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CHAOS CONTROL IN UNCERTAIN ECONOMIC SYSTEMS VIA QUASI SLIDING MODE METHOD

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Abstract

In this paper the method of quasi-sliding mode control is introduced for stabilizing unstable fixed points of uncertain economic systems which show chaotic behavior. The control algorithm is presented, and the performance of proposed approach is examined by applying it to two economic models with chaotic behavior. The Behrens-Feichtinger and Cournot-Puu duopoly models as two chaotic economic systems are selected for this purpose. Simulation results show the effectiveness and feasibility of the method for chaos control in uncertain chaotic markets.

Key words
Quasi sliding mode, Chaos control, Economic models.

1 Introduction

It has been accepted that many economic systems have nonlinear deterministic models [1,2] which show chaotic behavior in their time series [1,3]. It has also been shown that even oligopolistic markets may become chaotic under certain conditions [4-6]. Chaotic behavior in an economic system is often an undesirable phenomenon which prevents from prediction in long time period, and may threaten the safety of investment. In [4], it has been shown that the performance of a chaotic economics can be improved by controlling chaos [4]. Chaotic systems have infinitely many fixed points or periodic solutions in their chaotic attractors [7]. Stabilizing the periodic solutions of a chaotic market model may increase economic efficiency [6].

Since 1990, various control algorithms for chaos control by stabilizing the periodic orbits or fixed points of nonlinear dynamic systems have been proposed. Ott, Grebogi and Yorke [8] presented a perturbing method (OGY) for chaos control by linearizing the nonlinear map of the system. The OGY method was successfully applied for chaos control in some economic systems, e.g. see [9]. Also Pyragas [10] presented a method for chaos control by using a delayed feedback signal. Stabilizing the first order fixed point of the Cournot duopoly via delayed feedback control has been investigated in [11]. Other nonlinear control methods such as feedback linearization [12], continuous time sliding mode [13] and adaptive Lyapunov based control [14] have been greatly used for chaos suppression in numerous physical systems. Many models of economic systems are illustrated though nonlinear discrete maps [1]. If the governing equations of such systems are exactly known, then one can easily used the OGY or the Pyragas method to design stabilizing controllers for them. However due to uncertainties, determining an exact governing equation for a system is not possible, and the parameters of a system always have some uncertainties. So introducing a robust control strategy for chaos control in such systems seems to be vital. One of the famous nonlinear methods used as a robust control is the sliding mode. Sliding mode control initially developed for continuous time systems [15]. Due to some technical difficulties it can not be used directly for discrete dynamical systems generated by nonlinear maps. In [16, 17] some concepts for adapting the sliding mode method to use in discrete time systems was presented. The modified method is called the quasi sliding mode control. In this paper it is shown that how the quasi-sliding mode method can be used for stabilizing the fixed points of chaotic systems, and then the proposed method is applied for chaos elimination from two economic models. The main advantage of the new strategy is its ability to control chaos when the actual parameters of market are not available or have some uncertainties.

2 Chaos Control via Quasi Sliding Mode Control

In this part, the concept of sliding mode control in discrete systems, known as the quasi sliding mode, is introduced. Consider the discrete nonlinear single-input single-output (SISO) plant:

$$x_{k+1} = f_k\left(x_k, \ldots, x_{k-n}, u_{k-1}, \ldots, u_{k-b}\right) + g_k\left(x_k, \ldots, x_{k-c}, u_{k-1}, \ldots, u_{k-d}\right) u_k \tag{1}$$

where x_{k+1} is the output, u_k is the control input, $f_k(.)$ and $g_k(.)$ are smooth nonlinear functions of past values of the input and output, and the constants *n*, *b*, *c*, and *d* are all positive integers.

