Optimal control of a linear system subjected to external sinusoidal and white noise excitations

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Abstract. The paper discusses a problem of stochastic optimal control of a linear single-degree-of-freedom system subjected to external sinusoidal and white noise excitations. An external, bounded in magnitude control force is introduced into the system to reduce mean system response energy. The dynamic programming approach is used to derive the corresponding Hamilton-Jacobi-Bellman equation. Hybrid solution method is used to derive a solution to this equation, thereby found an optimal control policy.

1 Introduction

Problems of stochastic optimal control proved to be very challenging from both practical and theoretical points of view. Very few problems, among which the stochastic linear quadratic regulator (LQR), are known to have an exact analytical solution, whereas the dominant majority of problems are waiting to be solved. The Dynamic Programming approach [1, 2] converts a problems of stochastic optimal control to a problem of finding a solution to a Hamilton-Jacobi-Bellman (HJB) equation - partial differential equation of parabolic type, written for the Bellman function.

There are number of difficulties arises when dealing with the HJB equation, especially if one considers a stochastic mechanical system. First of all, the white noise excitation enters only in one out of two equations of motion, written in a state-space form. This leads to the appearance of only one term with the second derivative in the corresponding HJB equation, making it degenerate. Secondly, the asymptotic behavior of the Bellman function is unknown, which does not allow one to use classical numerical methods to solve it numerically. It must be mentioned that in certain problems, related to the probability of system's state, such boundary conditions are known from the problem statement. Finally, if the introduced control force is bounded in magnitude, as it is considered here, the corresponding HJB equation becomes nonlinear with a signum type nonlinearity. Finding an analytical solution to a nonlinear, degenerate, multidimensional equation of parabolic type becomes almost an impossible task.

Despite all these difficulties there is a hybrid solution method [3], which allows finding a solution to this type of HJB equation. The basic idea of this analytical-numerical method may be stated as following. First, an exact analytical solution is found for the corresponding HJB equation in a certain outer domain. Then the numerical approach is used to find a solution to the HJB equation within the inner domain. This method has been used to solve a number of stochastic optimal control problems (see references in [3]).

This paper offers a solution to a stochastic optimal control problem of a linear, single-degree-of-freedom (SDOF) system, subjected to combined periodic and stochastic external excitations. A bounded in magnitude control force is applied to the system to reduce system's mean response energy.

2 Problem statement

Consider a motion of a linear oscillator under white noise and sinusoidal excitations. An equation of motion of such a system in a state-space form may be written as:

$$\begin{cases} \dot{x}_1 = x_2, \qquad 0 < t \le T, \\ \dot{x}_2 = -\omega^2 x_1 + v + \sigma\xi(t) + \lambda sin(\nu t), \\ x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad |v| \le R \end{cases} \tag{1}$$

where $\xi = \xi(t)$ – Gaussian white noise, $v = v(t)$ – bounded in magnitude control force, σ^2 – a white noise intensity and T – given final time.

The purpose of the control is to minimize the following functional:

$$J_{x_1,x_2,t}(v) = E\left\{ \frac{a}{2}\left[\omega^2 x_1^2(T) + x_2^2(T)\right] + \right.$$
$$\left. + \int_0^T \frac{b}{2}\left[\omega^2 x_1^2(t) + x_2^2(t)\right] dt. \right. \tag{2}$$

where a, b are some nonnegative constants, moreover when $a = 1$ or $b = 1$ the functional represents the system mean response energy. Let $u(x_1, x_2, t)$ be the Bellman function, or minimum of (2):

$$u(x_1, x_2, t) = \inf\left\{J_{x_1,x_2,t}(v) : \ |v| \le R\right\}. \tag{3}$$

Then, function u satisfies the following HJB equation:

$$\frac{\partial u}{\partial t} + Lu + \inf_{|v|\le R}\left\{ v\frac{\partial u}{\partial x_2}\right\} + f = 0; \tag{4}$$

with terminal condition

$$u(x_1, x_2, T) = \frac{a}{2}(\omega^2 x_1^2 + x_2^2). \tag{5}$$

where

$$f(x_1, x_2) = \frac{b}{2}(\omega^2 x_1^2 + x_2^2),$$
$$Lu = x_2\frac{\partial u}{\partial x_1} - \omega^2 x_1\frac{\partial u}{\partial x_2} + \tag{6}$$
$$+ \lambda sin(\nu t)\frac{\partial u}{\partial x_2} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x_2^2}.$$

The optimal control law is determined, based on (4) as:

$$v = -Rsign\left(\frac{\partial u}{\partial x_2}\right). \tag{7}$$

Thus, solving the HJB equation (4) with (5),(6) completely defines the optimal control strategy (7).

3 Analytical solution

In the present problem one have to distinguish two cases: resonant ($\omega = \nu$) and non-resonant cases ($\omega \neq \nu$), which is considered in this paper. Moreover, this paper considers the Lagrange problem only ($a = 0$), ($b = 1$). From the practical point of view this problem is of more importance, since it considers, in the limiting case, a control problem of steady-state oscillations. To use the hybrid solution method, lets skip temporary the second derivative in the equation (4) and consider an outer domain, which does not contain switching lines. In fact, this requirement is equivalent to the following:

$$sign\left(\frac{\partial u}{\partial x_2}\right) = sign(x_2). \tag{8}$$

The optimal control policy in the outer domain is represented by the dry friction law. Taking into account mentioned above we will be looking for a solution to the following equation:

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} = \tilde{L}u - R\frac{\partial u}{\partial x_2}z + f(x_1, x_2); z = sign(x_2)\\
&f(x_1, x_2) = \frac{1}{2}(\omega^2 x_1^2 + x_2^2),\\
&u(x_1, x_2, \tau = 0) = 0,\\
&\tilde{L}u = x_2\frac{\partial u}{\partial x_1} - \omega^2 x_1\frac{\partial u}{\partial x_2} +\\
&+ \lambda sin[\nu(T-\tau)]\frac{\partial u}{\partial x_2}.
\end{aligned} \tag{9}$$

