

NUMERICAL ALGORITHMS FOR STATE-LINEAR OPTIMAL IMPULSIVE CONTROL PROBLEMS BASED ON FEEDBACK NECESSARY OPTIMALITY CONDITIONS

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Abstract

We propose and compare three numeric algorithms for optimal control of state-linear impulsive systems. The algorithms rely on the standard transformation of impulsive control problems through the discontinuous time rescaling, and the so-called “feedback”, direct and dual, maximum principles. The feedback maximum principles are variational necessary optimality conditions operating with feedback controls, which are designed through the usual constructions of the Pontryagin’s Maximum Principle (PMP); though these optimality conditions are formulated completely in the formalism of PMP, they essentially strengthen it. All the algorithms are non-local in the sense that they are aimed at improving non-optimal extrema of PMP (local minima), and, therefore, show the potential of global optimization.

Key words

Optimal control, impulsive control, feedback control, Maximum Principle, algorithms for optimal control.

1 Introduction

Our study lays in the vein of (relatively recent) works [Dykhta, 2014; Dykhta, 2015], where a new sort of necessary optimality conditions is developed for classical and non-smooth optimal control problems. Such conditions, based on the technique of modified Lagrangians, operate with a particular, “extremal” class of feedback controls, and appear to be much “closer” to sufficient conditions (and dynamic programming) than the classical PMP does.

We attempt to extend the mentioned ideas towards the framework of impulsive control [Arutyunov, Karamzin and Lobo Pereira, 2011; Bressan and Rampazzo, 1994;

Dykhta, 1990; Gurman, 1972; Karamzin et al., 2014; Krotov, 1996; Miller, 1996; Rishel, 1965; Warga, 1987]. This area of control theory deals with dynamic systems, whose states are discontinuous, while related extremal problems can not be treated by the tools of classical variational analysis; at the same time, such models have behind them rather solid practical motivation [Dykhta and Samsonyuk, 2000; Miller and Rubinovich, 2003; Zavalishchin and Sesekin, 1997].

This paper follows our recent works [Sorokin and Staritsyn, 2018; Sorokin and Staritsyn, 2017; Sorokin and Staritsyn, 2017; Staritsyn and Sorokin, 2019], where different versions of the so-called feedback maximum (or minimum) principle were obtained for some classes of impulsive (and connected to them continuous and discrete-time) control problems (see also [Dykhta and Samsonyuk, 2018; Sorokin, 2014]). Now, based on the previous theoretical results, we develop numeric algorithms for optimal control, which demonstrate a potential of global optimization techniques. Here, we concentrate on a state-linear case, which enjoys a sort of “duality”, enabling us to employ the dual necessary condition along with the direct one (in some cases, the dual approach seems to be advantageous compared to the direct one). The feedback optimality conditions, we use in this paper, are formulated for an auxiliary *continuous* optimal control problem, and require “post-discretization”. A similar approach for a nonlinear pre-discretized impulsive control problem had been presented in [Sorokin and Staritsyn, 2018].

1.1 Problem Statement

We address a class of optimal control problems, which are linear in state variable and involve both usual (measurable uniformly essentially bounded functions) and

impulsive (distributions or signed Borel measures) control inputs:

Minimize $\langle c, x(T) \rangle$ subject to

$$\dot{x} = [A(u)x + a(u)] + [B(u)x + b(u)]\dot{v}, \quad (1)$$

$$x(0) = x_0, \quad t \in \mathcal{T} \doteq [0, T], \quad (2)$$

$$u(t) \in U \text{ a.e. } t \in \mathcal{T}, \quad \text{Var}_{\mathcal{T}}v(\cdot) \leq M. \quad (3)$$

Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n ; $c, x_0 \in \mathbb{R}^n$ are given vectors; $U \subset \mathbb{R}^m$ is compact, and $A, B : U \rightarrow \mathbb{R}^{n \times n}$, $a, b : U \rightarrow \mathbb{R}^n$ are given matrix- and vector-valued functions, assumed to be Borel measurable. Controls u are functions of class $L_\infty(\mathcal{T}, U)$, while v are right continuous on $[0, T)$ functions $\mathcal{T} \mapsto \mathbb{R}$ of bounded variation ($BV_+(\mathcal{T}, \mathbb{R})$); the derivative \dot{v} shall be understood in the generalized sense, i.e., as a signed Borel measure (or rather, first order distribution); $\text{Var}_{\mathcal{T}}v(\cdot)$ is the total variation of v on \mathcal{T} .

