ON SIMULTANEOUS OPTIMIZATION OF PROGRAM AND DISTURBED MOTIONS AND OPTIMALITY CONDITIONS IN THE FORM OF PONTRYAGIN MAXIMUM PRINCIPLE

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Abstract

This paper considers the problem of simultaneous optimization of the program motion and a bundle of perturbed motions for a nonlinear controlled dynamic system. Such problems arise in the study of optimization of charged particle beam dynamics in accelerating and focusing structures. When solving the problem of stabilizing the program motion, we also encounter the significant dependence of the perturbed motions on the selected program motion and the need to take this into account. The sum of a pair of Bolza type functionals is considered. One is defined on the program motion, the other is defined on a bundle of trajectories. Necessary optimality conditions are given. Moreover, in the stationary case, the function that attains a maximum under optimal control is constant, just as under optimal control of an individual trajectory.

Key words

Simultaneous optimization, Pontryagin maximum principle, optimality conditions.

Introduction

Mathematical optimization methods are widely used in various fields of science and technology. In the study of controlled systems, the maximum principle of L. S. Pontryagin plays a significant role [Pontryagin et al., 1969]. A special place in the control of dynamic systems is occupied by problems related to the control of a beam (ensemble) of trajectories, when the initial state of the controlled system belongs to a certain set [Kurzhanskii, 1977; Ovsyannikov, 1980]. Trajectory beam control problems arise when optimizing complex electrophysical equipment, in particular, when studying the dynamics of charged particles in accelerators [Ovsyannikov, 1990; Ovsyannikov et al., 2006]. Various control problems of continuous and discrete [Bortakovskii, 2020; Golovkina and Ovsyannikov, 2020; Nikolskii and Belyaevskikh, 2018], deterministic and stochastic dynamic systems [Karane and Panteleev, 2022; Vladimirova et al., 2015], as well as feedback systems, are also considered in conditions of uncertainty in setting the initial conditions of motion and incomplete information about the parameters of the control object [Zavadskiy et al., 2023; Panteleev and Semenov, 1992]. The theory of beam control can be used, in particular, to solve problems of optical and non-optical flows [Kotina et al., 2022; Larochkin et al., 2025; Kotina et al., 2024].

The paper investigates a nonlinear controlled dynamic system. The problem of simultaneous optimization of a software and a beam of perturbed motions is considered [Ovsyannikov, 2000; Ovsyannikov, 2006; Ovsyannikov et al., 2009]. A new representation of the functional variation and, accordingly, a new form of the maximum principle is given. It is shown that the function, which reaches its maximum under optimal control, is constant in the stationary case. A similar result occurs when controlling a single trajectory [Pontryagin et al., 1969].

The problems of simultaneous optimization of program and perturbed motions arise when optimizing the dynamics of charged particles in accelerators, as well as when studying the problems of program motion stabilization.

When optimizing perturbed movements, it is reasonable simultaneously consider the program motion. Thus, when studying the dynamics of a beam of charged particles in accelerators, it turned out that it is most effective to optimize the dynamics of a beam of particles with the joint optimization of the program (motion of a synchronous particle) and an ensemble of perturbed motions [Ovsyannikov, 2013; Ovsyannikov, 2014].

The paper considers a functional that takes into account both the terminal characteristics of the program and the beam of perturbed motions, as well as their characteristics over the entire considered interval. In this case, the density of the particle distribution along the perturbed motions is also taken into account.

1 Problem statement

Let us consider a controlled dynamical system described by a system of ordinary differential equations with initial conditions of the following form:

$$\frac{dx}{dt} = f(t, x, u), \tag{1}$$

$$\frac{dy}{dt} = F(t, x, y, u), \tag{2}$$

$$x(0) = x_0, \tag{3}$$

$$y(0) = y_0 \in M_0. \tag{4}$$

Along with the system (1, 2), let us consider the equation for changing the density of the particle distribution $\rho = \rho_t = \rho(t, y(t))$ along the trajectories of subsystem (2) with the law $\rho_0(y_0)$ of the particle density distribution on the set M_0 specified at the initial moment:

