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A MECHANISM FOR ANOMALOUS TRANSPORT

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Abstract

In this paper we study how deterministic features presented by a system can be used to perform direct transport in a *quasi*-symmetric potential and weak dissipative system. We show that the presence of nonhyperbolic regions around acceleration areas of the phase space plays an important role in the acceleration of particles giving rise to direct transport in the system. Such effect can be observed for a large interval of the weak asymmetric potential parameter allowing the possibility to obtain useful work from unbiased nonequilibrium fluctuation in real systems even in a presence of a *quasi*-symmetric potential.

Key words

anomalous transport, nonhyberbolicity

1 Introduction

Anomalous transport is an emerging field in physics and, generally speaking, refers to nonequilibrium processes that cannot be described by using standard methods of statistical physics. Anomalous transport occurs in a wide realm of physical systems ranging from a microscopic level (such as conducting electrons) to a macroscopic scale (as in global atmospheric events). One of the phenomena in this category is anomalous diffusion, for which the mean-squared-displacement increases with time as a power-law t^{μ}, where $\mu \neq 1$ [1]. In some cases even analytical results for the anomalous diffusion can be obtained [2]

There is a growing interest in anomalous transport properties of nonlinear systems presenting nonequilibrium fluctuations, the *ratchet systems* [3; 4; 5; 6]. For these systems the second principle of thermodynamics does not prohibit the transport provided we do not have any space or time symmetries forbidding it [4; 7]. Ratchet systems occur in a variety of physical problems, like unidirectional transport in molecular motors [8; 9], micro particles segregation in colloidal solutions [10] and transport in quantum and nanoscale systems [4; 11; 8]. Recently transport in spatially periodic potential influenced by periodic un-biased external forces was proved to be possible for the cases of large (phase space without islands) and small external force amplitude(phase space with islands) [12].

In this paper we show that it is not necessary to have strong asymmetry, in the sense that sizeable ratchet currents can be obtained in weakly dissipative systems with *slightly asymmetric* potentials. In fact, we claim that the presence of ratchet currents is influenced not so much by the potential asymmetry, but rather by the existence of *strongly nonhyperbolic regions* in the phase space of weakly dissipative systems. By a hyperbolic region $\mathcal{S}$ we mean a set for which the tangent phase space in each point splits continuously into stable (SM) and an unstable (UM) manifolds which are invariant under the system dynamics: infinitesimal displacements in the stable (unstable) direction decay exponentially as time increases forward (backward) [13]. In addition, it is required that the angles between the stable and unstable directions are uniformly bounded away from zero. On the other hand, the nonhyperbolic term will be used here to denote regions where we observe (almost) tangencies between stable and unstable manifolds of saddle points embedded in the chaotic region.

2 The Model

Chaotic orbits of dissipative two-dimensional mappings are often nonhyperbolic since the SM and UM are tangent in infinitely many points. As a representative illustration of this effect we consider a periodically kicked rotor subjected to a harmonic potential function, whose dynamics is two-dimensional. The dynamics of a periodically kicked rotor with small dissipation and potential asymmetry can be described in a cylindrical phase space $(-\infty \times \infty) \times [0, 2\pi)$, whose discrete-time variables p_n and x_n are respectively the momentum and the angular position of the rotor just after the nth kick, with the dynamics given by the following dissipative asymmetric kicked rotor map (DAKRM) [4]:

$$p_{n+1} = (1-\gamma)p_n + K[\sin(x_n) + a\sin(2x_n + \pi/2)], \quad (1)$$
$$x_{n+1} = x_n + p_{n+1}, \quad (2)$$

where K is related to the kick strength, $0 \le \gamma \le 1$ is a dissipation coefficient, and a is the symmetry-breaking parameter of the system. The conservative ($\gamma = 0$) and symmetric ($a = 0$) limits yield the well-known Chirikov-Taylor map [14]. In the following we will keep the dissipation small enough (namely $\gamma = 2 \times 10^{-4}$) in order to highlight the effect of the periodic islands of the conservative case. Moreover, the asymmetry parameter a will be kept small so as to emphasize the role of the nonhyperbolic phase space regions on the anomalous transport.

The conservative and asymmetric ($a \neq 0$) case has two fixed points (we call P_1) given by

$$p^{R,L} = 0, \quad (3)$$
$$x^{R,L} = \pi - \sin^{-1}\Theta^{R,L}(a, K), \quad (4)$$

where

$$\Theta^{R,L}(a, K) = \left(1 - \sqrt{1 + 8a^2 \pm 16\pi a/K}\right)/4a. \quad (5)$$

In Eq. 5, $+$ and $-$ mean right and left respectively and are marginally stable centers in the following parameter intervals: $0 < a < 1/4$, and $6.40 < K < 7.20$. These two P_1 points are the centers of two resonant islands that are actually accelerator modes. There are also two (left and right) period-3 fixed points (P_3) related to secondary resonances around the P_1 islands.

