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OSCILLATORY STROBODYNAMICS — A NEW AREA IN NONLINEAR OSCILLATIONS THEORY, NONLINEAR DYNAMICS AND CYBERNETICAL PHYSICS

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Abstract

The definition of the "oscillatory strobodynamics" is given as an interdisciplinary field of knowledge, which explores the slow component of dynamics of a system in the presence of high frequency oscillations in engineering, natural science and sociology systems. The general approach to the solution of corresponding problems is formulated. Applications to generic and specific models employed in different sciences, particularly to oscillatory models, are considered.

Key words

Nonlinear dynamics, high frequency oscillations, separation of motions.

1 Introduction

A number of experimental and theoretical results have been accumulated by now with respect to the problem of relatively high frequency action on system dynamics in physics, chemistry, biology, physiology, economics and even sociology. Some of those investigations revealed a number of unusual and seemingly paradoxical phenomena, appearing to be of significant fundamental and applied value. Unfortunately, the theoretical studies were carried out by different methods and their results were treated differently. Many of the fundamental and applied problems considered there still remain to be solved. It is hard to indicate a dynamical problem in any of the above listed sciences for that the investigation of oscillatory behavior is meaningless.

In this paper the author attempts to formulate a general approach both to the solution of the above mentioned problems and to the analysis of the results obtained. The approach in question consists in transition from the description of complete ("true") dynamics of certain processes to the description of dynamics, which takes into account only one component of the processes, the slow one, which is of the utmost interest for applications. Such dynamics strongly differ from the conventional ones, they are simpler but rather paradoxical; in particular, the laws of conservation break down there, a non-conservative system might be transformed into a potential one and, conversely, additional terms may appear in the equations or parameter values may be changed. All aforesaid makes it expedient to single out the class of studies under review into a separate interdisciplinary field of knowledge called *oscillatory strobodynamics (OS)*. This field may be regarded as an area of the theory of non-linear oscillations or non-linear dynamics. One part of OS, namely the vibrational control of physical processes, can be regarded as an area of cybernetical physics [Fradkov, 2007]. Here we introduce the definition and the description of the OS approach and give a brief review of results obtained earlier and a number of new results.

2 OS approach and definition

The essence of OS approach can be described in the following way. Let the process dynamics be described by the following relation:

$$Z(x, a, t) = 0, \quad (1)$$

where x is the vector of the system state, a is the vector of parameters, t is the time, Z is an operator that may represent finite, differential, integral and other equations. In presence of fast oscillating actions the same relation will take the form:

$$Z\left[x + \psi_x(t, \omega t), a + \psi_a(t, \omega t), t\right] = F(t, \omega t), \quad (2)$$

where ψ_x, ψ_a and F are some functions periodic in "fast time" $\tau = \omega t$ and $\omega >> 1$ (Notions such as "fast" "slow" "high frequency" can be formalized [Blekhman, 2000], [Blekhman, 2004]).

Practically in all cases the change of the vector $x(t)$ under high frequency actions can be represented as

$$x(t) = X(t) + \psi(t, \omega t), \quad (3)$$

where X is slow component, ψ is fast 2π-periodic in time τ component, with average zero. The component X is of utmost interest here. By a certain mathematical procedure and under some assumptions in relation to operator Z characteristics it is possible to obtain a relation including only a slow component X:

$$Z^*(X, a^*, t) = 0. \quad (4)$$

The method of direct separation of motions appears to be the most appropriate one to be used in this procedure. Such method is widely used in solution of mechanics problems [Blekhman, 2000], [Blekhman, 2004] and can be easily extends to more general case considered here. Schematically the procedure is as follows. Since, according to (3), a single initial variable x is replaced by two variables — X and ψ, equation (2) in general case can be replaced by the following two integro-differential equations:

$$\langle Z\left[X + \psi + \psi_x, a + \psi_a, t\right]\rangle = \langle F\rangle\,, \quad (5)$$