For this system, it is assumed that the function $g_k(.)$ is bounded away from zero. Let us assume that a model for (1) exists, in the form of

$$\hat{x}_{k+1} = \hat{f}_k\left(x_k, \ldots, x_{k-n}, u_{k-1}, \ldots, u_{k-b}\right) + \hat{g}_k\left(x_k, \ldots, x_{k-c}, u_{k-1}, \ldots, u_{k-d}\right) u_k \tag{2}$$

where $\hat{f}_k(.)$ and $\hat{g}_k(.)$ are estimated functions of $f_k(.)$ and $g_k(.)$, respectively. The switching function is defined as a linear combination of the past tracking errors as:

$$S_k = e_k + \alpha_1 e_{k-1} + \ldots + \alpha_p e_{k-p} \tag{3}$$

The coefficients $\alpha_1, \ldots, \alpha_p$ in (3) are selected to make the switching function a stable linear combination of the past tracking errors. In other words, all of the roots of the polynomial $z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p$ should lie inside the unit circle. One can select α_1 such that $\left|\alpha_1\right| < 1$, and here it is assumed that $\left|\alpha_1\right| < 1$. If one can design the control action, u_k, in a way that the tracking error trajectories lie on the surface $S_k = 0$, then after a finite period of time, the tracking error will converge towards zero. This mode of trajectory motion is called the sliding mode.

A fixed point of (1) is denoted by x^f and satisfies the following relation;

$$x_{k+1}^f = x_k^f = f_k\left(x_k^f, x_{k-1}^f, \ldots, x_{k-n}^f, 0, \ldots, 0\right) \tag{4}$$

Thus, the tracking error is defined as:

$$e_k = x_k - x_k^f \tag{5}$$

In contrary to the continuous systems, “sliding mode” cannot be achieved for discrete systems and hence we have to try reaching a quasi sliding mode. Some concepts regarding this matter can be found in [16]. A system is said to be in a quasi-sliding mode when the dynamics of S_k meets the following set of conditions:

I) Starting from any state, the S_k sequence moves toward the quasi-sliding surface, defined by $S_k=0$, and crosses it in a finite period of time.

II) Once the surface is crossed by the first time, the S_k value changes around the surface in a zigzag way.

III) The zigzag motion is stable and stays inside a fixed band.

Under these conditions, the system trajectories approach to the fixed point but they may deviate around it in a zigzag way.

Conditions I, II, and III can be mathematically expressed as:

$$S_k\left(S_{k+1} - S_k\right) < 0, \tag{6}$$

and

$$if\ \mathrm{sgn}(S_{k+1}) = -\mathrm{sgn}(S_k) \Rightarrow \begin{cases} \mathrm{sgn}(S_{k+2}) = \mathrm{sgn}(S_k) \\ |S_{k+1}| < \xi \end{cases}, \tag{7}$$

where ξ is the fixed bound mentioned in the third condition.

Let us consider that, in closed loop, the switching function will exhibit the following behavior [17]:

$$S_{k+1} = \left(x_{k+1} - \hat{x}_{k+1}\right) - \varepsilon.\mathrm{sgn}\left(S_k\right), \ \varepsilon > 0 \tag{8}$$

Starting from (8) and using (1), (3), and (5) one can determine that the input signal must be:

$$u_k = -\frac{1}{\hat{g}_k}\begin{pmatrix} \left(\hat{f}_k - x_{k+1}^f\right) + \alpha_1 e_k + \ldots + \alpha_p e_{k-p+1} \\ + \varepsilon.\mathrm{sgn}\left(S_k\right) \end{pmatrix} \tag{9}$$

We can now readily check if such behavior for the switching function can guarantee the conditions in (6) and (7). Inserting (8) right into (6) we get:

$$\begin{aligned} S_k\left(S_{k+1} - S_k\right) &= S_k\left(\left(x_{k+1} - \hat{x}_{k+1}\right) - \varepsilon.\mathrm{sgn}\left(S_k\right) - S_k\right) \\ &\le \left|x_{k+1} - \hat{x}_{k+1}\right|\left|S_k\right| - \varepsilon\left|S_k\right| \end{aligned} \tag{10}$$

hence, it is sufficient to take

$$\varepsilon = \eta.H \tag{11}$$

where η is an arbitrary constant larger than unity and H is an upper bound for the modeling error,

$$H > \left|x_{k+1} - \hat{x}_{k+1}\right| \qquad , \qquad \forall k \tag{12}$$

The first condition in (7) is also easy to check. The expression for S_{k+2} can be directly derived from (8).

