QUANTUM CONTROL FOR BOSE EINSTEIN CONDENSATES

Quan-Fang Wang
Mechanical and Automation Engineering
The Chinese University of Hong Kong
Hong Kong, China
quanfangwang@yahoo.co.jp, qfwang@mae.cuhk.edu.hk

Viacheslav P. Belavkin
School of Mathematical Science
The University of Nottingham
United Kingdom
viacheslav.belavkin@nottingham.edu.uk

Abstract
Bose Einstein condensates (BEC) is regarded as the control system in this paper. In the viewpoint of mathematics and physics, a completely synthesis for controlling of particles in BEC status will be considered using fundamental analysis via variational framework in Hilbert space theoretically, although it's not clear to execute with present quantum optical equipments, such as laser cooling, optical lattices.

Key words
Quantum control, Bose Einstein Condensates, Optimal control theory.

1 Physical background

With the rapidly growth of quantum control study in a variety fields, the physical and chemical researches have made a great deal significant contributions. For example, optimal control theory apply to molecule formations in a BEC see [Sklarz, 2002]. Genetic-learning algorithm to atomic BEC is reported in [Potting, 2001]. The remote physical controlling using coefficient concerned with soliton see [Radha, 2008]. Coherent control see [Holthaus, 2001] in a double well for steering the self-trapping N particle at zero temperature, for single particle see [Abdullaev, 2003] in high dimensions. Theoretical study for BEC also refer to [Choi, 2005]. Overall physical investigation for BEC reported in [Morsch, 2006].

To boost the development of the control in quantum system, it is quiet interesting to consider the Bose Einstein Condensates as the control target.

In the viewpoint of physics, if an ultracold vapor of bosonic atoms are trapped in magnetic well, pure condensates will be created as they are cooled to a temperature below the BEC threshold. After that creation, these BEC are located into a optical lattice potential which can be realized experimentally by a far-detuned, retroreflected laser beam. The real BEC picture see Figure 1 (top panel).

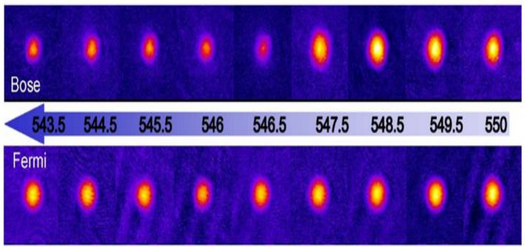

Figure 1. BEC image

This phenomenon of macroscopic quantum system consisting of ultracold atoms in unique in precision and flexibility for experimental control and manipulation.

What would be happen if external forcing acting at the particles in BEC? Did optical technology will provide the achievement of the controlling goal? Can laser pulse with high intensity drive the BEC to change their states and transfer energy during this control process. For this purpose, what kinds of ultra-fast (femisecond/attosecond) laser pulse should meet our satisfaction?

Mathematically, the BEC is usually modeled by the celebrated Gross-Pitaevskii equation, a cubically nonlinear Schrödinger equation (NLS), see [Pitaevskii, 2003],

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi_{\mathbf{xx}} + u(\mathbf{x})\psi + v(\mathbf{x})\psi + N\alpha|\psi|^2\psi, \quad (1)$$

where ψ denote the condensate wave function (i.e. probability amplitudes) of one particle in BEC, m denote the atomic mass, $\hbar$ is the Planck constant, N is the number of atoms in the condensate, and

$$\alpha = 4\pi\hbar^2 a/m,$$

with $a \in \mathbf{R}$ denoting the characteristic scattering length of the particles. The external potential $u(\mathbf{x})$ is confining in order to describe the electromagnetic trap needed for the experimental realization of a BEC. Typically it is assumed to be of harmonic form

$$u(\mathbf{x}) = m\omega_0^2 \frac{|x|^2}{2}, \quad \omega_0 \in \mathbf{R}. \tag{2}$$

A particular example for the periodic potentials used in physical experiments is then given by [Deconinck, 2002; Pitaevskii, 2003]

$$v(\mathbf{x}) = s \sum_{i=1}^{3} \frac{\hbar^2 \mathbf{x}^2}{m} \sin^2(\mathbf{x}_i \mathbf{x}_i), \quad \mathbf{x}_i \in \mathbf{R}, \tag{3}$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ denotes the wave vector of the applied laser field and $s > 0$ is a dimensionless parameter describing the depth of the optical lattice (expressed in terms of the recoil energy).

The GP equation (1) provides an interesting test case for NLS codes since it features high frequency oscillations, two-scale external potentials, and a (focusing or defocusing) nonlinearity. More precisely, the nonlinear cubic term in the GP equation accounts for the interatomic (many body) interactions, the coefficient in front of the nonlinear cubic term is proportional to the atomic scattering length. The size and sign of the atomic scattering length can be adjusted due to Feshbach resonances.