$$Z\left[X + \psi + \psi_x, a + \psi_a, t\right] - \\ - \langle Z\left[X + \psi + \psi_x, a + \psi_A, t\right]\rangle = F - \langle F\rangle\,, \quad (6)$$

where the angular brackets denote averaging at $\tau = \omega t$ in period 2π. Further, the equation of fast motions (6) is solved approximately with regard to the fast variable ψ; the main approximation consists in assumption that slow variables can be considered in this case as “frozen” i. e. constant parameters. Substitution of ψ into equation (5) of slow motions yields equation (4) (for details refer to [Blekhman, 2000], [Blekhman, 2004]). It is worth mentioning that approximation applied to solution of equation (6) will only slightly affect the accuracy of equation (4) because function ψ is included in this equation under the sign of averaging. Meanwhile, it is to be taken into account that the approximations assumed will hold true only with certain limitations in the system parameters.

In general, both operator Z^* and parameter a^* values in equation (4) may strongly differ from operator Z and parameter a, which describe change of the initial variable x. Usually relation (4) is much simpler then (2), whereas vectors X and operator Z may have much less dimensions. Let us call relation (4)*the equation of oscillatory strobodynamics and define the oscillatory strobodynamics as dynamics describing evolution of the slow motion component under the action of high frequency upon a system or a process.*

One can say that the slow component X is the result of observations made over x under stroboscopic light with the frequency of flashes equal to the frequency of oscillations ω. Hence the OS is the dynamics perceived by an observer watching a system subjected to high frequency influence under stroboscopic light or through special spectacles preventing him from seeing quick motions. It seems natural to suppose that the view of such observer is much simpler than the view of the observer watching the process x directly. It is to be emphasized that conservation laws in their usual form are not valid there.

3 General fundamental equation in natural sciences

3.1 Mechanics

The equation of dynamics

$$m\ddot{x} = F(\dot{x}, x, t) + \Phi(\dot{x}, x, t, \omega t), \quad (7)$$

where F is a slow force, and Φ is a fast force 2π periodic in relation to ωt, corresponds to the equation of OS (vibrational mechanics).

$$m\ddot{X} = F(\dot{X}, X, t) + V(\dot{X}, X, t), \quad (8)$$

where V is additional slow force called vibrational force (the term introduced by P.L. Kapitsa). The approach of the vibrational mechanics along with the solution of some applied problems was given in books [Blekhman, 2000], [Blekhman, 2004]. This paper can be considered as a generalization of that approach. Principal results are cited below. The following equations are related to the *Navier-Stokes equation* and the equation of continuity:

$$\rho\frac{\partial U}{\partial t} + \rho(U\cdot\nabla)U = -\nabla P + \mu\nabla^2 U + v,\ \nabla\cdot U = 0, \quad (9)$$

where

$$v = -\langle (u'\cdot\nabla)u'\rangle$$

while U and P are slow components of flow velocity and pressure , u' is a fast component of flow velocity, ρ — density, μ — viscosity coefficient, ∇ — Hamilton operator.

Note that the first equation (9) corresponds to well known *equation of Reynolds* describing turbulent motion of the viscous incompressible liquid while vector v represents so called turbulent stresses, and averaging is accomplished in relation to corresponding “large” period. It is to be noted that expressions for V and v in equations (8) and (9) depend both on right hand parts of the initial dynamics equations and on approximations taken in solution of fast motions equations. A wide scope of problems is associated with the influence of vibration on liquids, bulk solids and multiphase media in vibrating vessels. Of high interest appear to be the effects of sinking gas bubbles and floating denser particles, stability of heavy liquid layer floating over lighter liquid, generation of slow flowing streams (see books [Blekhman, 2000], [Blekhman, 2004], [Lubimov, *et al*, 2003], [Ganiev, Ukrainsky, 2008],paper [Fedotovsky, 2006] and collection [Blekhman, Sorokin, 2009b]. Almost in all cases those effects can be sufficiently described by OS equations.