Following the standard methodology [Miller and Rubinovich, 2003], one reduces impulsive system (1)–(3) to an ODE driven by *uniformly bounded* controls. This reduction, based on an appropriate Lipschitzian reparameterization of the time variable, is well-known and rather typical for impulsive control theory. For brevity, we drop the details and refer to [Miller and Rubinovich, 2003; Dykhta and Samsonyuk, 2000; Zavalishchin and Seseikin, 1997]. As a result of the transformation, we obtain the following terminally-constrained classical optimal control problem (P):

Minimize $I(\sigma) \doteq \langle c, x(T) \rangle$ subject to

$$\dot{x} = (1 - |v|)[A(u)x + a(u)] + v[B(u)x + b(u)], \quad x(0) = x_0, \quad (4)$$

$$\dot{y} = 1 - |v|, \quad y(0) = 0, \quad y(T) = y_T, \quad (5)$$

$$u(t) \in U, \quad |v(t)| \leq 1, \quad (6)$$

where *controls* are $w \doteq (u, v) \in L_\infty(\mathcal{T}, \times[-1, 1])$, and *trajectories* are $z \doteq (x, y) \in W^{1,1}(\mathcal{T}, \mathbb{R}^n \times \mathbb{R}_+)$.

Note that (P) is equivalent to the original problem, stated on solutions of (1)–(3), i.e., any minimizing sequence of controls in one problem produces a minimizing sequence in the other one. We shall stress that (P) is weighted by a scalar terminal constraint $y(T) = y_T$.

A collection $\sigma \doteq (z, w) \doteq (x, y, u, v)$ is a (control) process of system (4), (5). A process is called *admissible* as soon as it satisfies (4)–(6). Thanks to the linearity in x and Borel measurability of A, B, a, b , problem (P) has a minimizer within the class of admissible processes, which implies that the original impulsive control problem also has an optimal solution.

2 Theoretical background of the algorithms: Feedback Maximum Principles

Prior to presenting the announced numeric algorithms we shall introduce some necessary objects, and recall the basic theoretical background related to feedback necessary optimality conditions (further details can be found in [Sorokin and Staritsyn, 2017; Staritsyn and Sorokin, 2019]). We start with usual ingredients of PMP.

2.1 Adjoint System and Hamiltonians

The Pontryagin function (the non-maximized Hamiltonian) of problem (P) writes

$$H(x, \psi, \xi, u, v) = (1 - |v|)H_0(x, \psi, \xi, u) + vH_1(x, \psi, u),$$

where

$$H_0(x, \psi, \xi, u) \doteq \langle \psi, A(u)x + a(u) \rangle + \xi,$$

$$H_1(x, \psi, u) \doteq \langle \psi, B(u)x + b(u) \rangle$$

are the “partial Hamiltonians” (notice that H is independent of y). Then, the adjoint (dual) equation takes the form

$$\begin{aligned} \dot{\psi} &= -\frac{\partial H}{\partial x}(x, \psi, u, v) \\ &= -(1 - |v|)A^T(u)\psi - vB^T(u)\psi, \quad (7) \\ \psi(T) &= -c, \end{aligned}$$

where $\xi = \text{const}$ is dual of y (for ξ , there is no transversality condition, dictated by the Maximum Principle). The maximized Hamiltonian is easily calculated as

$$\begin{aligned} \mathcal{H}(x, \psi, \xi) &= \max_{u \in U} \max_{v \in [-1, 1]} H(x, \psi, \xi, u, v) \\ &= \max \{ \mathcal{H}_0(x, \psi, \xi), |\mathcal{H}_1(x, \psi)| \}, \end{aligned}$$

where $\mathcal{H}_{0,1} \doteq \max_{u \in U} H_{0,1}$ are maximized partial Hamiltonians. The maximizers of \mathcal{H} in u and v are the multi-functions

