SYNCHRONIZATION IN COUPLED ORGAN PIPES

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Abstract

We investigate synchronization in coupled organ pipes. Synchronization and reflection in the organ lead in special cases to undesired weakening of the sound. Experiments show that sound interaction is highly complex and nonlinear. As a model we consider two delay-coupled Van-der-Pol oscillators with distance-dependent coupling. Analytically, we investigate the synchronization frequency and bifurcation scenarios which occur at the boundaries of the Arnold tongues. We successfully compare our results to experimental data.

Key words

Coupled oscillators; Synchronization; Bifurcation

1 Introduction

The physics of organ pipes is an interdisciplinary topic where many fields of science meet. It includes elements of nonlinear models [Bader, 2013], aeroacoustic modeling [Howe, 2003] and synchronization theory [Pikovsky, 2001]. The focus of these different research areas is the “queen of instruments” which captivates through the grandeur of her sight and majesty of her sound. One interesting nonlinear effect is that organ pipes close to each other synchronize. In recent years this effect has been studied in experiments and theory [Abel, 2006; Fischer, 2014]. One cannot necessarily assume that synchronization effects are a desirable or an undesirable phenomenon. If synchronization yields a stable state for the pitch of special organ pipes, it is more desired than an amplitude minimum for the first harmonic, i.e. the pipes may weaken the sound, or a negative interaction between pipes during the actuating of the swell box, where the pipes stand close to each other. Sound generation in organ pipes can be described as a generator-resonator coupling. It is common to use the representation of an oscillating air sheet at the pipe mouth to describe the generation of the pipe sound. The oscillations of the jet exiting from the flue – the air sheet – is controlled by the airflow. Experimental and numerical investigations from the group of Abel [Abel, 2009] yield the conclusion that it is a justifiable approximation to compare the oscillating air sheet to a Van-der-Pol oscillator. Hence in this paper we investigate the bifurcation scenarios in the context of two delay-coupled Van-der-Pol oscillators which are supposed to represent two coupled organ pipes such as in the experimental setup from Bergweiler [Bergweiler, 2006]. In extension of previous work, we study Arnold tongues for the model parameters, in particular the time delay τ and the coupling strength κ, to find out how unwanted synchronization or chaotic behavior can be avoided.

2 A model of coupled organ pipes

To get a deeper insight into the synchronization phenomena of two coupled organ pipes we replace the pipes with Van-der-Pol oscillators using a model of direct coupling. In our numerical analysis we simulate a system in the following form:

$$\ddot{x}_i + \omega^2 x_i - \mu f_i(\mathbf{x}, \dot{x}_i) = 0, \quad i = 1, 2, \qquad (1)$$

where $\mathbf{x} = (x_1, x_2)$. These equations represent a harmonic oscillator with an intrinsic angular frequency ω, plus a nonlinear term $f_i(\mathbf{x}, \dot{x}_i)$, where the strength of the nonlinearity is measured by $\mu > 0$. In our case the nonlinear function is described by

$$\begin{aligned} f_1(\mathbf{x}, \dot{x}_i) &= (1 - \gamma x_1^2)\dot{x}_1 + \kappa x_2(t - \tau) - \Delta x_1, \\ f_2(\mathbf{x}, \dot{x}_i) &= (1 - \gamma x_2^2)\dot{x}_2 + \kappa x_1(t - \tau), \end{aligned} \qquad (2)$$

where $\Delta \in \mathbb{R}$ denotes the detuning between the two oscillators.

In Fig. 1 we can see the phenomenon of frequency locking in a numerical simulation. The picture shows the observed frequency ν versus the detuning Δ of two Van-der-Pol oscillators. A sharp transition to synchronization is observed in the synchronization region.