Remark 1.1. *For multi-species case located at optical wells, it need to consider the interspecies interactions between equations of 'ψ'. Our study is restricted to single specie case temporarily.*

Remark 1.2. *Chemically, the BEC experiment should be available for ^{7}Li, ^{85}Rb and ^{133}Cs, etc.*

This paper will be organized in following contents. After above introduction of physical model in Section 1, we will propose the BEC quantum system in the mathematical setting in Hilbert space. Section 3 is to state the control theory for BEC quantum dynamics. Section 4 summarize and draw some conclusions and discussions.

2 BEC quantum system

It is natural to specially consider the optimal control problem for BEC described by (1), it permit us to regard the problem into mathematical setting in Hilbert space.

Let Ω be an open bounded set of $\mathbf{R}^3$ and $Q = (0, T) \times \Omega$ for $T > 0$. Then $(\mathbf{x}, t) \in Q$. Regarding $u(\mathbf{x})$ and $v(\mathbf{x})$ are control variables. Introduce two Hilbert spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ with usual norm and inner products (cf. [Lions, 1971]). Then the embedding in Gelfand triple space $V \hookrightarrow H \hookrightarrow V'$ are continuous, dense and compact.

Suppose $\mathcal{U} = L^2(\Omega)$ is the space of laser controls u and v. Let $\mathcal{U}_{ad}$ be a closed and convex admissible set of $\mathcal{U}$. Assume initial ground states $\psi(u, v, 0) = \psi_0$. The objective function associated with (1) is given by

$$\begin{aligned} J(u, v) = \epsilon_1 \|\psi_f(u, v) - \psi_{\text{target}}\|_V^2 \\ + \epsilon_2 (u, u)_{\mathcal{U}} + \epsilon_3 (v, v)_{\mathcal{U}}. \end{aligned} \tag{4}$$

Here $u, v \in \mathcal{U}_{ad}$, ψ_{target} is target state, $\psi_f(u, v)$ is observed final state, respectively. Moreover, $\epsilon_i, i = 1, 2, 3$ are weighted coefficients for balancing the values of inherent and running costs.

Our goal is to find quantum optimal control u^* or v^* in GP system (1). Here u^* and v^* are called quantum optimal control for system (1) subject to objective function (4). We wish to drive the GP equation the optimality system for the OCT fields that allow efficient channeling of the condensate between given initial and desired states.

To do this, we define two basic concepts, weak solution and solution space, for preparation.

Definition 2.1. *For the theoretical control study for (1) with objective function (4), referring [Wang,ACC2006; Wang & Cao, CDC2007] to define weak solution's solution space by Hilbert space:*

$$\begin{aligned} W(0, T; V, V') = \Big\{ \psi \;\Big|\; \psi \in L^2(0, T; V), \\ \psi' \in L^2(0, T; V') \Big\}. \end{aligned}$$

Definition 2.2. *The function ψ is called weak solution of* (1) *if $\psi \in W(0, T; V, V')$ and satisfy*

$$\begin{aligned} \int_0^T \int_\Omega i\hbar \psi_t \, dt d\mathbf{x} = -\frac{\hbar^2}{2m} \int_0^T \int_\Omega \Delta \psi_{\mathbf{xx}} d\mathbf{x} dt \\ + \int_0^T \int_\Omega u(\mathbf{x}) \psi d\mathbf{x} dt + \int_0^T \int_\Omega v(\mathbf{x}) \psi d\mathbf{x} dt \\ + \int_0^T \int_\Omega N\alpha |\psi|^2 \psi d\mathbf{x}. \end{aligned} \tag{5}$$

3 Control theory for BEC

Previously established mathematical setting permit us to study the quantum system (1) in the framework of variational method and quantum mechanics theory. Therefore, using the same manipulation as in [Lions, 1971; Wang,ACC2006; Wang & Cao, CDC2007] and refer (5), it's easy to obtained the next theorems.

Theorem 3.1. *For initial given $\psi_0 \in V$, there exists weak solution $\psi \in W(0, T; V, V')$ for system* (1) *satisfy the weak form (5).*

Theorem 3.2. *For $\psi_0 \in V$, there exists at least one quantum optimal control pairing (u^*, v^*) for system* (1) *subject to objective function* (4).

