

# ON THE PROPERTIES OF AN AUTONOMOUS SYSTEM OF DIFFERENTIAL EQUATIONS

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## Abstract

This paper examines the shift operator of an autonomous system of ordinary differential equations. It is shown that the scalar product of the velocity field of the system under study and the normal of the boundary of the shift domain is constant along the corresponding trajectory. Positively invariant sets and asymptotic stability of absorbing sets are investigated.

## Key words

Dynamic system, shift operator, invariant set, set stability.

## 1 Introduction

Consider an autonomous system of differential equations

$$\frac{dx}{dt} = f(x), \quad (1)$$

where  $x = (x_1, \dots, x_n)^*$  is an  $n$ -dimensional vector, and  $f = (f_1, \dots, f_n)^*$  is an  $n$ -dimensional vector-function that is defined and continuous along with its derivatives  $\frac{\partial f_i(x)}{\partial x_j}$ ,  $i, j = 1, \dots, n \in R^n$ .

We assume that the solution of system (1)

$$x(t) = x(t, t_0, x_0), \quad (2)$$

with the initial condition  $x(t_0) = x_0$  exists and is unique for any  $t_0$  and  $x_0 \in R^n$  and is defined for all  $t \in (-\infty, +\infty)$ .

Solution (2) determines the shift of the point  $x_0$  along the trajectory (2) in time  $t - t_0$  to some point  $x_t = x(t) = x(t, t_0, x_0)$ . The operator  $U(t, t_0)$  of translation of the point  $x_0$  to the point  $x_t$  is called the shift operator along the trajectories of the system (1) and is defined by the equality [Krasnoselskii, 1966]

$$U(t, t_0)x_0 = x(t, t_0, x_0).$$

Let  $S_0$  be a smooth hypersurface in  $R^n$ , given by the equation

$$g(x) = g(x_1, \dots, x_n) = 0,$$

where  $g(x)$  is a scalar continuously differentiable function. As a result of shifting the surface  $S_0$  along the trajectories of system (1), we obtain the hypersurface  $S_t$ :

$$S_t = U(t, t_0)S_0.$$

The gradient  $\nabla g(x_0) = \left(\frac{\partial g(x_0)}{\partial x_0}\right)^*$  is the normal to the surface  $S_0$  at the point  $x_0 \in S_0$ . We will also denote this normal by  $\left(\frac{\partial S_0(x_0)}{\partial x_0}\right)^*$ . In what follows, we will be interested in the normal  $\left(\frac{\partial S_t(x_t)}{\partial x_t}\right)^*$  to the surface  $S_t$  at the point  $x_t = x(t, t_0, x_0)$ .

## 2 On the properties of shift operator

Let us consider the variational equation for system (1)

$$\frac{d\delta x}{dt} = \frac{\partial f(x(t))}{\partial x} \delta x. \quad (3)$$

The adjoint equation has the form

$$\frac{d\Psi}{dt} = - \left( \frac{\partial f(x(t))}{\partial x} \right)^* \Psi. \quad (4)$$

It is known that the scalar product  $\Psi(t)$  and  $\delta x(t)$ , defined by (3) and (4) on the trajectory  $x(t)$  is constant for all  $t$  and arbitrary initial data  $\delta x(t_0)$  and  $\Psi(t_0)$ , i.e.

$$\Psi^*(t)\delta x(t) \equiv \text{const.}$$

From this equality it follows that any hyperplane  $L_0$  with normal  $\Psi_0 = \Psi(t_0)$ , passing through point  $x_0$ , is transferred along the trajectory  $x(t)$  to the hyperplane  $L_t$  with normal  $\Psi(t)$ , passing through point  $x_t$  [Pontryagin, 1969]. Let  $L_0$  be the tangent hyperplane at point  $x_0$  to surface  $S_0$  with normal  $\left(\frac{\partial S_0(x_0)}{\partial x_0}\right)^*$ . The normal to the tangent hyperplane  $L_t$  of surface  $S_t$  at the point  $x_t$  is

$$\left(\frac{\partial S_t(x_t)}{\partial x_t}\right)^* = \left(\frac{\partial S_0(x_0)}{\partial x_0} \frac{\partial x_0}{\partial x_t}\right)^*.$$

It is easy to see that the vector function

$$\Psi(t) = \left(\frac{\partial S_t(x_t)}{\partial x_t}\right)^*,$$

due to the continuous differentiability of solutions (2) with respect to the initial data, satisfies the adjoint equation (4) with the initial condition

$$\Psi(t_0) = \left(\frac{\partial S_0(x_0)}{\partial x_0}\right)^* = \nabla g(x_0) \quad (5)$$

on the trajectory  $x(t) = x(t, t_0, x_0)$ . Thus, we have

**Lemma.** Let  $\Psi(t)$  be a solution of the adjoint equation (4) with the initial condition (5). Then the normal to the surface  $S_t$  at the point  $x_t = x(t, t_0, x_0)$  is  $\Psi(t)$ , i.e.

$$\left(\frac{\partial S_t(x_t)}{\partial x_t}\right)^* = \Psi(t).$$

Let us formulate the theorem, taking into account the given lemma.