$$S_{k+2} = \left(x_{k+2} - \hat{x}_{k+2}\right) - \varepsilon.\mathrm{sgn}\left(S_{k+1}\right) \tag{13}$$

Considering that S_k has just crossed the sliding surface, $\mathrm{sgn}(S_{k+1}) = -\mathrm{sgn}(S_k)$, thus:

$$S_{k+2} = \left(x_{k+2} - \hat{x}_{k+2}\right) + \varepsilon.\mathrm{sgn}\left(S_k\right) \tag{14}$$

Taking into account the conditions in (11) and (12), the first condition in (7) is guaranteed. The second requirement in (7) needs the consideration of the closed loop system in which one gets the following representation for the closed-loop dynamics of the switching surface:

$$\begin{aligned} S_{k+1} = {} & \left(1 - \frac{g_k}{\hat{g}_k}\right)\alpha_1 S_k + \left(f_k - \frac{g_k}{\hat{g}_k}\hat{f}_k\right) + \left(1 - \frac{g_k}{\hat{g}_k}\right) \\ & \cdot\begin{pmatrix} (\alpha_2 - \alpha_1^2)e_{k-1} + \cdots + (\alpha_p - \alpha_1\alpha_{p-1})e_{k-p+1} \\ -\alpha_1\alpha_p e_{k-p} \end{pmatrix} \\ & - \left(1 - \frac{g_k}{\hat{g}_k}\right)x_{k+1}^f - \varepsilon\frac{g_k}{\hat{g}_k}\mathrm{sgn}\left(S_k\right) \end{aligned} \tag{15}$$

The only recurrent terms in S_k are the first and the last terms on the right hand side. The latter is a bounded function of S_k, so considering that $\left|\alpha_1\right| < 1$, the switching function will remain stable only if :

$$\left|1 - \frac{g_k}{\hat{g}_k}\right| < 1 \tag{16}$$

One can see [17] for more details of proof.

The control form (9) satisfies the quasi sliding mode conditions if Eqs. (11) and (16) are satisfied.

3 Chaos Control of Behrens-Feichtinger Model

The Behrens-Feichtinger model denotes a simple micro-economical model [18] of two firms *X* and *Y* competing on the same market of goods. The firms perform active investment strategies, i.e. their temporary investments depend on their relative position on the market. The strategies are asymmetric. The firm *X* invests more when it has an advantage over the firm *Y* while the firm *Y* invests more if it is in a disadvantageous position to the firm *X*. The sales x_k and y_k of both firms are measured in discrete time periods $k = 1,2,3,\ldots$ and the system governing equations are modeled by [6]:

$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + \frac{a}{1+\exp[-c(x_k - y_k)]} \\ y_{k+1} &= (1-\beta)y_k + \frac{b}{1+\exp[-c(x_k - y_k)]} \end{aligned} \tag{17}$$

The constants α and β which $0 < \alpha, \beta < 1$ are the time rates at which the sales of both firms decay in absence of investments while the second terms in the right hand side of Eq. (17) describe the investment effect at the k^{th} time period on the sale quantities at $k+1^{\text{th}}$ time period. Parameters *a* and *b* define the investment effectiveness of the firms and *c* is an "elasticity" measure of the investment strategies. The regular or irregular behavior of dynamic system described by Eq. (17) depends on the values of parameters α, β, *a*, *b* and *c* [6,18]. For $\alpha = 0.46$, $\beta = 0.7$, $a = 0.16$, $b = 0.9$ and $c = 105$, system (17) shows chaotic behavior whose strange attractor has been presented in Fig. (1). The first order and the second order fixed points of the system are shown in Fig. (1), which are $x_k^f = 0.0120, y_k^f = 0.0438$ as the first order fixed point, and $x_k^f = 0.0333$, $y_k^f = 0.0135$ and $x_{k+1}^f = 0.01097$, $y_{k+1}^f = 0.0250$ as the second order fixed points.

