result with the result of the paper [Yershova, and Leonov, 1983], where the case of $z(0) = T\beta$, $s = 0.4$ is considered. For $\beta = 0.8$ the theorem proved in [Yershova, and Leonov, 1983] does not guarantee the gradient-like behavior. Meanwhile from the condition (85) we receive that for $\beta = 0.8$ and $T < 0.1$ the gradient-like behavior takes place and $k = 1$.

## 5 Conclusion

The paper is devoted to the problem of cycle-slipping for multidimensional systems of indirect control with periodic nonlinear input (phase-controlled systems). The case of differentiable nonlinearities is considered. The problem is investigated with the help of Lyapunov direct method and Yakubovich-Kalman-Popov lemma. In the paper various types of periodic Lyapunov functions are used. As a result a series of multiparametric frequency-algebraic estimates for the number of slipped cycles of the output of the system is established. The estimates depend on the parameters and initial data of nonlinear phase-controlled system. So by choosing appropriate values of initial data and parameters we can manage the transient mode of the system. In particular we can guarantee the small number of cycles slipped which is essential for mechanical, communication and electric engineering systems.

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