**Theorem 1.** Let  $\Psi(t)$  be a solution of equation (4) with initial condition (5). Then the following relation holds:

$$\Psi^*(t)f(x(t, t_0, x_0)) \equiv \text{const}, \quad t \in (-\infty, +\infty), \quad (6)$$

that is, the scalar product of the velocity field of equation (1) at point  $x_t$  and the normal to the boundary of the shift region  $S_t$  at point  $x_t$  is a constant function along the trajectory  $x(t) = x(t, t_0, x_0)$ . The proof is obvious; we need to differentiate relation (6) with respect to  $t$ .

### 3 On invariant and absorbing sets of a dynamical system

Note that, under the assumptions made earlier, system (1) defines a dynamical system in  $R^n$  [Nemytsky, 1949]. We further set  $t_0 = 0$ . We denote

$$\rho(M, S) = \inf_{x \in M, y \in S} \|x - y\|$$

the distance between sets  $M \subset R^n$  and  $S \subset R^n$ . Here  $\|z\|$  is the Euclidean norm of the vector  $z$ .

We say that a trajectory  $x(t)$  tends to a set  $A$  as  $t \rightarrow +\infty$  if for any  $\varepsilon > 0$  there exists  $T > t_0 = 0$  such that

$$\rho(x(t), A) < \varepsilon \text{ for } t > T.$$

A set  $A$  is called an invariant set with respect to the dynamic system (1) if the condition  $x(0) = x_0 \in A$  implies  $x(t) = x(t, x_0) \in A$  for all  $t \in (-\infty, +\infty)$ .

A set  $A$  is said to be positively invariant with respect to system (1) if the condition  $x(0) \in A$  implies  $x(t) \in A$  for all  $t \geq 0$ .

Let  $M_0 \subset R^n$  be a compact set of positive measure. Suppose that the set  $M_0$  has a boundary  $S_0 = \partial M_0$ , which is a smooth surface. The normal to the surface  $S_0$  at the point  $x_0$  is  $\left(\frac{\partial S_0(x_0)}{\partial x_0}\right)^*$ . From now on, we denote the largest invariant set in  $M_0$  by  $A_0$ . Since  $A_0$  is contained in a bounded set and the closure of an invariant set is an invariant set [Nemytsky, 1949],  $A_0$  is a compact set. Note that the set  $A_0$  may have a proper invariant subset.

Let us introduce set

$$M_t = U(t, 0)M_0, \quad S_t = U(t, 0)S_0, \quad t \in (-\infty, +\infty). \quad (7)$$

Here  $M_t$  and  $S_t$  are the shifts of the set  $M_0$  and, accordingly, its boundary  $S_0$  along the trajectories (2) of system (1). Note that the set  $A_0$  is the largest invariant set in  $M_t$ . We fix  $T \in (-\infty, +\infty)$ . Consider the sets

$$\begin{aligned} D_T &= M_T \setminus A_0; \quad B_t = M_T \setminus M_t \text{ for } t > T; \\ B_T &= \bigcup_{t > T} B_t, \\ B &= \bigcup_{t \geq 0} B_t. \end{aligned} \quad (8)$$

**Theorem 2.** Let the relation

$$\frac{\partial S_0(x_0)}{\partial x_0} f(x_0) \leq 0 \text{ hold for } \forall x_0 \in S_0. \quad (9)$$

Then

1. the set  $M_0$  contains a nonempty invariant set  $A_0$ ;
2. the set  $M_t$ , defined by equality (7), has positive measure for any  $t$  and is a compact positively invariant set;
3. the inclusions

$$A_0 \subseteq M_\tau \subseteq M_t \text{ hold for } \forall \tau, t \text{ such that } \tau > t; \quad (10)$$

4. the sets  $B_T$  and  $D_T$  coincide;
5. the distance  $\rho(x(t, x_0), A_0) \rightarrow 0$  as  $t \rightarrow \infty$  for  $\forall x_0 \in M_T$ ;
6. the measure of the set  $M_t$  tends to the measure of the set  $A_0$  as  $t \rightarrow +\infty$ , i.e.