In the weak dissipative case, the two P_1 points become stable foci, their basins of attraction present a complex structure. The chaotic region in the conservative map becomes a chaotic transient in the weakly dissipative situation. At $K \approx 6.92$ the right P_3 points collides with the right fixed point (p^R, x^R) by a bifurcation. At the bifurcation point the attraction basin of P_1 engulfs the SM of the P_3 and turns to be accessible to points in a large phase space region.

3 The nonhyperbolicity role in the transport

The vicinity of the fixed points plays a key role in the anomalous transport mechanism, in the same way as the islands do for the conservative case. More precisely, the wide accessibility of this vicinity near the bifurcation is responsible for large ratchet currents, just as the role of the accelerator modes in the non-dissipative map. Figs. 1(a-d) depict 2-D histograms for 5000 orbits (each orbit containing 10 points) of the DAKRM from initial conditions chosen in the phase plane region $0 < x < 2\pi$, $-\pi < p < \pi$, as well as the corresponding momentum probability distributions $\sigma(p)$ (Figs. 1(e-h)). For $K = 6.40$ there is a quasi-symmetric situation, the neighborhood of the two P_1 fixed points (left and right) being seldom visited [Fig. 1(a)]. Since the left (right) region is responsible for a positive (negative) increase of the transport, we observe that for this parameter value the momentum distribution function is nearly symmetric, with a Gaussian shape [Fig. 1(e)], resulting in a null transport.

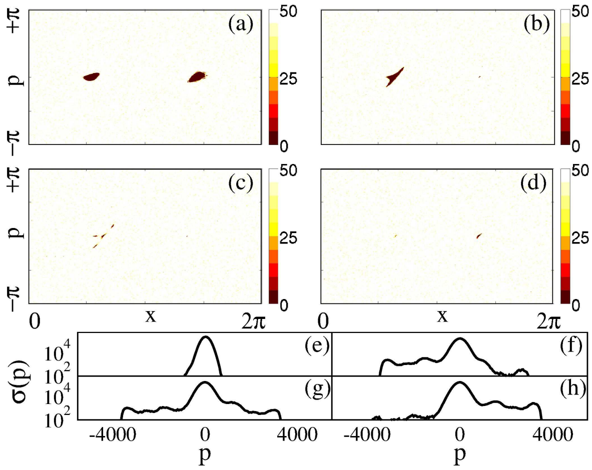

Figure 1. (color online) 2-D histograms (top) and momentum probability distributions (bottom) for the DAKRM with $\gamma = 0.0002$, $a = 0.005$, and $K = 6.40$ panels (a,e); $K = 6.92$ panels (b,f); $K = 6.96$ panels (c,g); $K = 7.00$ panels (d,h).

Symmetry-breaking effects start to be noticeable after $K = 6.40$ and reach its maximum at the bifurcation $K \approx 6.92$, where only the vicinity of the left P_1 is scarcely visited by orbits of the map [Fig. 1(b)]. This effect is triggered by the bifurcation whereby the right P_3 fixed point collides with the right P_1 point and turns its vicinity easily accessible (not shielded), what is reflected in the asymmetric left tail in the momentum distribution function [Fig. 1(f)]. The vicinity of the left P_1 is not yet affected since the collision process did not occur yet for left P_1 and P_3. As a result, a net transport current is generated. However, this is more an effect of the bifurcation (due to nonhyperbolicity) than of the symmetry-breaking itself. In other words, if there is weak symmetry-breaking but no bifurcation (and no shield process), the ratchet effect will not occur, at least with the magnitude we observed in this example.

Not too far from the bifurcation ($K = 6.96$) the vicinities of both P_1 fixed points become now almost equally visited [Fig. 1(c)] generating negative and positive currents, but no net currents. For this case the mo-

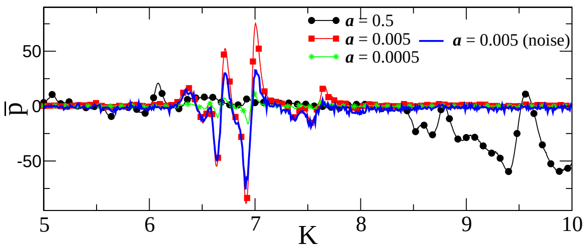

Figure 2. (color online) Ensemble averaged net current for different values of the nonlinearity and asymmetry parameters of the DAKRM with $\gamma = 0.0002$.

mentum distribution function is again approximately symmetric [Fig. 1(g)] but presenting right and left tails. The situation changes again after the left P_1 and P_3 fixed points collision [Fig. 1(d)], through the same bifurcation mechanism described for their right counterparts. The increase in the accessibility of the vicinity of the left P_1 point leads to an asymmetric momentum distribution function [Fig. 1(h)], restoring a net transport current. In this last case is noticeable that the seldom visited region is very small, nevertheless it is enough to inhibit negative currents. Once again the nonhyperbolicity of right region seems to be more important than the asymmetric situation itself.

