RESIDENCE TIME DISTRIBUTIONS FOR COEXISTING REGIMES OF BISTABLE DYNAMICAL SYSTEMS SUBJECTED TO NOISE INFLUENCE

Olga I. Moskalenko, Alexey A. Koronovskii
Faculty of Nonlinear Processes
Saratov State University
Russian Federation
o.i.moskalenko@gmail.com
alexey.koronovskii@gmail.com

Alexander E. Hramov
REC "Nonlinear Dynamics of Complex Systems"
Saratov State Technical University
Russian Federation
hramovae@gmail.com

Maksim O. Zhuravlev
Faculty of Nonlinear Processes
Saratov State University
Russian Federation
zhuravlevmo@gmail.com

Rider Jaimes-Reátegui
Universidad de Guadalajara
Centro Universitario de los Lagos
Mexico
rider.jaimes@gmail.com

Abstract

The behavior of bistable dynamical systems subjected to additional noise is studied both analytically and numerically. We show that such systems can demonstrate noise-induced intermittency characterized by exponential law distributions of the residence time. The main results are illustrated with the examples of the bistable energy model, erbium-doped fiber laser and mutually coupled Lorenz oscillators near the boundary of generalized synchronization.

Key words

Bistablity, stochastic systems, intermittency, residence time distribution, generalized synchronization.

1 Introduction

Intermittency is an ubiquitous phenomenon in nonlinear science [Berge et al., 1984]. It is observed in different systems including the physical, physiological and biological ones (see, e.g., [Kim et al., 1998; Perez Velazquez and et al., 1999; Kiss and Hudson, 2001; Boccaletti et al., 2002; Cabrera and Milton, 2002; Hramov et al., 2006b; Sitnikova et al., 2012]). It manifests itself as alternation of the episodes of periodic and chaotic regimes [Manneville and Pomeau, 1979] or different forms of the chaotic motion [Grebogi et al., 1987]. It can also be observed near the boundaries of different synchronous regimes demonstrating the interchange of the phases of synchronous and asynchronous behaviors (see, e.g. [Pikovsky et al., 1997; Boccaletti and Valladares, 2000; Hramov and Koronovskii, 2005; Hramov et al., 2006a; Moskalenko et al., 2011; Hramov et al., 2014]).

Among different types of the intermittent behavior, one can traditionally distinguish type I-III [Berge et al., 1984; Dubois et al., 1983], on-off [Heagy et al., 1994], eyelet [Pikovsky et al., 1997; Boccaletti et al., 2002] and ring [Hramov et al., 2006a] intermittencies and the coexistence of two or more types [Hramov et al., 2013; Moskalenko et al., 2014; Koronovskii et al., 2016]. Each type of intermittency mentioned above is characterized by its proper mechanism and own statistical characteristics. One can say that such characteristics allow the unambiguous definition of the type of intermittency realized in the system.

Recently the concept of intermittency has been extended to multistable systems. In such case the alternation between coexisting periodic or chaotic regimes regardless of the form of motion realized in the systems can also be observed [Pisarchik et al., 2012; Sevilla-Escoboza et al., 2015]. At that, the switches between coexisting regimes can be induced by noise. Therefore, the system under study demonstrates the so-called noise-induced intermittency or noise-induced attractor hopping [Arecchi et al., 1985; Wiesenfeld and Hadley, 1989; Kraut and Feudel, 2002; Pisarchik et al., 2011; Hramov et al., 2016].

Despite of a great interest to the problem of noise-induced intermittency (see, e.g. [Lai and Grebogi, 1995; Pisarchik and Pinto-Robledo, 2002; Hramov et al., 2016]) there is a number of questions demanding consideration and discussion. One of such problems consists in the fact that there is no appropriate theory allowing to obtain the characteristics of noised-induced intermittency even in the case of two different coexisting regime. In the present paper, we propose the theory of noise-induced intermittency in bistable dynamical systems. We will show that the residence time distributions for every coexisting regime obeys the exponential law.

2 Theory of Noised-induced Intermittency

A standard bistable system capable to demonstrate noised-induced intermittency can be described by the bistable energy model:

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \xi(t), \quad (1)$$

where $\xi(t)$ is zero mean δ-correlated Gaussian noise $[\langle\xi_n\rangle = 0, \langle\xi_n\xi_m\rangle = D\delta(n-m)]$, D is its variance,

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2} + bx \quad (2)$$

is the dimensionless energy function with two local minima, and b is the parameter of symmetry [Pisarchik et al., 2014; Moreno-Bote et al., 2007].