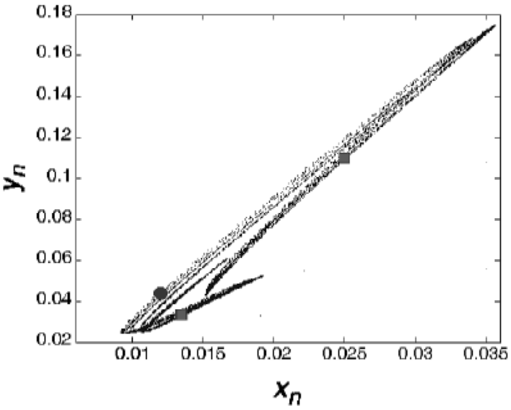

Figure 1. Chaotic attractor of the Behrens-Feichtinger model, its first order (circle) and second order fixed points (square).

For chaos elimination from the system, the first order fixed point of the system is selected to stabilize, and both firms achieve this objective by applying the control action to the parameters *a* and *b*, i.e. the investment effectiveness. In this case the controlled system is written as:

$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + \frac{a+u_{1k}}{1+\exp[-c(x_k - y_k)]} \\ y_{k+1} &= (1-\beta)y_k + \frac{b+u_{2k}}{1+\exp[-c(x_k - y_k)]} \end{aligned} \tag{18}$$

Comparing Eq. (18) and Eq. (1) yields:

$$\begin{aligned} x_{k+1} &= f_{1k}(x_k, y_k) + g_{1k}(x_k, y_k)u_{1k} \\ y_{k+1} &= f_{2k}(x_k, y_k) + g_{2k}(x_k, y_k)u_{2k} \end{aligned} \tag{19}$$

where

$$\begin{aligned} f_{1k}(x_k, y_k) &= (1-\alpha)x_k + \frac{a}{1+\exp[-c(x_k - y_k)]} \\ g_{1k}(x_k, y_k) &= \frac{1}{1+\exp[-c(x_k - y_k)]} \\ f_{2k}(x_k, y_k) &= (1-\beta)y_k + \frac{b}{1+\exp[-c(x_k - y_k)]} \\ g_{2k}(x_k, y_k) &= \frac{1}{1+\exp[-c(x_k - y_k)]} \end{aligned} \tag{20}$$

It is assumed that all of the state variables can be measured and fed back. Due to market uncertainties the system parameters are not known while their nominal or estimated values are assumed to be known which are denoted by $\hat{\alpha}$, $\hat{\beta}$, $\hat{a}$, $\hat{b}$ and $\hat{c}$. So the estimated functions $\hat{f}_{ik}$ and $\hat{g}_{ik}$ for $i = 1,2$ are obtained as:

$$\begin{aligned} \hat{f}_{1k}(x_k, y_k) &= (1-\hat{\alpha})x_k + \frac{\hat{a}}{1+\exp[-\hat{c}(x_k - y_k)]} \\ g_{1k}(x_k, y_k) &= \frac{1}{1+\exp[-\hat{c}(x_k - y_k)]} \\ \hat{f}_{2k}(x_k, y_k) &= (1-\hat{\beta})y_k + \frac{\hat{b}}{1+\exp[-\hat{c}(x_k - y_k)]} \\ \hat{g}_{2k}(x_k, y_k) &= \frac{1}{1+\exp[-\hat{c}(x_k - y_k)]} \end{aligned} \tag{21}$$