$$\frac{\rho}{dt} = -\rho \cdot div_y F(t, x, y, u), \tag{5}$$

$$\rho(0) = \rho(0, y(0)) = \rho_0(y_0), y_0 \in M_0. \tag{6}$$

Here $t \in T_0 = [0, T] \subset R^1$ – independent variable (as a rule, time); $x \in R^n$ and $y \in R^m$ – vectors of phase variables $x_1, x2, \ldots, x_n$ and $y_1, y2, \ldots, y_m$ sizes n and m respectively; $u \in R^r$ – r-dimensional control vector-function; T – fixed number. The vector-functions $f(t, x, u)$ and $F(t, x, y, u)$ are assumed to be definite and continuous with respect to the totality of arguments on the sets $T_0 \times \Omega_x \times U$ and $T_0 \times \Omega_x \times \Omega_y \times U$ together with its partial derivatives in x and y – $\frac{\partial f_i}{\partial x_k}$, where $i, k \in \{1, 2, \ldots, n\}$ and $\frac{\partial F_i}{\partial x_k}, \frac{\partial F_i}{\partial y_j}, \frac{\partial^2 F_i}{\partial x_k \partial y_j}, \frac{\partial^2 F_i}{\partial y_l \partial y_j}$ where $k \in \{1, 2, \ldots, n\}, i, j, l \in \{1, 2, \ldots, m\}$ The set of nonzero measure $M_0 \subset \Omega_y$ will be considered compact; the point $x_0 \in \Omega_x$; $\rho_0(y_0)$ is some non-negative continuous function.

Acceptable controls $u = u(t), t \in T_0$, form a certain class D of piecewise continuous functions on the interval $[0, T]$, that take values from a compact set U. *By piecewise continuous functions, we mean functions that have only a finite number of discontinuities of the first kind.*

Definition 1. By program motion, we will further understand the solution of subsystem (1) under the initial condition (3).

Definition 2. Solutions of subsystem (2) with initial conditions (4) for a fixed program motion will be called disturbed (perturbed) motions.

Remark. Controls for programmed and perturbed movements can be different. However, without losing generality, we can denote them by u, assuming that different components of this control may or may not simultaneously enter the right-hand sides of systems (1) and (2).

Let's say that subsystem (1) describes the dynamics of program motion, and subsystem (2) describes the dynamics of a bundle of trajectories or the dynamics of perturbed movements according to initial conditions.

The mathematical control model under study describes a fairly general situation encountered in practice. We can consider separately the task of optimizing program motion by selecting the appropriate control function. Subsystem (2), in particular, can be considered as an equation in deviations from the program motion. The choice of control and program motion affects the decisions of subsystem (2), which directly depends on them. It can be solved in stages: first, the task of finding a program motion, and then the tasks of stabilizing and optimizing transients processes, for example, by the deviation of the initial data from the specified ones. However, this is not always advisable, because the dependence of perturbed movements on program motion is not taken into account.

The problem arises of simultaneous optimization of program motion and an ensemble of perturbed motions or a bundle of trajectories originating from the set M_0, characterizing the area of permissible deviations of the initial data in y_0.

2 Functional of quality

We introduce a functional that simultaneously evaluates the dynamics of the program motion and the dynamics of the particle beam, taking into account their distribution density, for further joint optimization. This functional will be determined based on the solutions of system (1), (2) and equation (5) under the appropriate initial conditions (3), (4), (6) and the selected control $u(t)$:

$$I(u) = I_1(u) + I_2(u), \tag{7}$$

where

$$I_1(u) = \int_0^T \varphi_1(t, x(t), u(t))\, dt + g_1(x(T)), \tag{8}$$

$$I_2(u) = \int_0^T \int_{M_{t,u}} \varphi_2(t, x(t), y_t, \rho(t, y_t), u(t))\, dy_t\, dt + \\ + \int_{M_{T,u}} g_2(y_T, \rho(T, y_T))\, dy_T, \tag{9}$$

Here, the set $M_{t,u}$ is the cross-section at time t of the bundle of trajectories of subsystem (2) originating from the set M_0 under control of $u(t)$ and the corresponding program motion $x(t)$. The functions $\varphi_1, \varphi_2, g_1, g_2$ are non-negative continuously differentiable functions of their arguments.