$$\begin{aligned} &\mathbf{U}_\xi(x, \psi) = \\ &\text{Arg max}_{u \in U} \max \{ H_0(x, \psi, \xi, u), |H_1(x, \psi, u)| \}, \\ &\mathbf{V}_\xi(x, \psi) = \text{Arg max}_{v \in [-1, 1]} \{ (1 - |v|)\mathcal{H}_0(x, \psi, \xi) \\ &\quad + v\mathcal{H}_1(x, \psi) \} \\ &= \begin{cases} \{0\}, & \mathcal{H}_0(x, \psi, \xi) > |\mathcal{H}_1(x, \psi)|, \\ \text{Sign } \mathcal{H}_1(x, \psi), & \mathcal{H}_0(x, \psi, \xi) < |\mathcal{H}_1(x, \psi)|, \\ [0, 1], & \mathcal{H}_0(x, \psi, \xi) = \mathcal{H}_1(x, \psi) > 0, \\ [-1, 0], & \mathcal{H}_0(x, \psi, \xi) = -\mathcal{H}_1(x, \psi) > 0, \\ [-1, 1], & \text{otherwise.} \end{cases} \end{aligned}$$

Here, $\text{Sign } a = \text{sign } a$ if $a \neq 0$, and $\text{Sign } 0 = \{-1, 1\}$.

2.2 Feedbacks

Below, we shall deal with feedback controls

$$\mathbf{w}(t, z) = (\mathbf{u}, \mathbf{v})(t, x, y) : \mathcal{T} \times \mathbb{R}^{n+1} \mapsto U \times [-1, 1],$$

which are assume to be measurable in t . By $\mathcal{Z}(\mathbf{w})$ we denote the set of both Carathéodory and Krasovskii-Subbotin (sampling) solutions [Krasovskii and Subbotin, 1988; Clarke et al., 1998] of system (4), (5), associated to \mathbf{w} . Recall that at least one sampling solution exists for any feedback \mathbf{w} , which implies $\mathcal{Z}(\mathbf{w}) \neq \emptyset$.

As is obvious, functions $z \in \mathcal{Z}(\mathbf{w})$ generically loose to satisfy the terminal constraint $y(T) = y_T$. This requires addressing the ‘‘corrected’’ multifunctions, which guarantee the mentioned property [Sorokin and Staritsyn, 2017; Staritsyn and Sorokin, 2019; Sorokin and Staritsyn, 2018]:

$$\check{\mathbf{U}}_\xi(t, z, \psi) = \begin{cases} \text{Argmax}_{u \in U} H_0(x, \psi, \xi, u) & \text{on } \Omega_1, \\ \text{Argmax}_{u \in U} |H_1(x, \psi, u)| & \text{on } \Omega_2, \\ \mathbf{U}_\xi(x, \psi), & \text{on } \Omega_3, \end{cases} \quad (8)$$

$$\check{\mathbf{V}}_\xi(t, z, \psi) = \begin{cases} \{0\}, & \text{on } \Omega_1, \\ \text{Sign } \mathcal{H}_1(x, \psi), & \text{on } \Omega_2, \\ \mathbf{V}_\xi(x, \psi), & \text{on } \Omega_3. \end{cases} \quad (9)$$

Here, $\Omega_1 = \{(t, z, \psi) \mid y \leq t - T + y_T\}$, $\Omega_2 = \{(t, z, \psi) \mid y \geq y_T\}$, and $\Omega_3 = (\overline{\Omega_1 \cup \Omega_2})$.

2.3 Direct and Dual Feedback Maximum Principles

Now we shall fix an admissible process $\bar{\sigma} = (\bar{z} = (\bar{x}, \bar{y}), \bar{w})$, whose optimality is the question of interest.

Let \mathbf{W}_ξ , $\xi \in \mathbb{R}$, denote the ξ -parametric set of feedback controls $\mathbf{w} = (\mathbf{u}, \mathbf{v})$ being selections of multivalued maps (8), (9) contracted to the dual $\bar{\psi}$ of the reference trajectory \bar{x} , i.e., $\mathbf{u}(t, z) \in \check{\mathbf{U}}_\xi(t, z, \bar{\psi}(t))$, and $\mathbf{v}(t, z) \in \check{\mathbf{V}}_\xi(t, z, \bar{\psi}(t))$.

The direct feedback maximum principle is formulated as follows:

Theorem 2.1 ([Sorokin and Staritsyn, 2017]).