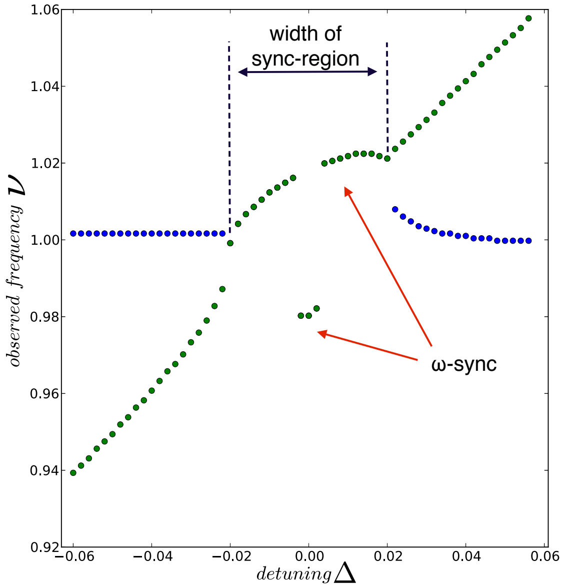

Figure 1. The plot shows the observed frequency ν of oscillator x_1 (blue symbols), and oscillator x_2 (green symbols) versus the detuning of the uncoupled oscillators Δ. Our purpose is to analyze the width of the synchronization region, the phase difference in the synchronization state, the bifurcation scenarios which occur at the boundaries and the synchronization frequency itself. In this plot the delay time is $\tau = 0.1\pi$, $\gamma = 1$, $\mu = 0.1$, $\omega = 1$, and the coupling strength is $\kappa = \frac{2}{5}$.

3 Analytic approaches

We try to find an analytic solution for the coupled system by a perturbation method analyzing the width of the synchronization region, the phase difference in the synchronization state, the bifurcation scenarios which occur at the boundaries. The method of averaging describes weakly nonlinear oscillations in terms of slowly varying amplitude and phase, representing the solution in the ideal form for phase model reduction.
For $\mu = 0$ the system reduces to $\ddot{x}_i + \omega^2 x_i = 0$ with solution

$$x_i = R_i \sin(\omega t + \phi_i). \tag{3}$$

For $1 \gg \mu > 0$ we look for a solution in the form Eq. (3) but assume that R_i and ϕ_i are time dependent, nonnegative functions:

$$x_i = R_i(t)\sin(\omega t + \phi_i(t)), \tag{4}$$
$$\dot{x}_i = R_i(t)\omega\cos(\omega t + \phi_i(t)). \tag{5}$$

Without loss of generality, we choose $\omega = 1$. For small μ we use the method of averaging, assuming that the product $\mu\tau$ is small, Taylor expand $R_i(t-\tau)$ and $\phi_i(t-\tau)$ in the following way:

$$R_i(t-\tau) = R_i(t) - \tau\dot{R}_i(t) + \frac{\tau^2}{2}\ddot{R}_i(t) + \dots, \tag{6}$$

We introduce the phase difference $\psi(t) = \phi_1(t) - \phi_2(t)$. Defining a new time scale $\tilde{t} = \frac{2t}{\mu}$ we formulate the slow equations which describe the system of two delay-coupled Van-der-Pol oscillators:

$$\begin{aligned}\dot{R}_{1/2}(\tilde{t}) = \; & R_{1/2}(\tilde{t})\left(1 - \frac{\gamma R_{1/2}(\tilde{t})^2}{4}\right)\\ & \mp\kappa R_{2/1}(\tilde{t})\sin(\psi(\tilde{t}) + \tau),\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{\psi}(\tilde{t}) = -\Delta + \kappa & \left[\frac{R_1(\tilde{t})}{R_2(\tilde{t})}\cos(\psi(\tilde{t}) - \tau)\right.\\ & \left. - \frac{R_2(\tilde{t})}{R_1(\tilde{t})}\cos(\psi(\tilde{t}) + \tau)\right].\end{aligned} \tag{8}$$

This method of averaging together with truncation of Taylor expansions reduces an infinite dimensional problem in functional analysis to a finite dimensional problem by assuming that the product $\mu\tau$ is small. This key step enables us to handle the original system, a delay differential equation, as a system of ordinary differential equations [Wirkus, 2002]. With Eqs. (7) we now have two dynamical equations for the amplitudes R_i of the two oscillators and one equation – Eq. (8) is called the generalized Adler equation – for the phase difference $\psi(t)$ which is also called slow phase.

4 Generalized Adler equation

The synchronization state of our system of two delay-coupled Van-der-Pol oscillators representing coupled organ pipes corresponds to frequency locking. In the case of slow-phase equations equilibria correspond to motions which are phase and frequency locked. To investigate the stability of the fixed points we take a further look at the generalized Adler equation (8):

$$\dot{\psi}(t) = -\Delta + \kappa q(\psi(t)) \tag{9}$$

where the averaged forcing $q(\psi(t))$ is the 2π-periodic function

$$q(\psi(t)) = \frac{R_1(t)}{R_2(t)}\cos(\psi(t) - \tau) - \frac{R_2(t)}{R_1(t)}\cos(\psi(t) + \tau). \tag{10}$$