The upper limits of the errors between the actual and nominal values of parameters are assumed to be specified and denoted by $\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_a, \varepsilon_b$ and ε_c,

$$\begin{aligned} &|\alpha - \hat{\alpha}| < \varepsilon_\alpha, \qquad |\beta - \hat{\beta}| < \varepsilon_\beta, \\ &|a - \hat{a}| < \varepsilon_a, \qquad |b - \hat{b}| < \varepsilon_b, \\ &|c - \hat{c}| < \varepsilon_c \end{aligned} \tag{22}$$

Two sliding surfaces are considered for designing the control actions,

$$S_{1k} = e_{1k} + \alpha_1 e_{1k-1}, \qquad S_{2k} = e_{2k} + \alpha_1 e_{2k-1} \tag{23}$$

where

$$e_{1k} = x_k - x_k^f, \qquad e_{2k} = y_k - y_k^f \tag{24}$$

and α_1 is selected such that $|\alpha_1| < 1$. Regarding Eq. (9) the control inputs, u_{1k} and u_{2k} are calculated as:

$$u_{1k} = -\frac{1}{\hat{g}_{1k}}\left(\left(\hat{f}_{1k} - x_k^f\right) + \alpha_1 e_{1k} + \varepsilon_1.\mathrm{sgn}\left(S_{1k}\right)\right)$$

$$u_{2k} = -\frac{1}{\hat{g}_{2k}}\left(\left(\hat{f}_{2k} - y_k^f\right) + \alpha_1 e_{2k} + \varepsilon_2.\mathrm{sgn}\left(S_{2k}\right)\right) \qquad (25)$$

where ε_i, $i = 1,2$ are obtained according to Eq. (11) as $\varepsilon_i = \eta H_i$, $i = 1,2$ where:

$$H_i \ge \left|f_{ik} - \hat{f}_{ik}\right| + \left|g_{ik} - \hat{g}_{ik}\right| U_{im}, \quad i = 1,2 \qquad (26)$$

U_{im} is the upper bound limit for control action u_i. It is to be noted that by applying a saturation limit for control action one should provide a bounded limit for U_{im}. In addition, manipulating some calculations, for sufficiently small ε_c, the upper bounds of $\left|f_{ik} - \hat{f}_{ik}\right|$ and $\left|g_{ik} - \hat{g}_{ik}\right|$ can be derived as:

$$\left|f_{1k} - \hat{f}_{1k}\right| \le \varepsilon_\alpha |x_k| + \varepsilon_a + |\hat{a}|\left|\frac{1}{1+\exp[-(\hat{c}-\varepsilon_c)|x_k - y_k|]} - \frac{1}{1+\exp[-\hat{c}|x_k - y_k|]}\right| \qquad (27\text{-a})$$

$$\left|f_{2k} - \hat{f}_{2k}\right| \le \varepsilon_\beta |y_k| + \varepsilon_b + |\hat{b}|\left|\frac{1}{1+\exp[-(\hat{c}-\varepsilon_c)|x_k - y_k|]} - \frac{1}{1+\exp[-\hat{c}|x_k - y_k|]}\right| \qquad (27\text{-b})$$

$$|g_{1k} - \hat{g}_{1k}| = |g_{2k} - \hat{g}_{2k}| \le \left|\frac{1}{1+\exp[-(\hat{c}-\varepsilon_c)|x_k - y_k|]} - \frac{1}{1+\exp[-\hat{c}|x_k - y_k|]}\right| \qquad (27\text{-c})$$

Therefore

$$H_1 = \varepsilon_\alpha |x_k| + \varepsilon_a + \left(|\hat{a}| + U_{1m}\right)\left|\frac{1}{1+\exp[-(\hat{c}-\varepsilon_c)|x_k - y_k|]} - \frac{1}{1+\exp[-\hat{c}|x_k - y_k|]}\right|$$

$$H_2 = \varepsilon_\beta |y_k| + \varepsilon_b + \left(|\hat{b}| + U_{2m}\right)\left|\frac{1}{1+\exp[-(\hat{c}-\varepsilon_c)|x_k - y_k|]} - \frac{1}{1+\exp[-\hat{c}|x_k - y_k|]}\right| \qquad (28)$$