The task of minimizing the functional (7) will be called the task of simultaneous (joint) optimization of the program motion and particle beam dynamics. The acceptable control $u^0(t) \in D$, which provides a minimum to the functional (7), will be called optimal control.

3 Auxiliary functions

Let the auxiliary (conjugate) functions $\psi(t)$, $\nu(t, y_t)$, $\mu(t, y_t)$, $\lambda(t, y_t)$ satisfy along the trajectories of the system (1), (2), (5) the following equations and conditions are at the right end:

$$\frac{d\psi}{dt} = -\left(\frac{\partial f}{\partial x}\right)^* \psi + \left(\frac{\partial \varphi_1}{\partial x}\right)^*, \quad (10)$$

$$\frac{d\nu}{dt} = -\left(\frac{\partial f}{\partial x}\right)^* \nu - \left(\frac{\partial F}{\partial x}\right)^* \mu + \rho_t \left(\frac{\partial div_y F}{\partial x}\right)^* \lambda + \frac{1}{\rho_t}\left(\frac{\partial \varphi_2}{\partial x}\right)^*, \quad (11)$$

$$\frac{d\mu}{dt} = -\left(\frac{\partial F}{\partial y}\right)^* \mu + \rho_t \left(\frac{\partial div_y F}{\partial y}\right)^* \lambda + \frac{1}{\rho_t}\left(\frac{\partial \varphi_2}{\partial y}\right)^*, \quad (12)$$

$$\frac{d\lambda}{dt} = (div_y F)\lambda - \frac{1}{(\rho_t)^2}\left(\varphi_2 - \frac{\partial \varphi_2}{\partial \rho}\rho_t\right), \quad (13)$$

$$\psi(T) = -\left(\frac{\partial g_1}{\partial x_T}\right)^*, \quad (14)$$

$$\nu(T) = -\frac{1}{\rho_T}\left(\frac{\partial g_2}{\partial x_T}\right)^*, \quad (15)$$

$$\mu(T) = -\frac{1}{\rho_T}\left(\frac{\partial g_2}{\partial y_T}\right)^*, \quad (16)$$

$$\lambda(T) = \frac{1}{(\rho_T)^2}\left(g_2 - \frac{\partial g_2}{\partial \rho}\rho_T\right). \quad (17)$$

Let's introduce the functions:

$$H_1 = \psi^* \cdot f(t, x, v) - \varphi_1(t, x, v), \quad (18)$$

$$H_2 = \nu^* \cdot f(t, x, v) + \mu^* \cdot F(t, x, y, v) - \lambda \cdot \rho div_y F(t, x, y, v) - \frac{1}{\rho}\varphi_2(t, x, y, \rho, v). \quad (19)$$

Note that the function H_1 can be considered as the Hamiltonian of the combined system (1), (10) for variables x and ψ, since the relations obviously hold:

$$\frac{dx}{dt} = \left(\frac{\partial H_1(t, x(t), \psi(t), u(t))}{\partial \psi}\right)^*, \quad (20)$$

$$\frac{d\psi}{dt} = -\left(\frac{\partial H_1(t, x(t), \psi(t), u(t))}{\partial x}\right)^*. \quad (21)$$

Variables ψ are called conjugate to variables x, and system (21) is called the conjugate system of equations for system (20).

Note also that the function H_2, in turn, is the Hamiltonian of the system obtained by combining the systems (1), (2), (5) and the corresponding auxiliary systems (3)-(13) for variables x, y, ρ and ν, μ, λ, respectively, since the relations are fulfilled

$$\begin{cases} \frac{dx}{dt} = \left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial \nu}\right)^* \\ \frac{dy}{dt} = \left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial \mu}\right)^* \\ \frac{d\rho}{dt} = \left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial \lambda}\right)^* \end{cases}, \quad (22)$$

$$\begin{cases} \frac{d\nu}{dt} = -\left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial x}\right)^* \\ \frac{d\mu}{dt} = -\left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial y}\right)^* \\ \frac{d\lambda}{dt} = -\left(\frac{\partial H_2(t,x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))}{\partial \rho}\right)^* \end{cases}. \quad (23)$$

The variables ν, μ, λ will be called conjugate to the variables x, y, ρ, and system (23) will be called the conjugate system of equations for system (22).