$$\text{mes}(M_t) \rightarrow \text{mes}(A_0) \text{ as } t \rightarrow +\infty. \quad (11)$$

**Proof.** By virtue of condition (9), the set  $M_0$  is a positively invariant set for system (1). Therefore, all solutions to (2) with initial conditions  $x_0 \in M_0$  are positively Lagrange stable and the set of  $\omega$ -limit points of these solutions is not empty [Nemytsky, 1949]. Consequently, the set  $A_0$ , which includes these points, is also not empty. If the largest invariant set  $A_0$  coincides with the set  $M_0$ , then the theorem is obvious. Let  $A_0$  not coincide with  $M_0$ . Under the made assumptions, the solutions to system (1) are continuous and continuously differentiable with respect to the initial data [Pontryagin, 1965]. Therefore, the transformation of the set  $M_0$  into the set  $M_t$ , determined by the trajectories  $x(t) = x(t, x_0)$  of system (1), is continuous and continuously differentiable. Consequently, the compactness of the set  $M_0$  implies the compactness of the set  $M_t$ .

The positivity of the measure  $M_t$  follows from the positivity of the measure  $M_0$  and the equalities

$$\text{mes}(M_t) = \int_{M_t} dx_t = \int_{M_0} \det \left( \frac{\partial x(t, x_0)}{\partial x_0} \right) dx_0,$$

where  $\det \left( \frac{\partial x(t, x_0)}{\partial x_0} \right)$  is the Jacobian of the specified transformation, representable using the Liouville formula [Pontryagin, 1965] in the form

$$\det \left( \frac{\partial x(t, x_0)}{\partial x_0} \right) = \exp \left( \int_0^t \text{tr} \left( \frac{\partial f(x(\tau, x_0))}{\partial x} \right) d\tau \right).$$

Here  $\text{tr} \left( \frac{\partial f}{\partial x} \right)$  is the trace of the Jacobian matrix  $\frac{\partial f}{\partial x}$ . By Theorem 1, taking into account condition (9), we have

$$\frac{\partial S_t(x_t)}{\partial x_t} f(x_t) = \frac{\partial S_0(x_0)}{\partial x_0} f(x_0) \leq 0 \text{ for } \forall x_0 \in S_0. \quad (12)$$

It follows that for  $\forall t \in (-\infty, +\infty)$ , the set  $M_t$  is a compact positively invariant set and inclusions (10) hold.

By construction, the set  $D_T$  includes the set  $B_T$ . Suppose the sets  $B_T$  and  $D_T$  do not coincide, then the set  $D_T \setminus B_T = G$  is not empty. Consider the trajectory  $x(t, x_0)$ ,  $x_0 \in G$ . This trajectory is defined and will remain inside the set  $M_T$  for all  $t \in (-\infty, +\infty)$  and therefore must belong to the set  $A_0$ , as the largest invariant set. We have obtained a contradiction. Consequently, the set  $G$  is empty and  $B_T = D_T$ . In this case, the equalities obviously hold

$$M_T = D_T \cup A_0; A_0 = M_T \setminus D_T = M_T \setminus B_T. \quad (13)$$

Let us show that the distance

$$\rho(x(t, x_0), A_0) \rightarrow 0 \text{ as } t \rightarrow +\infty \text{ for } \forall x_0 \in M_T.$$

Indeed,  $M_T$  is a compact positively invariant set. Therefore, all solutions of (2),  $x_0 \in M_T$ , are positively Lagrange stable and tend to their  $\omega$ -limit points [Nemytsky, 1949], which belong to  $A_0$ . Consequently, all solutions also tend to the invariant set  $A_0$ .

Let us consider the function  $\mu(t) = \text{mes}(M_t)$ . The derivative of the function  $\mu(t)$ , taking into account the formulas of Liouville and Ostrogradsky [Nikolsky, 1973], has the form

$$\frac{d\mu(t)}{dt} = \frac{d}{dt} \int_{M_t} \nabla \cdot f(x_t) dx_t = \int_{S_t} f^T(x_t) n(t, x_t) dS_t. \quad (14)$$

Here  $n(t, x_t)$  is the outward unit normal to the surface  $S_t$  at the point  $x_t$ .

By the lemma, we have

$$n(t, x_t) = \frac{\Psi(t, x_t)}{\|\Psi(t, x_t)\|}. \quad (15)$$

Thus, given the positivity of the measure  $M_0$  and relations (12), (14), (15),  $\mu(t)$  is a positive non-increasing function and, therefore, has a limit as  $t \rightarrow +\infty$ . Furthermore, from (9), it clearly follows that

$$\text{mes}(A_0) \leq \text{mes}(M_\tau) \leq \text{mes}(M_t) \text{ for } t > \tau.$$

Therefore, taking into account equalities (13), we obtain relation (11). The theorem is proved.