To satisfy the condition of Eq. (16) consider the following expression:

$$\left|1 - \frac{g_{ik}}{\hat{g}_{ik}}\right| = \left|1 - \frac{1+\exp[-\hat{c}(x_k - y_k)]}{1+\exp[-c(x_k - y_k)]}\right| \le \max\left\{\begin{matrix}\left|1-\exp\left(\varepsilon_c |x_k - y_k|\right)\right|, \\ \left|1-\exp\left(-\varepsilon_c |x_k - y_k|\right)\right|\end{matrix}\right\} \qquad (29)$$

So for sufficiently small ε_c, Eq. (16) will be satisfied.

Numerical simulation for $\alpha = 0.46$, $\beta = 0.7$, $a = 0.16$, $b = 0.9$, $c = 105$, $\hat{\alpha} = 0.42$, $\hat{\beta} = 0.74$, $\hat{a} = 0.19$, $\hat{b} = 0.86$, $\hat{c} = 101$, $\varepsilon_\alpha = 0.05$, $\varepsilon_\beta = 0.05$, $\varepsilon_a = 0.05$, $\varepsilon_b = 0.05$ and $\varepsilon_c = 5$ is shown in Fig. (2). The control parameters are set as $\alpha_1 = 0.1$, and $U_{im} = 0.13$. It is observed that the fixed point of the system is asymptotically stabilized in a zigzag manner.

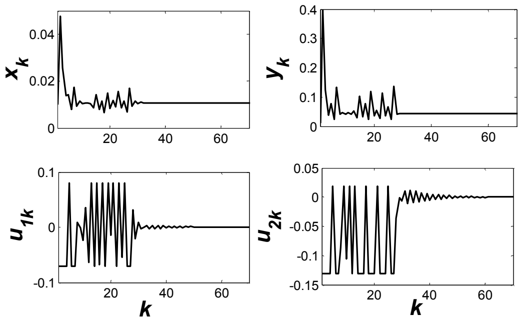

Figure 2. Time series of x_k, y_k, u_{1k} and u_{2k} in controlled Behrens-Feichtinger model.

4 Chaos Control of Cournot-Puu Model

The Cournot-Puu model denotes a duopoly in which two suppliers denoted by X and Y face many consumers on the demand side of the market. Production quantities of the two firms in time period k, are dented by x_k and y_k, and satisfy the following dynamic system:

$$x_{k+1} = \sqrt{\frac{y_k}{c_1}} + y_k$$
$$y_{k+1} = \sqrt{\frac{x_k}{c_2}} + x_k \qquad (30)$$

where c_1 and c_2 are the marginal costs of production for the two firms. The dynamic system (30) has two first order fixed points: one is the trivial point, $(0,0)$ and the other one is a non-trivial point, (x_k^f, y_k^f). Our concern is on the non-trivial fixed point and thus no further consideration is given to the trivial one. The non-trivial point, called the Cournot equilibrium (or Nash equilibrium), is the intersection of the reaction curves, given by:

$$x_k^f = \frac{c_2}{(c_1 + c_2)^2}, \quad y_k^f = \frac{c_1}{(c_1 + c_2)^2} \qquad (31)$$

Stability of the Cournot equilibrium points has been studied extensively [19]. The ratio of marginal costs is defined as $c_r = c_2 / c_1$. Here without loss of generality it is assumed that $c_2 \ge c_1$. The fixed point in Eq. (31) is asymptotically stable if and only if $1 \le c_r < 3 + \sqrt{8}$. For $3 + \sqrt{8} < c_r \le \frac{25}{4}$ the system lies in a period doubling bifurcation process which routes the system behavior toward chaos. Here $c_r \le \frac{25}{4}$ guarantees positive productions to both firms. Figure 3 shows the bifurcation diagram of the firm Y's production, y_k, with respect to c_r. It can be seen that after the first cycle there is a period doubling bifurcation cascade to chaos. Figure 3(b) is a zoomed part of the bifurcation diagram in Fig. 3(a).