The Adler equation is important for the bifurcation analysis. The next step is to eliminate the amplitudes $R_i(t)$ from Eq. (10). Therefore we express $R_2(t)$ by $R_1(t)$ in the case of an equilibrium point ($\dot{R}_i(t) = 0$). We achieve two relevant solutions for the amplitude. Inserting the values of R_i in Eq. (10) – according to the numerical results – we can plot the Adler equation in Fig. 2. The maximum and minimum in Fig. 2 correspond to the point where at the bifurcation point $\Delta = \Delta_{bif}$ an unstable fixed point exists. A change

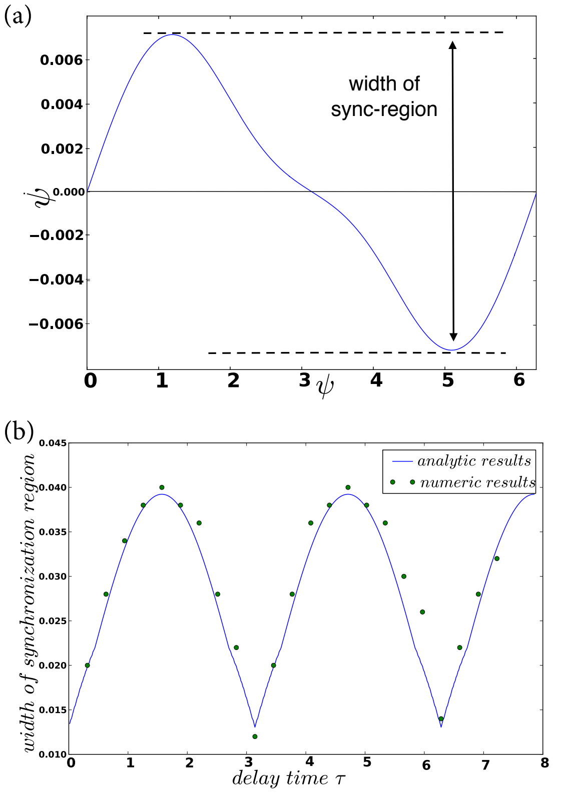

Figure 2. (a) The right-hand side of the Adler equation (9) versus the slow phase ψ. The difference between the maxima of the Adler equation gives us the border of the synchronization region. The frequency detuning is $\Delta = 0, \tau = 0.1\pi, \kappa = \frac{1}{5}, \gamma = 1, \omega = 1,$ and $\mu = 0.1$. (b) Analytic and numeric results of the dependence of the width of the synchronization region on the delay time τ for $\kappa = \frac{2}{5}, \gamma = 1, \omega = 1,$ and $\mu = 0.1$.

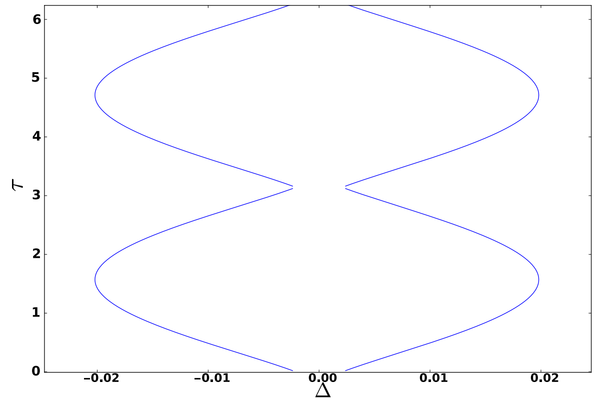

Figure 3. The synchronization region in the parameter plane of delay time τ and frequency detuning Δ. The plot shows an analytically computed Arnold tongue for the coupling strength $\kappa = \frac{1}{5}, \gamma = 1,$ $\omega = 1,$ and $\mu = 0.1$. One can see the symmetric, π-periodic boundaries of the tongue.