For one dimensional case the equation of *vibrational hydraulics* [Blekhman, 2004]] will correspond to *Bernoulli equation* [Blekhman, 2004]

$$\frac{1}{2}\rho(U_1^2 - U_2^2) + \Delta P + \Delta P_v = 0, \qquad (10)$$

where ΔP is a “usual” pressure difference, and ΔP_v is an additional, vibrational pressure difference.

Vibrational rheology, vibrational dynamic materials and composites. Suppose $\varepsilon = E(t) + \psi_\varepsilon(\omega t)$, $\sigma = \sum(t) + \psi_\sigma(\omega t)$, then the governing rheological equation $\sigma = f(\dot{\varepsilon}, \varepsilon)$ will be transformed in OS as $\sum = F(\dot{E}, E)$, where function F may be largely different from that of f[7]. Quite another dynamic effect involving transformation of solid body elasticity under the action of vibration we see in parametric mechanism of dynamic materials generation [8] (the effect of Indian Magic Rope, transformation of a string into a bar, increase in pipe rigidity under pulsed liquid flow and others (see below and [Blekhman, 2004], [Blekhman, Sorokin, 2009b], [Blekhman, 2007]).

In celestial mechanics problems the results of pulsating distance between the masses interacting by the law of gravitation can be interpreted with respect to slow motion in the relation either as increase of gravity constant or as emergence of additional force. This phenomenon was put into the basis of the invention of Gravilet [Blekhman, 2000], where dragging force in a spacecraft is generated by cyclic variation of distance between the weights of dump-bells placed in the space ship.

3.2 Heat conduction and diffusion

As for the linear equations describing *heat conduction and **diffusion*** neither the pulsation of the coefficient nor the pulsation of the source will entail any change in the oscillatory strobodynamics equation at first approximation though such change is certainly substantial when a non-linear term is present. Significant changes will be also caused by introducing the term $\partial^2 T/\partial t^2$, when the equation becomes hyperbolic one.

3.3 Maxwell’s electrodynamics equations

In their simplest case (flat wave) Maxwell’s equations can be reduced to a wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad (11)$$

for potential $u(x,t)$:

$$\frac{\partial u}{\partial t} = E, \quad \frac{\partial u}{\partial x} = B,$$

where $c = \sqrt{1/\mu\varepsilon}$ is velocity of light in a medium with dielectric permittivity ε and magnetic permeability, μ (in corresponding units), E is electric field intensity and B is magnetic induction. If A^2 varies in accordance with the law $A^2 = A_0^2(1 + \alpha \sin\omega t)$ the OS equation in comparison with (3) acquires an additional term $\frac{1}{2}(A_0^2\alpha/\omega)^2\partial^4 U/\partial x^4$ like it was observed in the string-bar case (see below).

3.4 Electromechanical systems

Speaking of such systems in particular electric machines one should say that mechanical processes run much slower than electromagnetic ones. This circumstance permits to pass from the classical *Gorev-Park system* of equations to simpler equations of slow motions i.e. OS equations [9]. The same can be said about the theory describing *the effects of bodies steady levitation* observed in a quickly oscillating field. In addition as it follows from the Earnshaw theorem, such effect is unlikely to occur in static condition [Fradkov, 2007], [Blekhman, 2004].

4 Some interdisciplinary problems

4.1 Vibrational displacement

By the effect of vibrational displacement we mean the emergence of the “directional on the average”, as a rule — slow, change (particularly of motion) at the expense of the undirected on the average (as a rule — fast, oscillatory) effects. Since much attention is attracted to the velocity of slow change of state (i. e. the velocity of vibrational displacement) it appears to be reasonable in finding this component to pass from the initial equations to those of OS. The problems associated with the theory of vibratory displacement are likely to appear apart from mechanics in other sciences such as chemistry, biology, biomechanics etc. A number of new problems in this field have been considered lately [Blekhman, 2000], [Blekhman, 2004], articles in [Blekhman, Sorokin, 2009b], [Blekhman, 2010].