A compact set  $M \subset R^n$  of positive measure will be called an absorbing set under the condition that if the trajectories (2) of system (1) for some  $t_0$  are included in the set  $M$ , then they are located there (remain) for all  $t > t_0$ .

A closed invariant set  $A$  is called Lyapunov stable if for any  $\varepsilon > 0$  it is possible to specify a value  $\delta > 0$  such that for  $\rho(x_0, A) < \delta$  there will be  $\rho(x(t, x_0), A) < \varepsilon$  for  $t \geq 0$ . If in this case  $\rho(x(t, x_0), A) \rightarrow 0$  for  $t \rightarrow +\infty$ , then the closed invariant set  $A$  is called asymptotically stable [Zubov, 1957].

**Theorem 3.** Let

$$\frac{\partial S_0(x_0)}{\partial x_0} f(x_0) < 0 \text{ for } \forall x_0 \in S_0. \quad (16)$$

Then

1. the set  $M_t$  is an absorbing set for any  $t \in (-\infty, +\infty)$ ;
2. the function  $\mu(t) = \text{mes}(M_t)$  is a strictly decreasing positive function and relation (11) holds;
3. the distance  $\rho(S_\tau, S_t) \neq 0$  if  $\tau \neq t$ ;  $\rho(S_t, A_0) \rightarrow 0$  as  $t \rightarrow +\infty$ ;
4. the compact invariant set  $A_0$  is asymptotically Lyapunov stable.

**Proof.** Obviously, under condition (16) of Theorem 3, all the statements of Theorem 2 also hold in this case. Therefore, the set  $M_t$  is a compact positively invariant set. Moreover, by Theorem 1 and taking into account (16), the following relations hold:

$$\frac{\partial S_t(x_t)}{\partial x_t} f(x_t) = \frac{\partial S_0(x_0)}{\partial x_0} f(x_0) < 0. \quad (17)$$

Therefore, for any  $\tau, t$  such that  $\tau > t$ ,  $M_\tau \subset M_t$  and the set  $M_t$  is an absorbing set. From formulas (14)–(17) it follows that the function  $\mu(t) = \text{mes}(M_t)$  is a strictly decreasing positive function. The assertion that the distance  $\rho(S_\tau, S_t) \neq 0$  if  $\tau \neq t$ , follows from the continuity of the distance function, the compactness of the sets  $S_\tau, S_t$  and the fact that they have no common points. Moreover,  $\rho(S_t, A_0) \rightarrow 0$  as  $t \rightarrow +\infty$ , since condition (5) of Theorem 2 is satisfied.

According to V. I. Zubov's theorem [Zubov, 1957], in order for a closed invariant set  $A$  with a sufficiently small compact neighborhood to be asymptotically stable in the sense of Lyapunov, it is necessary and sufficient that the following conditions be satisfied:

- a) there is no solution  $x(t) = x(t, x_0)$ ,  $x(0) = x_0 \notin A$  such that  $\rho(x(t, x_0), A) \rightarrow 0$  as  $t \rightarrow -\infty$ ;
- b) there is a sufficiently small neighborhood of the set  $A$  that does not contain entire trajectories of the dynamic system under study.

In this case, the compact set  $A_0$  has a sufficiently small compact neighborhood in  $R^n$ . Moreover, the neighborhood of  $A_0$  does not contain entire trajectories, since  $A_0$  is the largest closed invariant set in  $M_0$ . Condition (a) is satisfied by relations (17). Moreover, by Theorem 2, we have  $\rho(x(t, x_0), A_0) \rightarrow 0$ , as  $t \rightarrow +\infty$ , for  $\forall x_0 \in M_T$ .

Thus, the invariant set  $A_0$  is asymptotically stable, and the set  $B$  is an attraction domain. The theorem is completely proved.

From the proof of Theorem 3, the following statements clearly hold.

**Corollary 1.** Under conditions (9) and (16), the set  $M_0$  contains a nonempty invariant set.

**Corollary 2.** Under condition (16), the set  $M_t$  is an asymptotically stable absorbing set with basin of attraction  $B$ . Moreover, the set  $M_t$  contains an attractor. By an attractor we mean an asymptotically stable compact invariant set.

**Corollary 3.** Let  $f(0) = 0$  and the trivial solution  $x(t) \equiv 0$  of system (1) be the unique solution that always remains in the invariant set  $A_0$ . Then, under conditions (16), the equilibrium position  $x = 0$  is asymptotically Lyapunov stable and the set  $B$  is a basin of attraction.