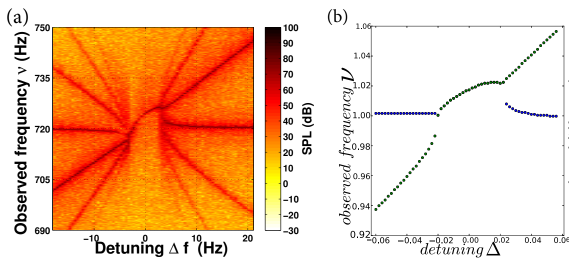

Figure 4. Comparison of (a) the experimental synchronization region [Fischer, 2014] and (b) the numerical one for $\tau = 1.1\pi$, $\kappa = \frac{2}{5}, \gamma = 1, \omega = 1,$ and $\mu = 0.1$.

of Δ shifts, according to Eq. (9), the curve in the y-direction but does not change its shape. Thus, the maximum and minimum correspond to the border of the synchronization regions when varying Δ as we can see in Fig. 2. In this way we can calculate the width of the synchronization region and compare these analytic results to numerical ones (see Fig. 2b). The accordance between the results is remarkable, even if we have an unavoidable deviation because of the limit of computational power during the numerical simulations. The transient times are very large at bifurcation points.

5 Bifurcation analysis

One important theoretical question is the transition to synchronization, usually characterized in the parameter plane of frequency detuning Δ and coupling strength κ. In the case of delayed coupling the time delay τ has the same importance as the coupling strength κ. The synchronization region in the (κ, Δ) or (τ, Δ) plane is generally called Arnold tongue, and it is one of the main characteristics of synchronizing nonlinear systems. We analytically calculate the synchronization region in the plane of the coupling strength κ and the detuning Δ. The delay time τ is kept constant but does not vanish.

The nonlinear boundaries of the Arnold tongue like in Fig. 5 (b) is a remarkable result. Already for a small delay time τ this behavior can be observed. Also the analytical calculation of the synchronization region in the plane of the delay time τ and the detuning Δ gives excellent agreement with numeric results. The coupling strength κ is kept constant but does not vanish. Instead of a monotonic nonlinear increase we have now a π-periodic behavior at the boundaries of the tongue.

6 Comparison with acoustic experiments

A comparison of a complex experiment with a simple oscillator model is an ambitious endeavor: On the one side there is a pipe with a whole spectrum of overtones and a complicated aeroacoustic behavior, and on the other side a Van-der-Pol oscillator. Nevertheless a simple model can provide a deeper comprehension of a complex system. We compare the synchronization region in Fig. 4. There are many similarities between the plots. Especially the behavior at the transition region between frequency detuning and locking is remarkable, as well as the convex curvature of the synchronization regions itself. In Fig. 5 we compare an experimental and an analytically calculated Arnold tongue. Exper-

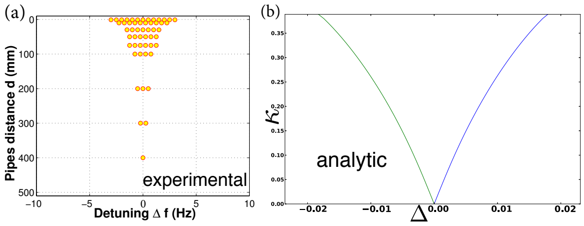

Figure 5. Arnold tongue in the plane of coupling strength vs. detuning: (a) experiment [Fischer, 2014] (b) analytic result for the delay time $\tau = \frac{1}{10}\pi, \gamma = 1, \omega = 1$, and $\mu = 0.1$.

imentally, the coupling strength is determined by the distance of the two organ pipes. The most considerable similarity is the nonlinear behavior of their boundaries. We are able to change the curvature of the boundaries by introducing a coupling strength function in the form $\kappa_{const} \rightarrow \kappa(\tau)$.

7 Conclusion

In this paper we have investigated the synchronization of organ pipes in the light of nonlinear dynamics. We have motivated the simplifying approach to represent organ pipes as nonlinear coupled oscillators. The distance between the pipes is reflected by a delayed coupling term in the equations of motion. We have studied bifurcation scenarios in this system of delay-coupled oscillators. In particular, we have considered two Van-der-Pol oscillators which interact by using a direct delayed coupling. The results of numerical investigations agree well with the experiments. Subsequently, we have changed the system parameters, namely the coupling strength κ and the delay time τ.
For a deeper understanding of the various bifurcations we have developed and extended an analytical approach. By the method of averaging we have derived a generalized Adler equation, a nonlinear ordinary first-order differential equation, which allows us to study the stability of fixed points corresponding to a frequency locking of the oscillators. This approach has offered us the possibility of a wide bifurcation analysis. In general we can see an excellent agreement of our analytic results with the numerical simulations.
A detailed bifurcation analysis has confirmed the existence of synchronization. Investigations about the Arnold tongues have shown that the behavior at their boundaries depends on the interplay of the coupling strength κ and the delay time τ. In general, it is non-linear in κ and τ which is also clearly confirmed by experimental data.

Acknowledgements

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