5 Some fundamental equations

By taking into account oscillatory action in differential equations which describe some fundamental phenomena, we can detect considerable change of the system behavior – this is reflected in corresponding OS equations. Take for example a transformed version of the well-known *Lorenz's equation system* (see [Neimark, Landa, 1992])

$$\ddot{\xi} + (\eta - 1)\xi + \xi^3 = -\mu\dot{\xi}, \qquad (12)$$
$$\dot{\eta} = -\frac{\mu}{\sigma + 1}\left[b\eta - (2\sigma - b)\xi^2\right] + A\cos\omega t,$$

where μ, σ and b are positive parameters, A and ω are the amplitude and the frequency of action correspondingly.

In this case we obtain the following OS equation for the slow component B of the variable ξ:

$$\ddot{B} + (\frac{1}{2}\frac{A^2}{\omega^4} - 1)B + B^3(1 + \frac{3}{2}\frac{A^2}{\omega^6}) = -\mu\dot{B}. \quad (13)$$

Hence, one can see that as $A > \sqrt{2}\omega^2$ the equilibrium position $B = 0$ becomes stable and no self-oscillations including chaotic ones will arise. The equation (13) is obtained after the simplest introduction of oscillating action according to (10) and based on approximate solution of the fast motion equations, when $\psi_1 = AB\sin\omega t/\omega^5$, $\psi_2 = A\sin\omega t/\omega$. (Supposing $\xi = B(t) + \psi_1(\omega t)$, $\eta = \psi_2(\omega t)$). Other ways of introducing the disturbance and more accurate solution of the equations are certain to yield more complete results.

The other example is the classic *Lotka–Volterra* system "*prey-predator*" [Neimark, Landa, 1992]. With periodic action this system takes the form:

$$\dot{n}_1 = n_1(\varepsilon_1 - \gamma_1 n_2) + A\sin\omega t \qquad (14)$$
$$\dot{n}_2 = -n_2(\varepsilon_2 - \gamma_2 n_1) + B\sin(\omega t + \delta),$$

where n_1 and n_2 denote animals population, substances concentration and so on and $\varepsilon_1, \varepsilon_2, \gamma_1, \gamma_2, A, B, \delta$ are positive constants. In the simplest approximation the solution of the fast motion equations $\dot{\psi}_1 = -A\sin\omega t$, $\dot{\psi}_2 = B\sin(\omega t + \delta)$ will yield the following OS equations:

$$\dot{N}_1 = N_1(\varepsilon_1 - \gamma_1 N_2) - \gamma_1 a,$$

$$\dot{N}_2 = -N_2(\varepsilon_2 - \gamma_2 N_1) + \gamma_2 a, \qquad (15)$$

$$(a = AB\cos\delta/\omega^2).$$

These equations are distinguished from the classic ones by the presence of constant terms in their right hand parts. This tends to change the behavior of the phase trajectories of the system which becomes not conservative any more. Quantities N_1 and N_2, depending on parameter a values, either approach some constant values or depart from those; the latter case corresponds to the "extinction" of populations. There exists a significant distinction of OS equations from initial ones in models of self-oscillatory chemical reactions (see [Neimark, Landa, 1992]) as well as in Lancaster's war model. The behavior of corresponding systems is to be changed under the influence of oscillatory action.

Wave equations are also subjected to such changes, the alterations being not only quantitative but qualitative too. If in the equation of string oscillation

$$m\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}. \qquad (16)$$

the string tension changes according to the rule $T = T_0(1 + \alpha\sin\omega t)$, we obtain the following OS equation:

$$m\frac{\partial^2 U}{\partial t^2} = T_0\frac{\partial^2 U}{\partial x^2} - (EY)_v\frac{\partial^4 U}{\partial x^4}, \qquad (17)$$

where

$$(EY)_v = \frac{1}{2m}(T_0\alpha/\omega)^2$$

is bending rigidity acquired by the string in its slow motion $U = \langle u\rangle$. To put it differently, the string appears to be transformed into a bar. Identical results can be obtained in solution of other problems such as the problem of Indian Magic Rope and the problem of pipe stability with some pulsating liquid flowing through it [Blekhman, 2000], [Blekhman, 2007] as well as the problems described by *Sine-Gordon non-linear wave equations*. Specification of above mentioned results with accompanying reference list is given in [Shishkina, *et al*, 2008].