The differential Eq. (1) with stochastic term $\xi(t)$ results in the stochastic differential equation:

$$dX = \frac{dU(x)}{dx}\,dt + dW, \quad (3)$$

where $X(t)$ is a stochastic process and $W(t)$ is a one-dimensional Winner process. Equation (3) is equivalent to the Fokker-Plank equation:

$$\frac{\partial\rho_X(x,t)}{\partial t} = \frac{\partial}{\partial x}[\frac{dU(x)}{dx}\rho_X(x,t)] + \frac{D}{2}\frac{\partial^2\rho_X(x,t)}{\partial x^2} \quad (4)$$

written for probability density $\rho(x,t)$ of the stochastic process $X(t)$.

Since in the regime of intermittency the coordinate of the system state stays for a long time in the vicinity of one of the local minima, we can assume that the solution of Eq. (4) should be searched in the form of the metastable distribution decaying slowly for a long period of time, i.e.

$$\rho(x,t) = A(t)r(x), \quad (5)$$

where $r(x)$ is the stationary probability density obtained from the solution of Eq. (4) in a stationary case and $A(t)$ is a coefficient decreasing slowly as the time increases. Equation (5) results in the following relation for the residence time distributions for both coexisting regimes:

$$p_{1,2}(t) = \frac{1}{T_{1,2}}\exp\left(-\frac{t}{T_{1,2}}\right). \quad (6)$$

Here

$$T_{1,2} = \frac{P_{1,2}}{kr(x^*)}, \quad (7)$$

x^* is a boundary point located at equal distances from two local minima of $U(x)$, $P_{1,2}$ are the probabilities of location of the representation point in the vicinity of the first or second local minima, and k is a normalization factor.

In other words, in the regime of noise-induced intermittency the residence time distributions obey the exponential law Eq. (6).

3 Numerical Verifications of the Proposed Theory

To confirm the results of theoretical predictions, we analyze numerically the behavior of two different systems capable to demonstrate the regime of noise-induced intermittency. As the first example, we consider the same energy model Eq. (1) with the same potential function Eq. (2) and characteristics of noise with its variance $D = 0.1$. In Fig. 1 the time realization (*a*) and statistical distributions (*b,c*) of the residence times corresponding to two coexisting regimes in symmetrical case ($b = 0$) are shown. The graph in Fig. 1,*b* corresponds to the first coexisting regime (marked by *1* in Fig. 1,*a*), i.e. the first local minimum of the potential function, whereas Fig. 1,*c* refers to the second coexisting state (marked by *2* in Fig. 1,*a*). The results of the numerical simulations are marked by symbols for both coexisting regimes and theoretical approximated by the exponential laws Eq. (6) (solid lines) with the parameters indicated in the figure caption. It is clearly seen the excellent agreement between the results of the numerical calculations and theoretical approximations. This confirms the validity of the proposed theory.

As the second example, we consider dynamics of an erbium-doped fiber laser which is known to demonstrate noised-induced intermittency [Pisarchik et al., 2005; Hramov et al., 2016]. The system under study is given by

$$\frac{dx}{dt} = \frac{2L}{T_r}x\left\{r_w\alpha_0\left[y\left(\xi_1-\xi_2\right)-1\right]-\alpha_{th}\right\}+P_{sp}, \quad (8)$$
$$\frac{dy}{dt} = -\frac{\sigma_{12}r_w x}{\pi r_0^2}(y\xi_1-1)-\frac{x}{\tau}+P_{pump},$$

where x is the intracavity laser power,

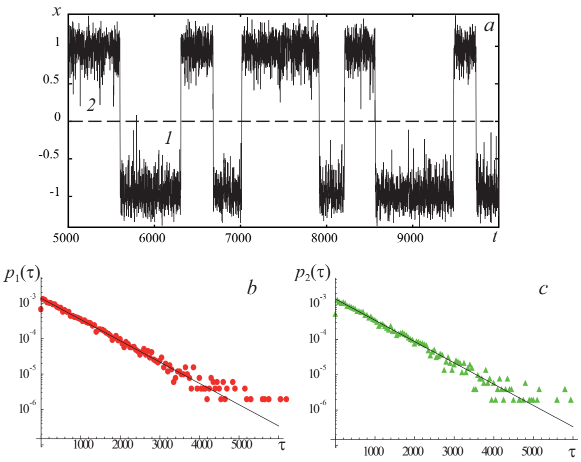

Figure 1. Time realization (a) and residence time distributions (b,c) for two coexisting regimes in bistable energy model Eq. (1) for $b = 0$. The results of numerical calculations are marked by symbols. Theoretical approximations by the regularity Eq. (6) are shown by solid lines. The parameters of approximations are (b) $T_1 = 722$ and (c) $T_2 = 721$.