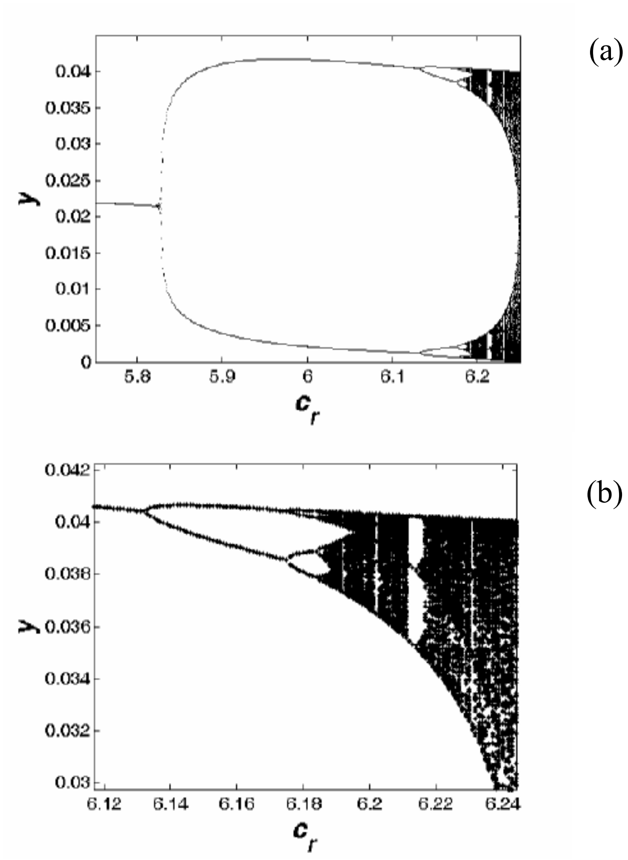

Figure 3. Bifurcation diagram of the second firm production versus the ratio of marginal costs, in Cournot-Puu model

For stabilizing the unstable fixed point of the system, when the system parameters have some uncertainties, the proposed quasi-sliding mode algorithm is used. In this case study, the control action is applied directly to the first firm:

$$\begin{aligned} x_{k+1} &= \sqrt{\frac{y_k}{c_1}} + y_k + u_k \\ y_{k+1} &= \sqrt{\frac{x_k}{c_2}} + x_k \end{aligned} \qquad (32)$$

Substituting the second relation of Eq. (32) into the first one, it is obtained that:

$$x_{k+1} = \sqrt{\frac{1}{c_1}\left(\sqrt{\frac{x_{k-1}}{c_2}} + x_{k-1}\right)} + \sqrt{\frac{x_{k-1}}{c_2}} + x_{k-1} + u_k \qquad (33)$$

By setting:

$$\begin{aligned} f_k(x_k, x_{k-1}) &= \sqrt{\frac{1}{c_1}\left(\sqrt{\frac{x_{k-1}}{c_2}} + x_{k-1}\right)} + \sqrt{\frac{x_{k-1}}{c_2}} + x_{k-1} \\ g_k &= 1 \end{aligned} \qquad (34)$$

Eq. (33) is rewritten as:

$$x_{k+1} = f_k(x_k, x_{k-1}) + g_k u_k \qquad (35)$$

which has the same form of Eq. (1). Again the actual values of the system parameters, c_1 and c_2 are assumed to be unknown, and their estimated or nominal values are denoted by $\hat{c}_1$ and $\hat{c}_2$. The upper limits of uncertainties are specified and denoted by ε_{c_1} and ε_{c_2}:

$$\left|c_1 - \hat{c}_1\right| < \varepsilon_{c_1} \;, \qquad \left|c_2 - \hat{c}_2\right| < \varepsilon_{c_2} \qquad (36)$$