4 Representation of a functional variation

Let us choose a certain control function $u(t)$. Let us consider corresponding to it the program motion $x(t)$, a bundle of perturbed motion trajectories y_t, and also ρ_t, which are the densities of the particle distribution along them. For shortness, the set of these values will be called the selected process.

Let's introduce the function $H(t, v)$ for some selected process:

$$H(t, v) = H_1(t, x(t), \psi(t), v) + \int_{M_{t,u}} H_2(t, x(t), y_t, \rho_t, \nu_t, \mu_t, v)\rho_t \, dy_t,$$

where, as all arguments (except the v) in the functions H_1 and H_2 the functions of the same name of the selected process and the corresponding auxiliary functions $\psi(t)$, $\nu_t = \nu(t, y_t)$, $\mu_t = \mu(t, y_t)$, $\lambda_t = \lambda(t, y_t)$ are substituted.

Then the variation of functional (7) can be written in the following form

$$\delta I = -\int_0^T \Delta_v H(t, v)|_{v=u(t)} \, dt = -\int_0^T \Delta_u H_1(t, x(t), \psi(t), u(t)) \, dt - \int_0^T \int_{M_{t,u}} \Delta_u H_2(t, x(t), y_t, \rho_t, \nu_t, \mu_t, \lambda_t, u(t)) \cdot \rho_t \, dy_t \, dt.$$

5 Necessary optimality conditions

Theorem 1. Let $u^0 = u^0(t)$ be the optimal control that corresponds to the optimal selected process. Then for all $t \in T_0 = [0, T]$ the following condition holds:

$$\max_{v \in U} H^0(t, v) = H^0(t, u^0(t)). \quad (24)$$

Here, the function $H^0(t,v)$ is the function $H(t,v)$ for the optimal selected process $x^0(t)$, y_t^0, ρ_t^0, $u^0(t)$ with the corresponding auxiliary functions $\psi^0(t)$, $\nu_t^0 = \nu^0(t,y_t)$, $\mu_t^0 = \mu^0(t,y_t)$, $\lambda_t^0 = \lambda^0(t,y_t)$.

Theorem 1 provides the necessary conditions for the optimality of the process and can also be understood as an extension of the Pontryagin maximum principle for the introduced generalized functional.

The selected process $x(t)$, y_t, ρ_t, $u(t)$ will be called a process satisfying the necessary optimality conditions if the condition of Theorem 1 is fulfilled for it:

$$\max_{v\in U} H(t,v) = H(t,u(t)).$$

6 The constancy of the function *H(t,u(t))* in the stationary case

Let us consider in more detail in the stationary case the properties of the function $H(t,v)$ for some selected process satisfying the necessary optimality conditions. By the stationary case, we will understand the case when the right-hand sides of the studied system of differential equations (1), (2) and integral functions in functionals (8) and (2) do not explicitly depend on time.

Then the functions $H_1 = H_1(x,\psi,v)$ and $H_2 = H_2(x,y,\rho,\nu,\mu,\lambda,v)$ can be introduced as follows and will not explicitly depend on time:

$$H_1 = \psi^* \cdot f(x,v) - \varphi_1(x,v),$$
$$H_2 = \nu^* \cdot f(x,v) + \mu^* \cdot F(x,y,v) -$$
$$-\lambda \cdot \rho div_y F(x,y,v) - \frac{1}{\rho}\varphi_2(x,y,\rho,v).$$

The function $H(t,v)$ will take the form:

$$H(t,v) = H_1(x(t),\psi(t),v) +$$
$$+\int_{M_{t,u}} H_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,v)\rho_t\, dy_t.$$

Note that all the properties and statements for the functions H_1, H_2 and $H(t,v)$ from the general (nonstationary) case remain valid.