$y = \dfrac{1}{n_0 L}\int\limits_0^L N_2(z)dz$ is the averaged (over the active fiber length L) population of the upper lasing level, N_2 is the upper level population at the z coordinate, n_0 is the refractive index of a "cold" erbium-doped fiber core, ξ_1 and ξ_2 are parameters defined by the relationship between cross sections of ground state absorption (σ_{12}), return stimulated transition (σ_{21}), and exited state absorption (σ_{23}). T_r is the photon intracavity round-trip time, α_0 is the small-signal absorption of the erbium fiber at the laser wavelength, α_{th} accounts for the intracavity losses on the threshold, τ is the lifetime of erbium ions in the excited state, r_0 is the fiber core radius, w_0 is the radius of the fundamental fiber mode, and r_w is the factor that conveys the match between the laser fundamental mode and erbium-doped core volumes inside the active fiber. The spontaneous emission into the fundamental laser mode is derived as

$$P_{sp} = y\frac{10^{-3}}{\tau T_r}\left(\frac{\lambda_g}{w_0}\right)^2 \frac{r_0^2\alpha_0 L}{4\pi^2\sigma_{12}}, \qquad (9)$$

where λ_g is the laser wavelength. The pump power is expressed as

$$P_{pump} = P_p\frac{1-\exp\left[-\alpha_0\beta L\left(1-y\right)\right]}{N_0\pi r_0^2 L}, \qquad (10)$$

where P_p is the pump power at the fiber entrance and β is a dimensionless coefficient. Similar to previous

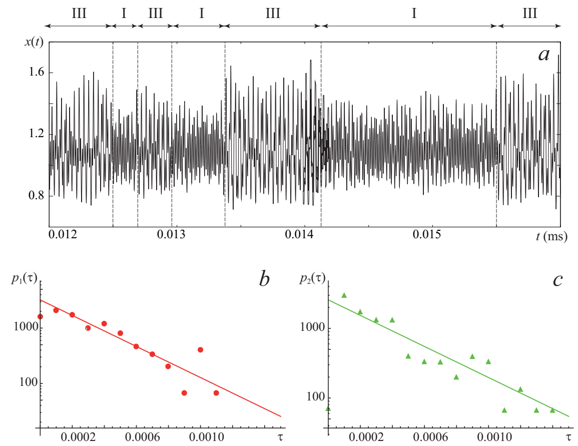

Figure 2. Time realization (a) and residence time distributions (b,c) for two coexisting regimes (b — A_1 (I), c — A_3 (III)) in erbium-doped fiber laser Eq. (8) for noise intensity $\eta = 0.11$ (symbols) and their analytical approximations by regularity Eq. (6) (solid lines). The parameters of approximations are (b) $T_1 = 3.1\times10^{-4}$ and (c) $T_2 = 3.9\times10^{-4}$.

works, the parameters are chosen as follows: $L = 0.88$ m, $T_r = 8.7$ ns, $r_w = 0.308$, $\alpha_0 = 40$ m^{-1}, $\xi_1 = 2$, $\xi_2 = 0.4$, $\alpha_{th} = 3.92\times10^{-2}$, $\sigma_{12} = 2.3\times10^{-17}$ m^2, $r_0 = 2.7\times10^{-6}$ m, $\tau = 10^{-2}$ s, $\lambda_g = 1.65\times10^{-6}$ m, $w_0 = 3.5\times10^{-6}$ m, $\beta = 0.5$, and $N_0 = 5.4\times10^{25}$ m^{-3}. These parameters correspond to real experimental conditions.

Under the harmonic and random modulations

$$P_p = p\left[1 - m_d\sin\left(2\pi f_d t\right) + \eta G(\zeta, f_n)\right], \qquad (11)$$

where p is the pump power, $m_d = 0.95$ and $f_d = 80$ kHz are the driving frequency and amplitude, respectively, η is the noise amplitude, and $G(\zeta, f_n)$ is the zero-mean noise function of a random number $\zeta \in [-1, 1]$ and noise low-pass cut-off frequency $f_n = 30$ kHz), the system Eq. (8) demonstrates noised-induced intermittency with up to four coexisting regimes A_i ($i = 1, 3, 4, 5$) with frequencies $f_i = f_d/i$ [Pisarchik et al., 2012; Hramov et al., 2016].