Using the estimated values of parameters, the estimated functions, $\hat{f}_k$ and $\hat{g}_k$ are calculated as:

$$\begin{aligned} \hat{f}_k(x_k, x_{k-1}) &= \sqrt{\frac{1}{\hat{c}_1}\left(\sqrt{\frac{x_{k-1}}{\hat{c}_2}} + x_{k-1}\right)} + \sqrt{\frac{x_{k-1}}{\hat{c}_2}} + x_{k-1} \\ \hat{g}_k &= 1 \end{aligned} \qquad (37)$$

Now by setting the tracking error and the sliding surface as:

$$e_k = x_k - x_k^f \qquad (38)$$

$$S_k = e_k + \alpha_1 e_{k-1} \qquad (39)$$

and regarding Eq. (9) the control action is obtained as:

$$u_k = -\left(\hat{f}_k - x_k^f + \alpha_1 e_k + \varepsilon . \operatorname{sgn}\left(S_k\right)\right) \qquad (40)$$

where $\varepsilon = \eta H$, η is an arbitrary number larger than unity and H is calculated regarding Eq. (12):

$$H \geq \left|f_k - \hat{f}_k\right| \qquad (41)$$

For evaluating a proper value for H, one may use the upper bounds ε_{c_1} and ε_{c_2} as following:

$$\begin{aligned} \left|f_k - \hat{f}_k\right| \leq & \left|\sqrt{\frac{1}{\hat{c}_1 - \varepsilon_{c_1}}\left(\sqrt{\frac{x_{k-1}}{\hat{c}_2 - \varepsilon_{c_2}}} - x_{k-1}\right)} - \sqrt{\frac{1}{\hat{c}_1}\left(\sqrt{\frac{x_{k-1}}{\hat{c}_2}} - x_{k-1}\right)}\right| \\ & + \left|\sqrt{\frac{x_{k-1}}{\hat{c}_2 - \varepsilon_{c_2}}} - \sqrt{\frac{x_{k-1}}{\hat{c}_2}}\right| \end{aligned} \qquad (42)$$

So it can be let:

$$\begin{aligned} H = & \left|\sqrt{\frac{1}{\hat{c}_1 - \varepsilon_{c_1}}\left(\sqrt{\frac{x_{k-1}}{\hat{c}_2 - \varepsilon_{c_2}}} - x_{k-1}\right)} - \sqrt{\frac{1}{\hat{c}_1}\left(\sqrt{\frac{x_{k-1}}{\hat{c}_2}} - x_{k-1}\right)}\right| \\ & + \left|\sqrt{\frac{x_{k-1}}{\hat{c}_2 - \varepsilon_{c_2}}} - \sqrt{\frac{x_{k-1}}{\hat{c}_2}}\right| \end{aligned} \qquad (43)$$

Considering g_k and $\hat{g}_k$ it is obvious that condition of Eq. (16) is trivially satisfied. So the control action of Eq. (40) will satisfy the quasi-sliding mode conditions. The system and control parameters are set to $c_1 = 1$, $c_2 = 6.20$, $\hat{c}_1 = 1.05$, $\hat{c}_2 = 6.15$, $\varepsilon_{c_1} = \varepsilon_{c_2} = 0.1$ and $\alpha_1 = 0.1$ for simulation. The results are shown in Fig. (4). As it is observed the fixed point of the system is stabilized and the control action converge to zero in a zigzag manner.

It must be noted that if x_k approaches x_k^f then y_k will approach y_k^f too. To show this, assume that $x_k = x_k^f$ and reconsider Eq. (30):

$$y_{k+1} = \sqrt{\frac{x_k^f}{c_2}} + x_k^f = const = y_k^f$$

So y_k will be also fixed and equals to y_k^f .

5 Conclusion

In this paper the problem of chaos control in uncertain economic models is investigated. To this purpose the quasi-sliding mode method is adapted and utilized for stabilization of unstable fixed points of system. The idea is examined by applying it to two chaotic models of market, the Behrens-Feichtinger and the Cournot-Puu models. Simulation results show that the presented algorithm may be successfully applied to chaotic markets to obtain regular and stable ones.

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