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CHAOS IN MULTI-VALUED DYNAMICAL SYSTEMS

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Abstract
This contribution addresses a possible description of the chaotic behavior in multi-valued dynamical systems. An important area leading to description via the multi-valued dynamical systems is the non-smooth dynamical systems theory and their applications. Examples of such applications are mechanics with dry friction, electric circuits with small conductivity, systems with small inertia, economy, biology, control theory, game theory, optimization, etc. The phenomenon of the chaos in multi-valued systems is even more complicated issue than in case of the single-valued ones and deserves to be intensively studied. The most of the existing results proves the existence of chaos in multi-valued systems via an appropriate construction of a homeomorphism between one selected solution from the set of them and the bi-directional full shift of symbols. The approach presented here does not require construction of a selector on the set of solutions and uses a more intuitive and descriptive definition of the chaos. This novel concept is demonstrated on several examples of multi-valued dynamical systems determining the conditions leading to the chaotic behavior.

Key words
Multi-valued dynamical systems, chaos, differential inclusions

1 Introduction

Satisfactory solution of numerous particular problems in mechanics, engineering sciences and other related fields are often heavily influenced by non-smooth phenomena. Imagine the noise of a squeaking chalk on a black-board or, sometimes more pleasantly, the sounds of stringed instruments like a violin. More relevant applications include noise generation in railway wheels, the drilling machines, etc. Physically speaking, these effects often are due to the fact that there are rigid bodies, which are in contact (they "stick"), whereas these contact phases are interrupted by "slip" phase during which one of the bodies moves relative to another. In addition to such behavior mainly induced by friction, there may also be impacts between different parts of the system.

From mathematical viewpoint, problems of this kind are not easy to handle, since the resulting models are dynamical systems whose right-hand sides are non-smooth or even discontinuous. In many cases the solution have to observe additional restrictions that frequently appear in the form of inequality constraints. Since many concepts from classical dynamical systems theory do rely on the smoothness of the underlying system or (semi-) flow, it was necessary to generalize those concepts to cover non-smooth dynamical systems as well, and it turned out that almost always such generalization is a non-trivial issue.

One can found a huge amount of result concerning chaotic behavior in the case of single-valued nonlinear mappings or in the case of single-valued nonlinear differential equations. To our best knowledge, the only one serious result concerning analysis of chaos in the case of multi-valued systems is in [Fečkan, 1999]. In [Fečkan, 1999], a more general case of Coulomb friction description has been studied. The author considered a perturbed system and he assumed that the unperturbed system is periodic and has a homoclinic trajectory. Based on these assumptions he shows that perturbed system has a solution, which is topologically equivalent to the bi-directional shift on a set of symbols.

The goal of this contribution is to generalize the results dealing with chaotic behavior of single valued flows to the case of the so-called *generalized semi-flows*, see precise definition later on. Briefly, the main result of this paper claims that there exists *at least one* trajectory of the generalized semi-flow such that for arbitrary covering of the solution set, possessing certain surjection-like property defined later on, such a trajectory connects mutually all subsets of that covering in a finite time. As a consequence, the trajectory of the generalized dynamical semi-flow can be described (in fact, coded) via the index set of the covering of the solution set and, subsequently, the methods of symbolic dynamics can be used to analyze its dynamics. To compare with [Fečkan, 1999], our results presented in the current paper will show the existence of chaotic behavior as well. Moreover, no *a priori* hypothesis concerning the existence of homo/heteroclical trajectory will be needed here.

To perform this idea in detail, the recent theoretical result of [Beran, 2009] and techniques developed there will be used. The corresponding conditions obtained in [Beran, 2009] are rather general, therefore, *more specific conditions will be developed in this paper that can be used to analyze particular dynamical systems in a much more comfortable way.* Several examples will illustrate these new theoretical results.

The rest of the paper is organized as follows. The next section summarizes some preliminaries and terminology to be used later on. Main results are collected in Section 3, while illustrative examples are given in Section 4. The last short section draws conclusions and provides some outlooks for the future research.

2 Preliminaries

Some basic notions are repeated in this section. Interested reader is referred to [Beran, 2009] for further details.

Definition 1. A *generalized semiflow* $\mathcal{G}$ on metric space (not necessarily complete) X is a family of maps $\varphi : [0,\infty) \longrightarrow X$ (called *solutions*) satisfying the hypotheses:

(H1) (*Existence*). For each $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$.