If the noise intensity η is small enough, Eq. (8) exhibits the coexistence of two different regimes (see Fig. 2,a) that allows us to apply the theory proposed in Sec. 2 to the system under study. In Fig. 2,b,c the numerically obtained distributions of the residence times corresponding to the regimes A_1 (I) and A_3 (III) for the value of the noise intensity $\eta = 0.11$ and their theoretical approximations by the regularities Eq. (6) are presented. A good agreement between the theoretical and numerical results are clearly seen for both considered regimes. Therefore, we can conclude that for a small noise intensity the residence time distributions corresponding to two coexisting regimes in erbium-doped fiber laser obey the exponential law.

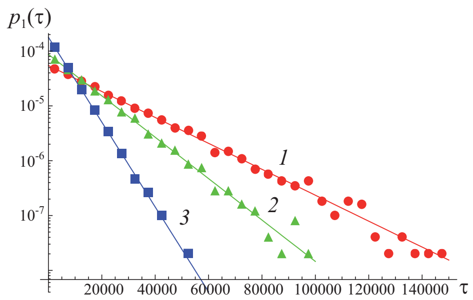

Figure 3. Distributions of laminar phase lengths for two mutually coupled Lorenz systems Eq. (12) being in the regime of intermittent generalized synchronization for different values of the coupling parameter (symbols) and their analytical approximations by regularity Eq. (6) (straight lines) for *(1)* $\varepsilon = 5.8$ and $T_1 = 18357$, *(2)* $\varepsilon = 5.5$ and $T_1 = 11519$, and *(3)* $\varepsilon = 5.0$ and $T_1 = 5612$.

4 Intermittent Generalized Synchronization

Let us now apply the proposed theory to the analysis of intermittent generalized synchronization in two mutually coupled Lorenz systems [Zheng et al., 2002; Moskalenko et al., 2012] with two coexisting chaotic attractors. The system under study is described as

$$\begin{aligned} \dot{x}_{1,2} &= \sigma(y_{1,2} - x_{1,2}) + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= r_{1,2}x_{1,2} - y_{1,2} - x_{1,2}z_{1,2}, \\ \dot{z}_{1,2} &= -bz_{1,2} + x_{1,2}y_{1,2}, \end{aligned} \qquad (12)$$

where $(x_{1,2}, y_{1,2}, z_{1,2})^T$ are the state vectors of the interacting systems, $\sigma = 10.0$, $b = 8/3$, $r_1 = 40.0$ and $r_2 = 35.0$ are the control parameters, and ε is the coupling strength.

In our previous papers [Koronovskii et al., 2011; Moskalenko et al., 2012] we have shown that the onset of generalized synchronization in mutually coupled dynamical systems is connected with a sign change in the second initially positive Lyapunov exponent. This occurs in the system Eq. (12) for $\varepsilon > \varepsilon_{GS} = 5.9$. Bellow the boundary of generalized synchronization, the intermittent behavior is observed as the presence of short-term time intervals with the divergence of phase trajectories on the different chaotic attractor sheets, whereas most of time the phase trajectories of interacting systems are almost completely synchronized (see Fig. 5 in [Moskalenko et al., 2012] for details). The trajectory divergence corresponds to *turbulent phases*, whereas the time intervals of the synchronous behavior are called by *laminar phases* in complete agreement with the theory of intermittency.

Thus, the appearance of intermittency near the boundary of chaotic generalized synchronization is connected with the jumps of representation points from one chaotic sheet to another, similar to the behavior of a bistable dynamical system in the presence of noise. Therefore, one can assume that the theory of noised-induced intermittency proposed in Sec. 2 can be applied to the system Eq. (12).

In order to verify the made assumption, we analyze the main statistical characteristics of the laminar phase lengths, i.e. the distributions of the laminar phase lengths for the fixed values of the control parameters, and compare them with the theoretical prediction Eq. (6). Due to the symmetry of the chaotic attractors of these two interacting systems, we have not distinguished the moments of time corresponding to the presence of representation points on the left and right shifts of the chaotic attractors and calculated only one distribution for fixed parameter values. The numerically obtained distributions of the laminar phase lengths for different values of the coupling parameter ε in mutually coupled Lorenz systems Eq. (12) are presented in Fig. 3 by symbols. Their theoretical approximations by the regularity Eq. (6) are shown by solid lines (the parameters of approximation are given in the caption). It is clearly seen the excellent agreement between the numerically obtained data and the results of theoretical predictions.

Thus, the results obtained with the coupled Lorenz oscillators confirm that the theory of noise-induced intermittency can also be applied for characterization of the intermittent behavior near the onset of generalized synchronization.

5 Conclusion

In the present paper, the theory of noise-induced intermittency in bistable dynamical systems has been proposed. We have shown that the residence time distributions for every coexisting regime obey the exponential law. The main results have been illustrated using the examples of the bistable energy model, erbium-doped fiber laser and coupled chaotic Lorenz oscillators.

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