Remark 3.3. *It is worth noting that, for real physical meaningful u and v, how to adjust the parameters (e.g. wire currents or radio-frequency fields) in (2) and (3), it needs to treat carefully in real physical experiments. In here, what we discussed is that such a optimal control pairing would be existed theoretically in the framework of variational method.*

Theorem 3.4. *For initial state $\psi_0 \in V$ and control problem for system (1) associated with (4), the optimality system is given by*

$$\begin{cases} i\hbar\psi_t = -\dfrac{\hbar^2}{2M}\Delta\psi + u^*(\mathbf{x})\psi \\ \qquad\quad +v^*(\mathbf{x})\psi + N\alpha|\psi|^2\psi \quad \text{in Q}, \\ \psi(u^*, v^*, 0) = \psi_0 \quad \text{in } \Omega, \end{cases} \tag{6}$$

$$\begin{cases} i\hbar p_t = -\dfrac{\hbar^2}{2M}\Delta p + 2|\psi|\psi p + |\psi|^2 p \;\text{ in Q}, \\ ip_f = \psi_f(u^*, v^*) - \psi_{\text{target}} \quad \text{in } \Omega, \end{cases} \tag{7}$$

$$(u^*, u - u^*)_{\mathcal{U}} + \int_Q p(u^*)(u - u^*)\, d\mathbf{x}dt$$

$$+(v^*, v - v^*)_{\mathcal{U}} + \int_Q p(v^*)(v - v^*)\, d\mathbf{x}dt \geq 0 \tag{8}$$

for all $u, v \in \mathcal{U}_{ad}$. In here, $p \in W(0, T; V, V')$ is solution of the adjoint systems (7) corresponding to ψ in state systems (6) respectively. As is well known that the inequality (8) is necessary optimality condition for (u^*, v^*).

By considering [Wang & Cao, CDC2007; Wang, GRC2007], quantum optimal control $u^*$and v^* can be found efficiently.

Remark 3.5. *It's easy to characterize the optimal control solution with above systems and inequality theoretically. For computational study to solve such quantum optimal solution pairing (u^*, v^*), it needs to employ the numerical approach methodologies to minimize the objective cost function (4).*

Remark 3.6. *Particularly, to select optimal laser pulse in real lab experiment, it's quite incredible to find the exactly optimal solution (u^*, v^*), besides to obtain the set of 'optimal' solution which much better than others.*

4 Conclusions and Discussions

In summary, the controlling for BEC has been solved regarding the quantum dynamics to seek the optimal solution (cf. [Wang, 2008; Wang, ESF2008]). This research exploration extremely acquire the real laboratory evidence for quantum controlling achievement. The attempt progress would become the promising research direction, see [Wang & Cao, ECC2009; Wang, ACS2009; Wang, 2009; Wang & Belavkin, PhysCon2009] and [Wang Cao & Luo, 2009].

Observing the literatures of researches on controlling of BEC in physical and chemical fields, see relevant contributed papers [Chacon, 2008; Deconinck, 2002; Hohenester,2007; Parker, 2004; Perez-Garcia, 2007; Bulatov, 1999; Roberts, 2001; Rodas, 2005; Stickney, 2007; Trotzky, 2008]. What we interested is controlling the BEC theoretically and computationally.

On the other hand, decoherence effects, which also play a role in atom condensate, can be naturally incorporated into OCT calculations. It has been quested in PhysCon 2009 conference.

Future perspective will be combining the predictions with real laboratory experiments with toiled advanced optical technologies.

Acknowledgements

The authors would like to express full gratitude to orginazing committee of 4th International Scientific Conference on Physics and Control (PhysCon 2009) for giving the opportunity to the paper to join the conference.

References

Abdullaev F. K., Caputo, J. G., Kraenkel R. A. and Malomed B.A. (2003) Controlling collapse in Bose-Einstein condensates by temporal modulation of the scattering length, *Physical Review A* **67**, p. 013605.

Bulatov, A., Vugmeister, B. E. and Rabitz, H. (1999) Nonadiabatic control of Bose-Einstein condensation in optical traps, *Physical Review A* **60**(6), pp. 4875–4881.

Chacon,R., D. Bote, D., and Carretero-Gonzalez, R. (2008) Controlling chaos of a Bose-Einstein condensate loaded into a moving optical Fourier-synthesized lattice, *Physical Review E* **78**, p.036215.

Choi, S. and Bigelow, N. P. (2005) Initial steps towards quantum control of atomic Bose-Einstein condensates, *Journal of Optical B: Quantum and Semiclassical Optics* **7**, p. 413–420.

Deconinck, B., Frigyik, B. A. and Kutz, J. N. (2002) Dynamics and stability of Bose-Einstein condensates: The nonlinear Schrödinger equation with periodic potential, *J. Nonlinear Sci.*, **12**, p. 169.

Hohenester, U., Rekdal, P. K., Borzi, A. and Schmiedmayer J. (2007) Optimal quantum control of Bose-Einstein condensates in magnetic microtraps, *Physical Review A* **75**, p. 023602.