(H2) (*Translates of solutions are solutions*). If $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then $\varphi^{\tau} \in \mathcal{G}$, where $\varphi^{\tau}(t) \triangleq \varphi(t+\tau), t \in [0,\infty)$

(H3) (*Concatenation*). If $\varphi, \psi \in \mathcal{G}, t \geq 0$, with $\psi(0) = \varphi(t)$ then $\theta \in \mathcal{G}$, where

$$\theta(\tau) \triangleq \begin{cases} \varphi(\tau) \text{ for } 0 \leq \tau \leq t, \\ \psi(\tau - t) \text{ for } t < \tau \end{cases}$$

(H4) (*Upper-semicontinuity with respect to initial data*). Consider the sequence of flows $\{\varphi_j\}_{j=1}^{\infty} \in \mathcal{G}$, with $\varphi_j(0) \to z$ as $j \to \infty$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Remark. Let $\mathcal{G}$ be a generalized semiflow and let $E \subset X$. Define for $t \geq 0$

$$T(t)\,E \triangleq \{\varphi(t)\,|\varphi \in \mathcal{G} \; with \; \varphi(0) \in E\},$$

so that $T(t) : 2^X \longrightarrow 2^X$, where 2^X is the space of all subsets of X. It follows from (H2), (H3) that $\{T(t)\}_{t\geq 0}$ defines a semigroup on 2^X. Note that (H4) implies that $T(t)\{z\}$ is compact for each $z \in X, t \geq 0$.

Notation. The expression $\varphi(\cdot) \in \mathcal{G}(x)$ means the solution $\varphi(\cdot)$ that starts at $x \in X$.

If for each $z \in X$ there is exactly one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ then $\mathcal{G}$ is called a *semiflow*.

Definition 2. The generalized semiflow $\mathcal{G}$ is said to be *upper-semicompact* from X to $\mathcal{C}([0,\infty); X)$ ($\mathcal{C}$ means a space of continuous mappings from $[0,\infty)$ into X) if for any solution $\varphi_n \in X$ converging to $x \in X$ and for any generalized semiflow $\varphi_n(\cdot) \in \mathcal{G}$ starting at φ_n, there exists a subsequence of $\varphi_n(\cdot)$ converging to a generalized semiflow $\varphi(\cdot) \in \mathcal{G}$ uniformly on compact intervals.

Definition 3. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$, $n \in \mathbb{N} \cup 0$, $\mathcal{S} = \{S_n\}$, such that $S_n \cap S_{n+1} \neq \emptyset$. Let $\varphi(\cdot) \in \mathcal{G}(x)$ be a solution. We say that $\mathcal{S}$ *forms a* $\varphi(\cdot)$*-chain* when there exists a nondecreasing sequence of times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots$ such that for all $n \geq 0$, for any $t \in [t_n, t_{n+1}], \varphi(t) \in S_n$ and $\varphi(t_{n+1}) \in S_{n+1}$.

Definition 4. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$ and we assume that there exists $T < +\infty$ such that for each nonnegative n and for each $z \in S_{n+1}$ there exists $x \in S_n$ with solution $\varphi_n(\cdot) \in S_n$ and exists $\tau \in [0,T)$ with $\varphi(\tau) = z$, then the system $\mathcal{S} = \{S_n\}$ is called to be $T - surjective\ under\ \mathcal{G}$. When $T \longrightarrow +\infty$, then the system $\mathcal{S} = \{S_n\}$ is called to be $surjective\ under\ \mathcal{G}$

Definition 5. Let $D \subset X$ be a constrained set. A solution $\varphi(\cdot)$ is *locally positively D-invariant* when there exists $T > 0$ such that for each $t \in [0,T]$ we have $\varphi(t) \in D$. When $T = +\infty$ we call $\varphi(\cdot)$ *positively D-invariant.* When all $\varphi(\cdot) \in \mathcal{G}$ are (locally) positively D-invariant, we say that generalized semiflow $\mathcal{G}$ is (locally) positively D-invariant.

Definition 6. The generalized semiflow $\mathcal{G}$ possesses the *chaotic behaviour* on the compact set D, if for any its at most countable closed covering $\mathcal{S} = \{S_m\}_{m\in\mathcal{I}}$, $D \subset \bigcup_{m\in\mathcal{I}} S_m$, $\mathcal{I}$ being a suitable index set, and any sequence $\{m_0, m_1, \ldots, m_n, \ldots\} \subset \mathcal{I}$ there exists *at least one* solution $\varphi(\cdot) \in \mathcal{G}$ and a nondecreasing sequence $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots$ such that system $\overline{\mathcal{S}} := \{\mathcal{S}_m\}_{m=m_0,m_1,m_2\ldots}$ is surjective under $\varphi(\cdot) \in \mathcal{G}$ with $\varphi(t_i) \in \mathcal{S}_{\Updownarrow_i}, i = 0, 1, \ldots$.