Assume that $\bar{\sigma} = (\bar{z}, \bar{w})$ is optimal for (P). Then

$$I(\bar{\sigma}) \leq \langle c, x(T) \rangle \quad \forall z = (x, y) \in \mathcal{Z}(\mathbf{w}), \\ \mathbf{w} \in \mathbf{W}_\xi, \quad \xi \in \mathbb{R}.$$

To formulate the (nonstandard) dual necessary optimality condition for problem (P), we are to introduce an extra adjoint variable $\eta(\cdot)$ as a solution to the ODE

$$\dot{\eta} = \langle \psi, (1 - |v|) a(u) + v b(u) \rangle, \quad \eta(T) = 0, \quad (10)$$

and address the optimal control problem

$$\text{Maximize } K(\psi, \eta, y, w) = \eta(0) + \langle \psi(0), x_0 \rangle \\ \text{subject to (7), (10), and (5).}$$

Again, we introduce the extremal multifunctions

$$\mathbf{U}_\xi^*(t, z, \psi) = \begin{cases} \text{Argmax}_{u \in U} H_0(x, \psi, \xi, u) & \text{on } \Omega_1^*, \\ \text{Argmax}_{u \in U} |H_1(x, \psi, u)| & \text{on } \Omega_2^*, \\ \mathbf{U}_\xi(x, \psi) & \text{on } \Omega_3^*, \end{cases} \quad (11)$$

$$\mathbf{V}_\xi^*(t, z, \psi) = \begin{cases} \{0\} & \text{on } \Omega_1^*, \\ \text{Sign } \mathcal{H}_1(x, \psi) & \text{on } \Omega_2^*, \\ \mathbf{V}_\xi(x, \psi) & \text{on } \Omega_3^*, \end{cases} \quad (12)$$

where $\Omega_1^* = \{(t, z, \psi) \mid y \geq t\}$, $\Omega_2^* = \{(t, z, \psi) \mid y \leq 0\}$, and $\Omega_3^* = (\overline{\Omega_1^* \cup \Omega_2^*})$.

Fixed $\xi \in \mathbb{R}$, \mathcal{W}_ξ denotes the set of (‘‘dual’’) feedback controls, i.e., single-valued selections $\omega = (v, \nu)$ of multivalued maps (11), (12) contracted to the reference state trajectory \bar{x} :

$$(v, \nu) \in (\mathbf{U}_\xi^*, \mathbf{V}_\xi^*)|_{x=\bar{x}(t)}.$$

Let $\mathcal{Z}^*(\omega)$ be the set of all Carathéodory and Krasovskii-Subbotin feedback solutions of (7), (10), (5), produced by $\omega \in \mathcal{W}_\xi$. The dual feedback maximum principle than takes the form:

Theorem 2.2 ([Sorokin and Staritsyn, 2017]).

Assume that $\bar{\sigma} = (\bar{z}, \bar{w})$ is optimal for (P). Then

$$I(\bar{\sigma}) = \langle c, \bar{x}(T) \rangle \leq -K(\psi, \eta, y, \omega) \\ \forall (\psi, \eta, y) \in \mathcal{Z}^*(\omega), \quad \omega \in \mathcal{W}_\xi, \quad \xi \in \mathbb{R}.$$

Now we are going to turn Theorems 2.1, 2.2 into numeric algorithms for optimal control.

3 Numeric Algorithms

In this section, we consider an explicit Euler discretization of dynamical systems (4), (5), (7), (10) with a uniform partition $\{0, 1, 2, \dots, N\}$ of the time interval $[0, T]$. The time lag is $h = T/N$. All control, state and adjoint functions are assumed to be defined at the nodes of the partition grid.

3.1 Direct Algorithm A

The direct algorithm is based on Theorem 2.1.

Step A0 (initialization). Fix the accuracy $\varepsilon > 0$ (this parameter of the algorithm measures the depth of control improvement, and defines the exit of the iterative process).