The variation of the average net transport current $\overline{p}$ with the nonlinearity parameter K is depicted in Fig. 2 for different values of the asymmetry parameter. The net current is ensemble-averaged over a large number (10^6) of initial conditions, each of them being followed by a short time ($t = 10^3$) to prevent the system to settle down into any fixed point. On varying K we obtain a series of positive and negative net transport currents resulting from the ratchet effect. Curiously the net current fluctuates less for both very small and large asymmetry, being more sensitive to K for intermediate values of a. As we have seen, the appearance of net currents is due to the fact that the left and right P_3 fixed points (that bifurcate in pairs for the symmetric case) start to bifurcate at *different* values of K. For the interval $6.40 < K < 7.00$, corresponding to Fig. 1, the right P_3 fixed point bifurcates before the left one, generating a sequence of negative and positive net currents with very large peak values. For $K = 6.92$ the maximum amplitude of the net transport current for $a = 0.005$ is at least three times larger than for the $a = 0.5$ (high asymmetry case). Nevertheless appreciable net currents can also be acquired for asymmetry parameter as small as $a = 0.0005$ or even smaller, although the peak values decrease considerably. Such decrease in the net transport current for small values of a is expected since for the case of $a = 0$ no transport can be observed due to the symmetry of the standard map. We emphasize that the transport mechanism is a property of the phase space and is not related to any asymptotic state of the system.

By the way of contrast, with a higher asymmetry value as those in Ref. [4] such large net currents are observed only for larger nonlinearities ($8.5 < K < 10$), hence they are not primarily related to the bifurcations we present. In such case the role of the potential asymmetry overcomes the bifurcation mechanism presented here. The sensitive dependence of the net transport current on the nonlinearity parameter in the weak asymmetry case reminds us of a similar behavior for the diffusion coefficient of the conservative and symmetric (Chirikov-Taylor) map, caused by the existence of accelerator modes [2].

In order to test the robustness of the problem we have simulated the effect of noise in the dynamics. A small amount of white noise distributed in the interval $(0, 0.1)$ was introduced in the x dynamics. The results are displayed in Fig. 2(a). The effect of the nonhyperbolicity in the transport is still observed.

The existence of nonhyperbolic regions in phase space, however small they may be, constitutes a *deterministic mechanism underlying anomalous transport* in the production of net currents through a ratchet effect. In order to quantify the degree of nonhyperbolicity related to the phenomena we describe in this paper, let us consider an initial condition (p_0, x_0) and a unit vector $\boldsymbol{v}$, whose temporal evolution is given by $\boldsymbol{v}_{n+1} = \mathcal{J}(p_n, x_n)\boldsymbol{v}_n/|\mathcal{J}(p_n, x_n)\boldsymbol{v}_n|$, where $\mathcal{J}(p_n, x_n)$ is the Jacobian matrix of the DAKRM. For n large enough, $\boldsymbol{v}$ is parallel to the Lyapunov vector $\boldsymbol{u}(p, x)$ associated to the maximum Lyapunov exponent λ_u of the map orbit beginning with (p_0, x_0). Similarly a backward iteration of the same orbit gives us a new vector $\boldsymbol{v}_n$ that is parallel to the direction $\boldsymbol{s}(p, x)$, the Lyapunov vector associated to the minimum Lyapunov exponent λ_s [15; 16]. For regions where $\lambda_s < 0 < \lambda_u$ the vectors $\boldsymbol{u}(p, x)$ and $\boldsymbol{s}(p, x)$ are tangent to the UM and SM, respectively, of a point (p, x).

The nonhyperbolic degree of a region $\mathcal{S}$ can be studied computing the angles between the two manifolds $\theta(p, x) = \cos^{-1}(|\boldsymbol{u} \cdot \boldsymbol{s}|)$, for $(p, x) \in \mathcal{S}$ [16]. Therefore, $\theta(p, x) = 0$ denotes a tangency between UM and SM at (p, x). Let $S_\epsilon^{R,L} = \{(p, x) \in \Omega : |(p - x) - (p^{R,L}, x^{R,L})| < \epsilon\}$ be a ϵ-radius neighborhood of right and left P_1 fixed points. Results for $\theta(p, x)$ and its distribution function $\rho(\theta)$ calculated in both regions are shown in Fig. 3 for four values of the nonlinear parameter K. The dark region in Fig. 3 correspond to strongly nonhyperbolic region bounding the acceleration region.