Theorem A [Beran, 2009]. *Let D be a compact subset. Assume:*

1. *generalized semiflow $\mathcal{G}$ is positively D-invariant and upper semicompact,*
2. *let any covering $\mathcal{S}$ be T-surjective under $\mathcal{G}$ for some $T < +\infty$.*

Then the generalized semiflow $\mathcal{G}$ possesses the chaotic behaviour.

3 Main results

In this section, we formulate the main results of our contribution. To start with, some basic definitions are repeated, [Aubin, Cellina, 1984].

Definition 7. Let $J = [a, b] \subset R$. Then we denote by $L^1(J)$ the Banach space of Lebesgue integrable $\psi : J \to \overline{R} = R \cup \{-\infty, \infty\}$ with norm $|\psi|_1 = \int_J |\psi|\, dt$.

Definition 8. Given $X = R^n, J = [0, a] \subset R$, a closed $D \subset X$, a multivalued mapping $F : J \times D \to 2^X \backslash \{\emptyset\}$, we define a norm $\|F(t, x)\| = sup\, \{|y| : y \in F(t, x)\}$.

Definition 9. Tangent cone $T_D(x),\ x \in D$, where $\rho(x, D) = inf_{d\in D}\, |x - d|$ is defined as follows
$T_D(x) = \left\{ y \in X : \underline{lim}_{\lambda\to 0+} \lambda^{-1}\rho\,(x + \lambda y, D) = 0 \right\}$.

To present the main results of the current paper, we restrict ourself to the case of finite-dimensional $X = R^n$. The main object of our investigation is initial value problem

$$\left.\begin{array}{l} \dot{u} = \frac{du}{dt} \in F(t, u) \\ a.e.\ on\ J = [0, a] \\ u(0) = x \in D \subset X \end{array}\right\} \qquad (1)$$

with $F : J \times D \to 2^X \backslash \{\emptyset\}$. Solution of (1) means the Filippov solution, that means following [Filippov, 1988]: $u : [0, \tau[\to R^n$ is an absolutely continuous function so that, for almost all $t \in [0, \tau[$ we have $\frac{du}{dt} \in \tilde{F}(t, u)$, where $\tilde{F}(t, u)$ is a convex regularization of $F(t, u)$, see [Filippov, 1988].

The purpose of the following theorem is to give technical conditions to guarantee the valididty of the Assumption 1. of the Theorem A.

Theorem 1. Let $Y = \{u \in C_X(J) : u(t) \in D\ on\ J\}$ with $|u|_0 = max_J\, |u|$ and be $Sol(x) \subset Y$ the solution set of (1). Let $F(t, \cdot)$ is u.s.c., $F(\cdot, x)$ is measurable, let $\|F(t, x)\| \le c(t)(1+|x|)$ on $J \times D$ and $c \in L^1(J)$. If $F(t, x) \cap T_D(x) \ne \emptyset$ on $[0, a) \times D$, then (1) has an a.c. solution for every $u_0 \in D$ and $Sol(x)$ is compact and $Sol(\cdot) : D \to 2^Y \backslash \{\emptyset\}$ is u.s.c.

Proof. Due to length of the proof, we give here only a sketch. The proof consists from a chain of steps:

1. We assume that D is compact.
2. Using the Gronwall's Lemma, it can be shown that $\|F(t, x)\| \le c(t)(1 + |x|)$ on $J \times D$ implies that $\|F(t, x)\| \le 1$. It is only a technical result.
3. The solution set $Sol(.)$ is upper semi continuous.
4. We define a mapping $p_t : C_X(J) \to X$ by $p_t(u) = u(t),\ t \in J$, That mapping is continuous.
5. We define a set $P_t(x) = p_t \circ Sol(x) = \{u(t) \in X : u \in Sol(x)\},\ t \in J$
6. Then it can be shown that $P_t : D \to 2^D \backslash \{\emptyset\}$ is upper semi continuous and has compact values $\forall\, t \in J$ □

In order to guarantee that also the Assumptions 2 of the Theorem A is satisfied, and also the existence of at least one covering in Definition 6, one needs the global compactness and the connectedness of the solution set Sol. These prerequisites are provided by the following theorem.