Choose a reference (admissible) control input

$$\bar{w} = (\bar{u}, \bar{v}) = \{(\bar{u}(t), \bar{v}(t)) \mid t \in \overline{0, N-1}\}.$$

Step A1. Calculate

$$\bar{z} = z(\bar{u}) = (\bar{x}, \bar{y}) = \{(\bar{x}(t), \bar{y}(t)) \mid t \in \overline{0, N}\}$$

as the corresponding solution of the discrete system

$$\begin{aligned} x(t+1) &= x(t) + h(1 - |\bar{v}(t)|) \\ &\quad \cdot \left[A(\bar{u}(t))x(t) + a(\bar{u}(t)) \right] \\ &\quad + h\bar{v}(t) \left[B(\bar{u}(t))x(t) + b(\bar{u}(t)) \right], \quad x(0) = x_0, \\ y(t+1) &= y(t) + h \left[1 - |\bar{v}(t)| \right], \quad y(0) = 0, \\ &\quad t = 0, 1, \dots, N-1. \end{aligned}$$

Set $\bar{\sigma} = (\bar{z}, \bar{w})$ and $I^{rec} := I(\bar{\sigma}) = \langle c, \bar{x}(N) \rangle$.

Step A2. Calculate the adjoint state $\bar{\psi} = \psi(\bar{\sigma})$ by iterating the discretized dual system along $\bar{\sigma}$:

$$\begin{aligned} \psi(t) &= \psi(t+1) + h \frac{\partial H}{\partial x} \left(\bar{x}(t), \psi(t+1), \bar{u}(t), \bar{v}(t) \right) \\ &= \psi(t+1) + h(1 - |\bar{v}(t)|) A^T(\bar{u}(t)) \psi(t+1) \\ &\quad + h\bar{v}(t) B^T(\bar{u}(t)) \psi(t+1), \\ &\quad t = N-1, N-2, \dots, 1, \\ &\quad \psi(T) = -c. \end{aligned}$$

Step A3. Choose $\xi \in \mathbb{R}$.

Step A4 (simultaneous calculation of the feedback control $w = (u, v)$ and respective trajectory $z^w = (x^w, y^w)$).

Set $z^w(0) = (x_0, 0)$;
For $t = 0, N-1$, given $z^w(t) := (x^w(t), y^w(t))$,
choose the values

$$\begin{aligned} \mathbf{u}(t, z^w(t)) &\in \check{\mathbf{U}}_\xi(t, x^w(t), y^w(t), \bar{\psi}(t+1)), \\ \mathbf{v}(t, z^w(t)) &\in \check{\mathbf{V}}_\xi(t, x^w(t), y^w(t), \bar{\psi}(t+1)), \end{aligned}$$

and compute $z^w(t+1) := (x^w(t+1), y^w(t+1))$
as follows:

$$\begin{aligned} x^w(t+1) &= x^w(t) + h(1 - |v^w(t)|) \\ &\quad \cdot \left[A(u^w(t))x^w(t) + a(u^w(t)) \right] \\ &\quad + hv^w(t) \left[B(u^w(t))x^w(t) + b(u^w(t)) \right], \\ y^w(t+1) &= y^w(t) + h \left[1 - |v^w(t)| \right], \end{aligned}$$

where we use short notation $u^w(t) := \mathbf{u}(t, z^w(t))$
and $v^w(t) := \mathbf{v}(t, z^w(t))$.

The outcome of this cycle is a control process $\sigma^w := (z^w, w^w)$.

Step A5. If $I(\sigma^w) = \langle c, x^w(N) \rangle \leq \langle c, \bar{x}(N) \rangle = I^{rec}$, then set $\bar{w} := w^w$, $\bar{z} := z^w$, $\bar{\sigma} := (\bar{z}, \bar{w})$, $I^{rec} := I(\bar{\sigma})$, and return to **Step A2**. Otherwise, go to **Step A3**.

The iterations terminate when $|I(\sigma^w) - I^{rec}| < \varepsilon$.

3.2 Dual Algorithm B

Steps B0 and **B1** coincide with **Steps A0** and **A1**, respectively.

Step B2 is the same as **Step A3**.

Step B3 (simultaneous calculation of the feedback control $\omega = (v, \nu)$ and respective adjoint trajectory $\zeta^\omega = (\psi^\omega, \eta^\omega, \gamma^\omega)$).