For $K = 6.40$, near the fixed points $(p^{R,L}, x^{R,L})$ there is a strong nonhyperbolic region which shields the acceleration area [Figs. 3(a-b)]. In fact, almost all θ values for the right area (dotted red (gray) curve) and left (solid blue (dark gray) curve) are confined in the interval $\theta < \pi/16$, characterizing a strongly nonhyperbolic region around both fixed points [Fig. 3(c)]. By way of contrast, when $K = 6.92$, only the left fixed point neighborhood is shielded, resulting in a large negative transport since only the right acceleration region is regularly visited [Figs. 3(d-e)]. The right P_1 and P_3 fixed points suffer a bifurcation and all tangencies

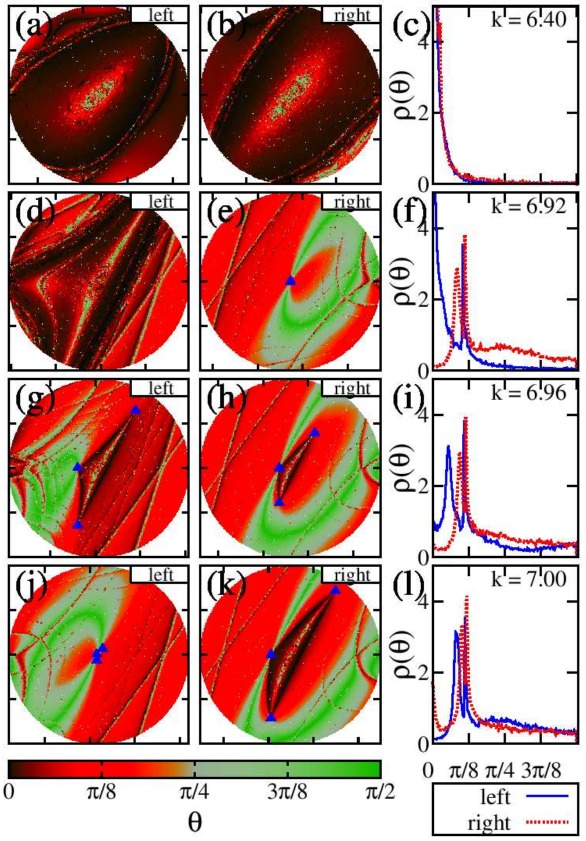

Figure 3. (color online) $\theta(p, x)$ values evaluated from 10^5 initial conditions uniformly distributed around (0.2 radius) $\mathcal{S}_{\epsilon}^{R,L}$ and the distribution function of θ. Blue (dark gay) triangles are right and left P_3 fixed points (inexistent in (a),(b) and (d) panels).

of SM and UM disappear from the right area, allowing the trajectories to visit the acceleration region. Accordingly Fig. 3(f) presents different distributions of θ values for left and right regions. The dotted red (gray) curve (right) presents a distribution peak around $\theta = \pi/8$ considerably greater than the solid blue (dark gray) one, leading to absence of shielding in the right region.

In the case of $K = 6.96$ neither of the areas surrounding the fixed points are *effectively* shielded by the tangencies of manifolds, resulting in large positive and negative transport currents, but no net current at all [Figs. 3(g-i)]. This situation, however, is different from the one displayed by Figs. 3(a-c), where the regions surrounding both fixed points were scarcely visited, hence there is no net current since the positive and negative currents are very small. Finally, for a higher value of K, only the right acceleration region is shielded, resulting in a positive transport current [Figs. 3(j-k)]. The distribution $\rho(\theta)$ presents a peak near zero for the dotted red (gray) curve, confirming the existence of a shielded right area [Fig. 3(l)].

The source of nonhyperbolicity in these regions is the tangencies between SM and UM of saddle orbits embedded in the chaotic region therein. For perpendicular manifold crossings an area near the acceleration region (P_1 vicinity) will map another area inside the acceleration region. However if we have tangencies of manifolds, the outer area (outer P_3 vicinity) that maps an inner one tends to zero forbidding the trajectory to visit the acceleration area [15]. The scenario can be regarded as a counterpart of the Poincaré-Birkhoff' theorem that describes the torus breakdown of a conservative two-degree of freedom map.

4 Conclusions

In conclusion we have shown that anomalous transport displayed by a *quasi*-symmetric potential and weakly dissipative system is strongly related to the topology of the acceleration regions around fixed points displayed by the system. The presence of nonhyperbolic regions caused by almost parallel UM and SM can inhibit a chaotic trajectory to visit the neighborhood of the acceleration region surrounding fixed points of the system. This mechanism is closely related to the scenario described by the Poincaré-Birkhoff theorem in area-preserving two-dimensional maps. This dynamical phenomenon yields large net transport current in some direction even though the potential has an extremely small degree of symmetry-breaking. Hence such net currents can yield useful work from unbiased nonequilibrium fluctuation even with *quasi*-symmetric potentials, which enlarges the realm of dynamical systems displaying the ratchet effect.

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