Theorem 2. Let $X = R^n, J = [0, a] \subset R$, $D \subset X$ be closed convex, $F : J \times D \to 2^X \backslash \{\emptyset\}$ have closed convex values and be such that $F(\cdot, x)$ has a measurable selection, $F(t, \cdot)$ is u.s.c., $F(t, x) \subset T_D(x)$ on $[0, a) \times D$ and $\|F(t, x)\| \le c(t)(1 + |x|)$ on $J \times D$ with $c \in L^1(J)$. Let $x \in D$. The the set $Sol = \bigcup_{x\in M} Sol(x)$ is compact and connected.

Proof. Proof of that theorem due to its length will be also only sketched:

1. We assume that $D = X$.
2. Using the Gronwall's Lemma, it can be shown that $\|F(t, x)\| \le c(t)(1 + |x|)$ on $J \times D$ implies that $\|F(t, x)\| \le 1$. It is only a technical result.
3. Let $\{\phi_\lambda\}$ is a locally Lipschitz partition of unity subordinate to some locally finite refinement $U_\lambda,\ \lambda \in \Lambda$ of $\{B_{r_n}(x) : x \in X\}$ with $r_n = 3^{-n}$ and x_λ is such that $U_\lambda \subset B_{r_n}(x_\lambda)$.
4. We define approximation $F_n(t, x) = \sum_{\lambda\in\Lambda} \phi_\lambda(x) C_\lambda(t)$ with $C_\lambda(t) = \overline{conv} F(f, B_{2r_n}(x))$ on $J \times X$.
5. It can be shown that $F(t, x) \subset F_{n+1}(t, x) \subset F_n(t, x) \subset \overline{conv} F(f, B_{3r_n}(x))$ on $J \times X$.
6. We have $S(x) \subset S_{n+1} \subset S_n$ and S_n is compact.
7. It can be shown that S_n is contractible.
8. Sequence (u_n) with $u_n \in S_n$ for $n \ge 1$ has a uniformly convergent subsequence with limit in $S9x)$ and we have $dist(S_n,\ S(x)) \to 0$ as $n \to \infty$ because $S(x) \subset S_n$ is compact. As the sets S_n are connected then $S(x)$ cannot be the union of two nonempty disjoint compact subsets. □

In the case of maximal monotone multivalued maps in Hilbert space, which will be useful in our example, e.g. [Aubin, Cellina, 1984], is the situation much more simple.

Definition 9. A multivalued map A from Hilbert space X to Hilbert space Y is called *monotone* if and only if $\forall x_1, x_2 \in Dom(A),\ \forall v_i \in A(X_i),\ i = 1, 2 \implies \langle v_1 - v_2, x_1 - x_2\rangle \ge 0$.

Definition 10. A monotone multivalued map is *maximal* if there is no other monotone multivalued map $\tilde{A}$ whose graph contains strictly the graph of A.

We point out some remarks.

A multivalued map is monotone (maximal monotone) if and only if its inverse A^{-1} is monotone (maximal monotone).

As a consequence of Zorn's lemma is the graph of any monotone multivalued map contained in the graph of maximal monotone multivalued map, because the union of an increasing family of graphs of monotone multivalued maps is obviously the graph of a monotone multi-valued map.

As a direct consequence of the definition, we have the following useful criterion to check if u belongs to $A(x)$.

Theorem 3. A multivalued map A is maximal monotone if and only if the following statements are equivalent:

$$\left.\begin{array}{l}\forall (y,v)\in Graph(A), \langle u-v, x-y\rangle \geq 0\\ u\in A(x).\end{array}\right\} \quad (2)$$

The next theorem guaranties the fulfillment of the conditions of the Theorem 1 and Theorem 2 in the case of maximal monotone multivalued maps:

Theorem 4. Let A be maximal monotone multivalued map. Then:

a Its images are closed and convex

b Its graph is strongly-weakly closed in the sense that if x_n converges to x and if u_n converges weakly to u, then $u\in A(x)$.