Holthaus, M. (2001) Toward coherent control of Bose-Einstein condensate in a double well, *Physical Review A* **64**, p. 011601.

Lions, J. L. (1971). *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin-Heidelberg-New York.

Morsch, O. and Oberthaler M. (2006) Dynamics of Bose-Einstein condensates in optical lattices, *Reviews of Modern Physics* **78**, pp. 179–215.

Parker,N. G., Proukakis, N. P., Barenghi, C. F. and Adams, C. S. (2004) Controlled Vortex-Sound Interactions in Atomic Bose-Einstein Condensates, *Physical Review Letters* **92**(6), p.160403.

Perez-Garcia, V. M. and Garcia-March, M. A. (2007) Symmetry-assisted vorticity control in Bose-Einstein condensates, *Physical Review A* **75**, p.033618.

Pitaevskii, L. and Stringari S. (2003). *Bose-Einstein Condensation*, Internat. Ser. Monogr. Phys. 116, Clarendon Press. Oxford UK.

Pötting, S., Cramer, M. and Meystre, P. (2001) Momentum-state engineering and control in Bose-Einstein condensates, *Physical Review A* **64**, p. 063613.

Radha, R., Kumar, V R. and Porsezian, K. (2008) Remote controlling the dynamics of Bose-Einstein condensates through time-dependent atomic feeding and trap, *Journal of Physics A: Mathematical and Theoretical* **41** p. 315209.

Roberts, J. L., Claussen, N. R., Cornish, S. L., Donley, E. A., Cornell, E. A. and Wieman, C. E. (2001) Controlled Collapse of a Bose-Einstein Condensate, *Physical Review Letters* **86** (19), pp. 4211–4214.

Rodas, M.I., Michinel H. and Perez-Garcia, V. M. (2005) Controllable Soliton Emission from a Bose-Einstein Condensate, *Physical Review Letters* **95**, p.153903.

Sklarz, S. E. and Tannor D. J. (2002) Loading a Bose-Einstein condensate onto an optical lattice: An application of optimal control theory to the nonlinear Schrödinger equation, *Physical Review A* **66**, p. 053619.

Stickney,J. A., Anderson, D. Z. and Alex A. Zozulya A. A. (2007) Increasing the coherence time of Bose-Einstein-condensate interferometers with optical control of dynamics, *Physical Review A* **75**, p. 063603.

Trotzky, S. et al, (2008) Time-Resolved Observation and Control of Superexchange Interactions with Ultracold Atoms in Optical Lattices, *Science* **319**, pp. 295–299.

Wang, Q. F. (2006). Quantum optimal control of nonlinear dynamics systems described by Klein-Gordon-Schrödinger equations, *Proceeding of American Control Conference*, Minneapolis, Minnesota, USA, Jun. 14-16. pp. 1032–1037.

Wang, Q. F. and Cao, C. (2007). Control problem for nonlinear systems given by Klein-Gordon-Maxwell equations with electromagnetic field, *46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, Dec.12-14. pp. 6370–6375.

Wang, Q. F. and Rabitz, H. A. (2007). Quantum optimal control for the Klein-Gordon-Schrödinger dynamics system in the presence of disturbances and uncertainties, *Gordon Research Conference 'Quantum Control of Light and Matter'*, Newport, PI, USA, Aug. 12-17. Poster.

Wang, Q. F. (2008) Theoretical issue of controlling nucleus in Klein-Gordon Schrödinger dynamics with perturbation in control field, *Applied Mathematics and Computation* **206**, pp. 276–289.

Wang, Q. F. (2008). Single particle quantum controlling in Yukawa interaction, *European Science Foundation conference 'Chemical Control with Electrons and Photons'* Obergurgl, Austria, Nov. 22-27. Poster.

Wang, Q. F. and Cao, C. (2009). Quantum numerical optimal control of nucleon and meson dissipative dynamics model, *Proceeding of the European Control Conference*, Budapest, Hungary, Aug. 23-26. pp. 168–172.

Wang, Q. F. (2009). Quantum control in ring/corral at matter surface, *238th American Chemistry Society National Meeting (Fall)*, Washington, USA, Aug. 16-20. Poster.

Wang, Q. F. (2009) Quantum optimal control of nuclei in the presence of perturbation in electric field, *IET Control Theory & Applications* **3**(9), pp. 1175–1182.

Wang, Q. F. and Belavkin, V. P. (2009). Quantum control for Bose Einstein condensates, *4th International Scientific Conference on Physics and Control*, Catania, Italy, Sept. 1-4. Oral presentation.

Wang, Q. F., Cao, C. and Luo, L. (2009). Quantum optimal control on meson exchange effect, *7th International Conference on Control and Automation*, Christchurch, New Zealand, Dec. 9-11. Oral presentation.