Let's introduce the function $M(t) = H(t,u(t))$:

$$M(t) = H_1(x(t),\psi(t),u(t)) +$$
$$+\int_{M_{t,u}} H_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,u(t))\rho_t\, dy_t.$$

Here $u(t)$ is the control that corresponds to the selected process.

Theorem 2. Let us consider a process satisfying the necessary optimality condition. Then for all $t \in T_0 = [0,T]$, the following condition holds:

$$M(t) = H(t,u(t)) \equiv const. \tag{25}$$

Theorem 2 provides the necessary conditions for the optimality of the process in the stationary case. The following lemma is used to prove Theorem 2.

Lemma. Let some function $H(t,v)$ defined for $t \in T_0 = [0,T]$ and $v \in U$ be a continuous function with respect to the totality of its arguments, as well as its partial derivative $\frac{\partial}{\partial t}H(t,v)$. Let also $u(t)$ be a piecewise continuous function defined at $t \in T_0$, taking values from the set U. If the maximum condition is fulfilled for them for all t: $\max_{v\in U} H(t,u(t))$ (or a weaker condition $H(t,u(s)) \le H(t,u(t))$ for all s and t), and at the points of continuity of the function $u(t)$ in addition $\frac{\partial H(t,v)}{\partial t}|_{v=u(t)} = 0$, then $H(t,u(t)) \equiv const$ for all $t \in T_0$.

The proof of the lemma is carried out similarly to the proof of the theorem from Boltyansky's book [Boltyanskii, 1966] on the constancy of the function H.

Now let's check that the function $M(t) = H(t,u(t))$ from Theorem 2 meets the conditions of the lemma. This obviously follows from its definition and properties discussed earlier. Note, in particular, that the following equalities will be true:

$$\left.\frac{dH_1(x(t),\psi(t),v)}{dt}\right|_{v=u(t)} =$$
$$= \left(\frac{\partial H_1(x(t),\psi(t),v)}{\partial x}\frac{dx(t)}{dt} + \right.$$
$$\left.\left. + \frac{\partial H_1(x(t),\psi(t),v)}{\partial \psi}\frac{d\psi(t)}{dt}\right)\right|_{v=u(t)} =$$
$$= \left(\frac{\partial H_1(x(t),\psi(t),v)}{\partial x}\left(\frac{\partial H_1(x(t),\psi(t),u(t))}{\partial \psi}\right)^* - \right.$$
$$\left.\left. - \frac{\partial H_1(x(t),\psi(t),v)}{\partial \psi}\left(\frac{\partial H_1(x(t),\psi(t),u(t))}{\partial x}\right)^*\right)\right|_{v=u(t)} =$$
$$= 0.$$

There are also equalities:

$$\left.\frac{dH_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,v)}{dt}\right|_{v=u(t)} = 0,$$
$$\frac{d}{dt}\int_{M_{t,u}} H_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,v)\rho_t\, dy_t =$$
$$= \int_{M_{t,u}} \frac{dH_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,v)}{dt}\rho_t\, dy_t,$$
$$\left.\frac{\partial H(t,v)}{\partial t}\right|_{v=u(t)} = \left.\frac{dH_1(x(t),\psi(t),v)}{dt}\right|_{v=u(t)} +$$
$$+\int_{M_{t,u}} \left(\left.\frac{dH_2(x(t),y_t,\rho_t,\nu_t,\mu_t,\lambda_t,v)}{dt}\right|_{v=u(t)}\right)\rho_t\, dy_t,$$

and therefore $\left.\frac{\partial H(t,v)}{\partial t}\right|_{v=u(t)} = 0$.

Thus, the conditions of the lemma are fulfilled, and Theorem 2 is proved.

Conclusion

A nonlinear controlled dynamic system is investigated. The problem of simultaneous optimization of the program and the beam of perturbed motions is set. For the introduced functional, a special representation is obtained for its variation. The necessary optimality conditions have been obtained that extend the Pontryagin maximum principle for the problem under consideration.

It is shown that the function, which reaches its maximum under optimal control, is constant in the stationary case. The obtained statement generalizes the result (known for controlling a single trajectory) to the case of controlling a bundle (beam) of trajectories and the case of simultaneous optimization of program and disturbed motions.

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