Proof. To prove the statement a), one can see that $A(x)$ is the intersection of the closed half-spaces $\{u\in: \langle u-v, x-y\rangle \geq 0\}$ in the case when $(y,v)\in Graph(A)$. So, $A(x)$ is closed and convex.
To prove the statement b), we suppose that x_n converge to x and simultaneously $u_n \in A(x_n)$ converge weakly to u. Let $(y,v)\in Graph(A)$. Then $\langle u_n-v, x_n-y\rangle \geq 0$ implies, by the limiting process, that $\langle u-v, x-y\rangle \geq 0$. Thus, $u\in A(x)$ using the Theorem 3. □

4 Example

In this section, we will apply the theoretical results to two examples.

Example 1 We consider a generalized Lorenz system with discontinuous right hand side $(\dot{x},\dot{y},\dot{z})^\top = f(x,y,z)$ of the form:

$$\begin{array}{l}\dot{x}=-\sigma x+\sigma y\\ \dot{y}=rx-y-Sign(y)\left|x\right|z\\ \dot{z}=-bz+\left|xy\right|\end{array} \quad (3)$$

The differential inclusion associated with this discontinuous system is $(\dot{x},\dot{y},\dot{z})^T \in F(x,y,z)$ where $F(x,y,z)$ is the convex regularization of $f(x,y,z)$, which has the following form:

$$F(x,y,z)=\begin{bmatrix}-\sigma x+\sigma y\\ rx-y-Sign(y)\left|x\right|z\\ -bz+\left|xy\right|\end{bmatrix},\ y\neq 0,$$

$$F(x,y,z)=\begin{bmatrix}-\sigma x\\ \left[rx-\left|xz\right|, rx+\left|xz\right|\right]\\ -bz\end{bmatrix},\ y=0.$$

We try to estimate the domain of existence of the chaotic attractor. In order to do the estimate, we will construct a Lyapunov function to the system. We start with a general form of the Lyapunov function:

$$V(x,y,z)=\alpha(\delta x+\xi)^2+\beta(\epsilon y+\rho)^2+\gamma(\mu z+\tau)^2$$

where $\alpha>0, \beta>0, \gamma>0,\ \delta,\ \epsilon,\ \mu,\ \xi,\ \rho,\ \tau$ are parameters to determine. We evaluate the gradient of V for both cases $y\neq 0,\ y=0$ and we asses a maximum of that gradient from above. For both cases one has the following:

$\max \frac{1}{2}\dot{V}(x,y,z) \leq -\alpha\sigma\delta^2x^2-\beta\epsilon^2y^2-\gamma b\mu^2z^2+\alpha\sigma\delta^2xy+(\beta\epsilon^2 r+\gamma\mu\tau)\left|xy\right|+\beta\epsilon\rho\left|xz\right|+(\gamma\mu^2-\beta\epsilon^2)\left|xy\right|z-(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x-(\beta\epsilon\rho-\alpha\sigma\delta\xi)y-\gamma\mu b\tau z.$

Now, if we choose $\tau=-\beta\epsilon^2 r/\gamma\mu$ and $\gamma\mu^2=\beta\epsilon^2$, the right hand side $c(x,y,z)$ of the above inequality takes the form

$-\frac{1}{2}c(x,y,z) = \alpha\sigma\delta^2x^2+\beta\epsilon^2y^2+\gamma\mu^2bz^2-\alpha\sigma\delta^2xy-\beta\epsilon\rho\left|xz\right|+(\alpha\sigma\delta\xi-\beta r\epsilon\rho)x+(\beta\epsilon\rho-\alpha\sigma\delta\xi)y+\gamma\mu b\tau z.$

One can easily see that the function $c(x,y,z)$ has the form of general quadric, so we can use a set of transformations in order to simplify the form of the function $c(x,y,z)$. After a lengthy calculations and final choice of parameters $\alpha=\sigma,\ \beta=\gamma=\epsilon=\mu=1,\ \xi=\rho=0,\ \tau=-r, \delta=1/\sigma$ the function $c(x,y,z)$ takes the form

$$-\frac{1}{2}c(x,y,z)=x^2+y^2+bz^2-xy-rbz$$

The last evaluation that is needed is to find out the

$$\sup_{\{(x,y,z):c(x,y,z)<0\}} V(x,y,z).$$

To do that, we utilize the very well known Lagrange multipliers method. After a rather long calculations, we get the result

$$\sup_{\{(x,y,z):c(x,y,z)<0\}} V(x,y,z)=r^2.$$

Consequently, we have estimated the domain of existence of the chaotic attractor by the manifold

$$\frac{1}{\sigma}x^2+y^2+(z-r)^2\leq r^2.$$

Due to compactness of that domain and due to the convex regularization of the equation (3), we can conclude that all the conditions of Theorem A are satisfied thus we have proved the existence of a chaotic solution in the above domain. In fact, if we choose $\sigma=10,\ r=28.5,\ b=2.5$, we obtain the traditional chaotic solution of the Lorenz system. We can conclude that the traditional chaotic solution of the Lorenz system is one of the more possible chaotic solutions of the general differential inclusion (3).