Set $\zeta^\omega(N) = (-c, 0, y_T)$;

For $t = N-1, N-2, \dots, 0$, given $\zeta^\omega(t+1) := (\psi^\omega(t+1), \eta^\omega(t+1), \gamma^\omega(t+1))$, choose the values

$$\begin{aligned} v(t, \zeta^\omega(t)) &\in \mathbf{U}_\xi^*(t, \bar{x}(t), \gamma^\omega(t+1), \psi^\omega(t+1)), \\ \nu(t, \zeta^\omega(t)) &\in \mathbf{V}_\xi^*(t, \bar{x}(t), \gamma^\omega(t+1), \psi^\omega(t+1)), \end{aligned}$$

and compute $\zeta^\omega(t) := (\psi^\omega(t), \eta^\omega(t), \gamma^\omega(t))$ as follows:

$$\begin{aligned} \psi^\omega(t) &= \psi^\omega(t+1) \\ &\quad + h \frac{\partial H}{\partial x} \left(\bar{x}(t), \psi^\omega(t+1), u^\omega(t), v^\omega(t) \right) \\ &= \psi^\omega(t+1) + h(1 - |v^\omega(t)|) A^T(u^\omega(t)) \psi^\omega(t+1) \\ &\quad + hv^\omega(t) B^T(u^\omega(t)) \psi^\omega(t+1), \\ \eta^\omega(t) &= \eta^\omega(t+1) - \langle \psi^\omega(t), (1 - |v^\omega(t)|) a(u^\omega(t)) \\ &\quad + v^\omega(t) b(u^\omega(t)) \rangle, \\ \gamma^\omega(t) &= \gamma^\omega(t+1) - h \left[1 - |v^\omega(t)| \right], \end{aligned}$$

where we abbreviate $u^\omega(t) := v(t, \zeta^\omega(t))$ and
 $v^\omega(t) := \nu(t, \zeta^\omega(t))$.

The outcome of this cycle is a pair $\delta = (\zeta^\omega, w^\omega)$.

Step B4. If

$$-K(\delta) = -\eta^\omega(0) - \langle \psi^\omega(0), x_0 \rangle \leq \langle c, \bar{x}(N) \rangle = I^{rec},$$

then set $\bar{w} := w^\omega$, and return to **Step B1**. Otherwise, go to **Step B2**.

3.3 Mixed Algorithm C

Steps C0, C1, C2, and **C3** are the same as **Steps B0, B1, B2,** and **B3**.

Step C4. If $-K(\delta) = -\eta^\omega(0) - \langle \psi^\omega(0), x_0 \rangle \leq \langle c, \bar{x}(N) \rangle = I^{rec}$, then set $\bar{w} := w^w$. Otherwise, go to **Step C2**.

Step C5 coincides with **Step A1**.

Step C6 is the same as **Step A3**.

Step C7 is equivalent to **Step A4** with ψ^ω from **Step C3** instead of $\bar{\psi}$.

Step C8. If $I(\sigma^w) = \langle c, x^w(N) \rangle \leq \langle c, \bar{x}(N) \rangle = I^{rec}$, then set $\bar{w} := w^w$ and return to **Step C1**. Otherwise, go to **Step C6**.

The “mixed” algorithm combines the direct and dual maximum principles: **Steps C2, C3** correspond to construction of feedback controls $\omega = (v, v)$ from Theorem 2.2, while **Steps C6, C7** involve feedback controls $w = (u, v)$ of the type, we met in Theorem 2.1.

4 Examples

The variational problem, addressed in this paper, presents the simplest class of *nonconvex* optimal impulsive control problems with states of bounded variation, from which the theory of dynamic optimization with discontinuous solutions actually starts. Meanwhile, problems of this class arise in different models of physical processes, some of which can be found, e.g., in [Dykhta and Samsonyuk, 2000; Zavalishchin and Sesekin, 1997]. As an example, we discuss below a simple model from the laser technology.

Example 1: Maximize the excitation of two-level atom

Consider the following singular bilinear problem:

$$I = \int_{\mathcal{T}} x_2(t) dt \rightarrow \max,$$

$$\dot{x}_1 = -a x_1 + x_2 v, \quad x_1(0) = 0,$$

$$\dot{x}_2 = -b(x_2 - 1) - x_1 v, \quad x_2(0) = 1,$$

$$v \in BV_+(\mathcal{T}, \mathbb{R}) \cap C(\mathcal{T}, \mathbb{R}), \quad \text{Var}_{\mathcal{T}} v(\cdot) \leq M.$$

Here, a, b are parameters, $0 < b \leq 2a$. The system describes the dynamics of a resonant approximation of an atom, whose state can vary between the basic and the excited levels, subject to a control resonant electromagnetic field. The input v is a linear function of the amplitude of a polarized light wave (for details, we refer to [Dykhta and Samsonyuk, 2000]), and the performance criterion presents an averaged population of the upper atomic level.