Solution to this equation may be obtained by the method of characteristics. Two characteristics of the equations (9) for the case of $\lambda = 0$ were found earlier [3] and are written as:

$$\begin{cases} \xi_1 = \frac{1}{2}(\omega^2 x_1^2 + x_2^2) + Rzx_1;\\ \xi_2 = \tau + \frac{1}{\omega}\arcsin\left[\frac{\omega(x_1 + Rz/\omega^2)}{\sqrt{2\xi_1 + R^2/\omega^2}}\right]. \end{cases} \tag{10}$$

Integration, using these characteristics, leads to the following result, which satisfies original HJB equation (4) within the outer domain, where (8):

$$\begin{aligned}
&u(x_1, x_2, \tau) = \frac{1}{2}(x_1^2\omega^2 + x_2^2)\tau +\\
&+ x_1 f_1(\tau) + x_2 f_2(\tau) + f_3(\tau).\\
&f_1(\tau) = \frac{Rz}{\omega^2}(\omega^2\tau - \omega sin(\omega\tau)) +\\
&+ \frac{\lambda\omega}{2}\left(\frac{4\omega\nu cos\phi_1}{(\omega^2-\nu^2)^2} - \frac{4\omega\nu cos\phi_2}{(\omega^2-\nu^2)^2} +\right.\\
&\left.+ \frac{2\omega\tau sin\phi_2}{\omega^2-\nu^2}\right),\\
&\phi_1 = \nu T + \omega\tau,\ \phi_2 = \nu(T-\tau),\\
&f_2(\tau) = \frac{Rz}{\omega^2}(cos(\omega\tau) - 1) +\\
&+ \frac{\lambda}{2}\left(\frac{2(\omega^2+\nu^2)cos\phi_2}{(\omega^2-\nu^2)^2} - \frac{2\nu\tau cos\phi_2}{\omega^2-\nu^2} -\right.\\
&- \frac{4\omega\nu cos(\nu T)sin(\omega\tau)}{(\omega^2-\nu^2)^2} +\\
&\left.\frac{2(\omega^2+\nu^2)sin(\nu T)cos(\omega\tau)}{(\omega^2-\nu^2)^2}\right).
\end{aligned} \tag{11}$$

Since the expression for $f_3(\tau)$, which is actually a function of other parameters (ω, R, ν, σ), is too long it is not given here.

It should be stressed that the boundaries of the outer domain should be found from

(8), which may be described as:

$$|x_2| \geq \left| \frac{f_2(\tau)}{\tau} \right| \qquad (12)$$

Consider an important case of steady state oscillations, when T and τ are very large. In this case the outer domain is defined as:

$$|x_2| \geq \frac{\lambda \nu}{|\omega^2 - \nu^2|} \qquad (13)$$

This inequality establishes the size of the outer domain and the parameters, which influences its size. The inner domain has a form of a strip with a finite width in x_2 direction and infinite in x_1 direction. Obviously, in the case of $\nu >> \omega$ or $\nu << \omega$ the inner domain size will be proportional to the excitation amplitude λ multiplied a small parameter. Taking into account the fact that the original equation of motion (1) may be scaled to λ, one may assume $\lambda = 1$. Thus, in the above cases it is possible to select a finite computational domain and solve the corresponding HJB equation numerically within the inner domain. In the case of ν close, but not equal to ω the inner domain size increases. In its turn it leads to a large size computational domain, which requires more computational power. A numerical solution to the HJB equation within the inner domain is to be conducted later. It should be stressed that in the case of purely white noise excitation ($\lambda = 0$) the inner domain shrinks to a line $x_2 = 0$ [4], which resulted in quasioptimality of dry friction law [5].

In 1 values of mean response amplitude are presented as a function of dimensionless frequency ratio ν/ω for different values of R and $\sigma^2 = 1$. Results for the case of $\lambda = 0$, which correspond to pure white noise excitation, were found earlier in [5]. Results presented by solid lines and symbols are obtained for $R = 0.1$ and $R = 0.5$ correspondingly. One may see, that an increase of control parameter R reduces mean response amplitude, as expected. It should also be emphasized that a presence of harmonic excitation increases the value of mean response amplitude as opposed to a pure white noise excitation.

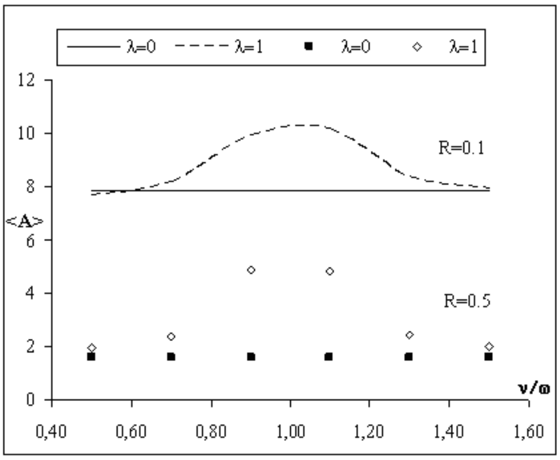

Figure 1: Mean response amplitude of the controlled system.

4 Conclusions

In this paper the author considered a problem of optimal control of a SDOF system, subjected to external periodic and white noise excitations. The stated problem is handled by the Dynamic Programming approach, which leads to a problem of finding a solution to the corresponding Hamilton-Jacobi-Bellman equation. The solution to this equation is found by the hybrid solution method. It is planned to obtain a solution to the stated problem in the resonant case.

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