6 Oscillators

Non-linear and parameterically excited oscillators with single degree of freedom are widely used as models of many oscillatory processes. A great number of studies accomplished mainly by digital simulation technique have been devoted to investigation of "controlling" influence on oscillators [Fradkov, 2007], [Neimark, Landa, 1992]. In this paper this task is treated by OS approach. *Mathieu's oscillator* is described by the equation:

$$m\ddot{x} + (c_0 + c_v\sin\omega t)x = 0, \qquad (18)$$

where m, c_0 and c_v are constants. When $\omega >> \lambda = \sqrt{c_0/m}$, OS equation has the form:

$$\ddot{x} + (c_0 + c_v)x = 0, \qquad (19)$$

where $c_v = c_0^2/2\omega^2$. Hence, equilibrium position $x = 0$ unstable if $c_0 < 0$, $c_\approx = 0$ becomes stable for $X = 0$, if $c_v > -c_0$. This explains in particular the effect of stabilization an inverted pendulum with a vibrating suspension axis [Blekhman, 2000; **?**].

Fidlin's oscillator [Blekhman, Sorokin, 2009a; **?**] is described by the following equation:

$$\ddot{x} + (\beta - a\omega \cos \omega t)\dot{x} + x = 0, \qquad (20)$$

where $a \sim O(1)$, $\beta \sim O(1)$, $\omega >> 1$. Corresponding OS equation takes the form:

$$\ddot{X} + \beta\dot{X} + I_0^2(a)X = 0, \qquad (21)$$

where $I_0(a)$ is modified Bessel's function. It is worthwhile to note that the strong oscillation of the damping coefficient in equation (20) leads to the essential change of the rigidity coefficient in equation (21). Double-frequency excitation in *Duffing oscillator* leads to the effect of conjugate resonances and bifurcations [Blekhman, 2004]. For the oscillator with a *positional – viscous resistance* $(a_0 + a_1 x + ...)\dot{x}$ the OS equation describes the phenomena of *asynchronous excitation and asynchronous depression of self-excited oscillations* as results of resistance characteristics transformation. This case includes *Van-der-Pol's* oscillator as well [Blekhman, 2000].

At first sight *oscillation of an independent variable* t seems to be exotic one. Such oscillation implies transition from time t to a variable $\tau = t + \alpha \sin \omega t$, where $\alpha << t$. For a linear oscillator $\ddot{x} + \lambda^2 x = 0$ the OS equation in corresponding approximation looks like:

$$\frac{d^2X}{d\tau^2} + \frac{1}{2}(\alpha\omega)^2\frac{dX}{d\tau} + \lambda^2(1 + 2\alpha^2\lambda^2)X = 0 \qquad (22)$$

i. e. time pulsation entails the appearance of dissipation!

It is to be noted that the employment of the pulsating independent variable – called "true anomaly" is characteristic for a number of problems in celestial mechanics. As it was shown by V.S. Sorokin, the reverse transition to time t in the known Beletsky's equation, describing the travel of a satellite with respect to the center of mass [Beletsky, 1978], transforms this equation into one devoid of the term depending on $\dot{x}$. The corresponding OS equation will be also in line with the conservative system.

7 Conclusion

The above-cited examples taken from various sciences provide evidences for consideration of the problems concerning slow motions of dynamical systems under high frequency oscillating actions within a new separate interdisciplinary field of knowledge. This new area of science may be called *oscillatory strobodynamics* (or *vibrational dynamics*. The problems of the OS allow to make use of a general mathematical approach and a clear interpretation of results that seem to be paradoxical at first glance.

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