In the absence of the constraint on the total variation of control v , a complete analysis of this model is carried out in [Dykhta and Samsonyuk, 2000] by the *variational* maximum principle, which is not more formally applicable in our case.

To investigate the problem numerically, we apply algorithms **A–C**. Passing to the notation of problem (P) the above model rewrites:

$$x_3(T + M) \rightarrow \max,$$

$$\dot{x}_1 = a(|v| - 1)x_1 + vx_2, \quad x_1(0) = 0,$$

$$\dot{x}_2 = b(|v| - 1)(x_2 - 1) - vx_1, \quad x_2(0) = 1,$$

$$\dot{x}_3 = (1 - |v|)x_2, \quad x_3(0) = 0,$$

$$\dot{y} = 1 - |v|, \quad y(0) = 0, \quad y(T + M) = T,$$

$$|v| \leq 1.$$

Taken different values of the parameters a, b, T, M , we obtain, as a result of a series of computations, a similar qualitative picture, whose profile corresponds to a single impulse at the initial time moment, which agrees with the analytical solution [Dykhta and Samsonyuk, 2000]. For $a = b = T = 1, M = 3$, and $v \equiv 0.75$, chosen as the initial (admissible) control, the resulted process is plotted on Figs 1–3. The control sequences, generated by all algorithms **A–C** converge to the same solution, but the number of iterations essentially depends on the value of the parameter ξ (see, e.g., the table below).

ξ	Number of iterations of algorithm A
0	50
1000	10
$\geq 10^{17}$	2

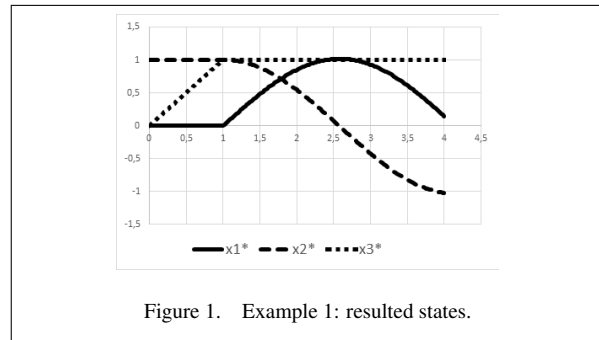


Figure 1. Example 1: resulted states.

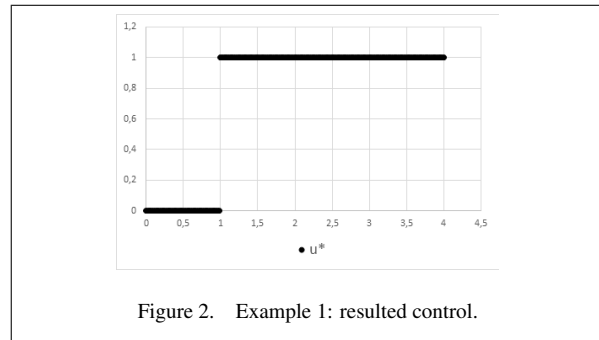


Figure 2. Example 1: resulted control.

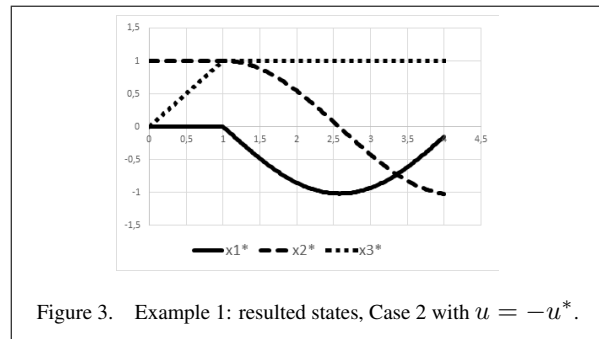


Figure 3. Example 1: resulted states, Case 2 with $u = -u^*$.