Example 2 Now, we will apply previous theoretical results to a practical problem. We have chosen a problem of modeling of the pendulum with friction, [Awrejcewicz, Lamarque, 2003].

Let us consider a forced pendulum with a viscous damping and Coulomb friction. This pendulum corresponds to the model:

$$\ddot{x} + a\dot{x} + \lambda sin(x) + \alpha Sign(\dot{x}) - f(t) \ni 0$$

where $\dot{x} := \frac{dx}{dt},\ \ddot{x} := \frac{d^2x}{dt^2},\ \lambda \in R,\ \alpha \in R^+$ and $Sign(\cdot)$ denotes the graph of the function $Sign(u) = -1$ if $u < 0, Sign(u) = +1$ if $u > 0, Sign(u) = [-1, 1]$ if $u = 0$. This model has to be understood as a differential inclusion. $\alpha Sign(\dot{x})$ is the expression of a Coulomb friction applied to the pendulum. Hereafter, we choose a particular expression of the external forcing $f(t) = f\sin(\omega t),\ f \in R,\ \omega \in R$.

The previous model can be written in an obvious way in the form of a first order differential inclusion:

$$\dot{Y} + \begin{bmatrix} y_2 \\ ay_2 + \lambda\sin(y_1) - f(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha Sign(y_2) \end{bmatrix} \ni \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This relation gives that

$$\dot{Y} + F(Y,t) + H(Y) \ni 0, \quad (4)$$

where $Y = (y_1, y_2)^\top = (x, \dot{x})^\top$. The initial conditions are $Y(t_0) = Y_0 = (x_0, \dot{x_0})^\top$ and $x_0 \in [-\pi, \pi] \ni \dot{x_0}$.

Moreover, F is clearly a Lipschitz-continuous map: for usual Euclidean scalar product $(\cdot,\cdot)$ and its associated norm $\|\cdot\|$ of R^2 it holds that: $\forall t \in R,\ \forall (Y,Z) \in R^2 \times R^2,\ Y = (y_1,y_2)^T,\ Z = (z_1,z_2)^T$ we have $\|F(Y,t) - F(Z,t)\| \le (1 + |a| + |\lambda|)\,\|Y - Z\|$.

Further, let us show that H is a monotone operator. Let $\forall Y = (y_1,y_2)^T \in R^2,\ \forall Z = (z_1, z2)^T \in R^2,\ \forall U = (u_1,u_2) \in R^2,\ \forall V = (v_1,v_2) \in R^2$. Then for $U \in H(Y),\ V \in H(Z)$ we have $(V - U, Z - Y) = \alpha(v_2 - u_2, z_2 - y_2) \ge 0$ because the function $Sign$ is monotone.

It is easy to show that H is maximal because $(I + \mu H)$ is invertible for every real $\mu \ge 0$. Due to [Brezis, 1973], there exists unique solution of differential inclusion (3).

As a result, the operator $A = F + H$ meets the implication of the Theorem 4. So, the conditions of the Theorem 2 are fulfilled. Accordingly to the Theorem A, one can stated that the chaotic solution exists for the differential inclusion (6). This result coincides with the results of [Awrejcewicz, Lamarque, 2003] where chaotic behavior has been observed by numerical experiments in the case of parameters $a = 0.052,\ \lambda = 0.87,\ f = 0.586,\ \omega = 0.666,\ \alpha = 0.144$.

5 Conclusion and outlooks

The analysis of chaos in the case of multi-valued nonlinear dynamical systems, modeled by differential inclusions, has been presented here. Chaotic phenomena in the case of multi-valued dynamical systems are still at the beginning of their analysis, even the notion of the chaos itself is still open and discussed. Similarly to previously existing research, the results presented in the current paper shows the existence of chaotic behavior. Moreover, no *a priori* hypothesis concerning the existence of homo/heteroclical trajectory is needed here. Conditions are illustrated on several examples. On the other side, design of an efficient numerical method to compute a particular chaotic solution remains an open problem yet to be investigated.

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