#### 4 Example

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= -x(x^2 + y^2 - 1)^2, \\ \frac{dy}{dt} &= -y(x^2 + y^2 - 1)^2. \end{aligned}$$

The equilibrium points for this system are the points on the unit circle  $C_1$  and the origin  $O$  (see Figure 1). Any disk centered at the origin with a radius greater or less than unity can be considered an absorbing set  $M_0$  as per Theorem 3; condition (16) holds on its boundary. It

should also be noted that both outside the unit disk and inside it (excluding the center), the system possesses a first integral of the form (allowing for both finite values of  $k$  and  $\pm\infty$ ):

$$\frac{y}{x} = k,$$

which determines the shape of the system's trajectories in the two aforementioned regions of the phase plane (see Figure 1). Figure 1 also shows circles  $C_R$  and  $C_r$  with radii  $R$  and  $r$ , respectively. In the first case, where the boundary  $S_0$  of the absorbing set corresponds to the circle  $C_R$  ( $R > 1$ ), the unit disk corresponds to the largest invariant set  $A_0$  from Theorem 3, and its region of attraction is the entire exterior of the unit disk. In the second case, where the boundary  $S_0$  of the absorbing set corresponds to the circle  $C_r$  ( $0 < r < 1$ ), the largest invariant set  $A_0$  consists of the single point  $O$ , and its region of attraction is the interior of the unit disk.

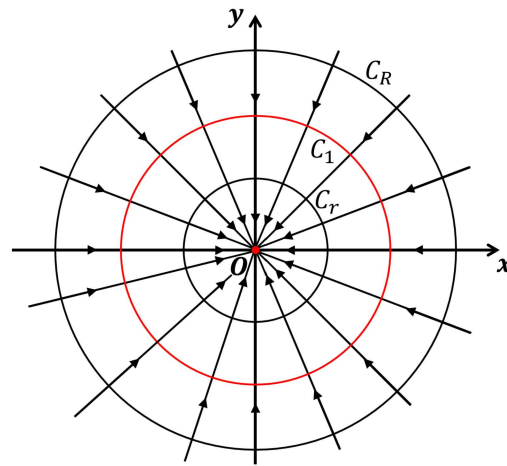


Figure 1. To Theorem 3.

Figure 2 illustrates Theorem 2. The set  $M_0$  from Theorem 2 is the set bounded by the ellipse  $C$ ; condition (9) holds on its boundary. The set  $A_0$  consists of the points within the sector  $ONL$  of the unit circle.

#### 5 Conclusion

This paper investigates the properties of the shift operator. It is shown that the scalar product of the velocity field of an autonomous system of differential equations and the normal to the boundary of the shift domain is constant and, consequently, preserves its sign along the corresponding trajectory. Conditions for the asymptotic stability of the largest invariant set contained in a certain compact set of nonzero measure are given. These conditions differ from the asymptotic stability conditions of E. A. Barbashin, N. N. Krasovsky, and V. I. Zubov [Zubov, 1957, Barbashin, 1967, Krasovsky, 1959]. The construction of the Lyapunov function is

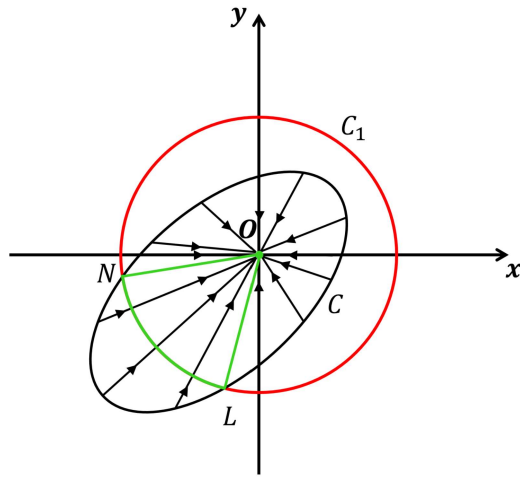


Figure 2. To Theorem 2.

not used here. We also note that in the La Saal theorem, when studying positively invariant sets, the largest invariant set contained among the points where the derivative of the Lyapunov function is zero is considered [Khalil, 2009]. In this paper, we study the largest invariant set contained in the compact set under consideration. It seems interesting to construct the basins of attraction of the invariant set, bounded by hypersurfaces or hyperplanes. The results obtained in this paper can be applied to problems involving the optimization and control of trajectory bundles [Ovsyannikov, 1980, Ovsyannikov, 2004, Ovsyannikov, 2025], as well as to the analysis of dynamical system stability and the study of the shift operator for the purpose of finding periodic solutions.

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