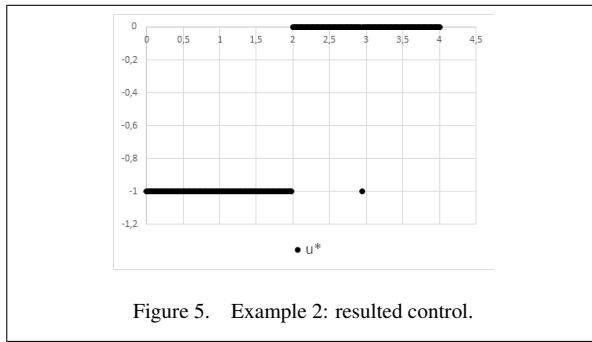


Figure 5. Example 2: resulted control.

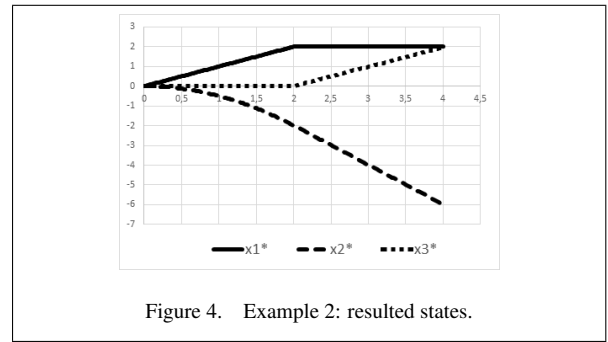


Figure 4. Example 2: resulted states.

The following academic example is aimed at demonstrating the global optimization potential of algorithms A–C, in comparison with the direct method, involving popular solvers such as IPOPT, APOPT and BPOPT.

Example 2: Discarding of a strict local extremum

In [Staritsyn and Sorokin, 2019], we find the following non-convex variational problem

$$\begin{aligned} x_2(4) &\rightarrow \min, \\ \dot{x}_1 &= (y-1)v, \quad \dot{x}_2 = x_1(v+|v|-1), \quad \dot{y} = 1-|v|, \\ (x_1, x_2)(0) &= 0, \quad y(0) = 0, \quad y(4) = 2, \\ v &\in [-1, 1/2], \end{aligned}$$

which is equivalent to the pre-impulsive model

$$\begin{aligned} x_2(2) &\rightarrow \min, \\ \dot{x}_1 &= (t-1)v, \quad \dot{x}_2 = x_1(v-1), \\ x_1(0) &= x_2(0) = 0, \\ \text{Var}_{[0,2]} v(\cdot) &\leq 2, \quad \int_0^t v dt \leq t \quad \forall t \in [0, 2]. \end{aligned}$$

As a matter of comparison, we applied the free academic package GEKKO for Python 3 [Beal, Hill, Martin and Hedengren, 2018], which automatically reduces an optimal control problem to NLP through discretization in time. The terminal constraint $y(T+M) = T$ is handled by quadratic penalization. The computations were carried out in the remote mode (APMonitor, Version 0.9.2), involving the IPOPT, APOPT and BPOPT as internal NLP solvers. As an outcome of multiple numeric experiments, the same control $v \equiv 1/2$ was found, which is known to be a local Pontryagin extremal with the cost $I = 0$.

Starting from this local solution, all three algorithms A–C produce, in a single iteration (!), a process with the cost $I = -6$ and states/controls presented on Figs 4, 5.

5 Conclusion

It is important to stress that the direct and dual feedback maximum principles are independent one of another in the sense that a process satisfying one of them shall not satisfy another one [Sorokin and Staritsyn, 2017] (see also [Dykhta, 2014; Sorokin, 2014]). As some examples show, this feature is inherited by the respective Algorithms A and B. Thus, a combination of the direct and dual approaches in the spirit of Algorithm C could be a promising way. Furthermore, such a combination is natural from the very “machinery” viewpoint. Indeed, given the initial control process, Theorem 2.1 produces a state of a new, “better” process σ , which can be used as the initial data of Theorem 2.2. Next, the outcome of the dual feedback maximum principle, i.e., the adjoint state of an “improving” process, could be used as an input of the direct feedback maximum principle.

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