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REDUCED-ORDER MODELING OF ELECTROSTATICALLY-ACTUATED MICRO-BEAMS

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Abstract
This paper addresses the compact modeling of MEMS devices with nonlinear electromechanical forces. Ideally, the reduced-order model should only involve one or two generalized coordinates. We show that a projection method based on the first eigenmode of the linearized unactuated structure is a good choice in terms of simplicity and accuracy. To take into account the influence of distributed electrostatic forces, it is necessary to approximate the nonlinear force in terms of the reduced coordinate. It is shown that a Padé approximation of order 2 is very attractive in terms of accuracy and numerical efficiency.

Key words
Model reduction, electromechanical coupling, nonlinear dynamics, MEMS.

1 Introduction

MEMS are very small devices in which electric as well as mechanical, thermal and fluid phenomena appear and interact. Because of their microscopic scale, strong coupling effects arise between the different physical fields, and some forces, which were negligible at macroscopic scale, have to be taken into account. In order to accurately design such micro-electro-mechanical systems, it is important to handle the coupling between the electric and the mechanical fields.

This work concerns the development of reduced-order models for MEMS devices actuated by nonlinear electromechanical forces. The concept of model order reduction in the framework of nonlinear systems remains an important challenge in the scientific community. Dimensionality reduction covers a wide number of applications and constitutes today a sensible subject of research.

Many authors deal with reduction of MEMS models in the literature. In [Zavracky et al., 1997], the static behavior of a clamped micro-beam is modelled using a single stiffness element to which the electrostatic force is applied. This type of model exhibits large errors regarding to the static pull-in value. In [Swart et al., 1998], an automatic tool called "AutoMM is developed and coupled to the finite element software MEMCAD [Senturia et al., 1992] to generate a reduced model of a three-dimensional MEMS structure. Parameters such as the electrostatic stiffness or the mechanical stiffness are approximated by polynomials in function of the state variables for different responses corresponding to different excitations of the structure. In [Younis et al., 2003] and [Gabbay, 1998], linear normal modes are used to represent the dynamic behavior of the structure submitted to non-linear forces. In [Nayfeh et al., 2005], different reduction methods based on the reduction of nodes of the discretized system and on the reduction of domains are presented. In [Gabbay and Senturia, 1998; Gabbay and Senturia, 2000], a methodology is described to generate models on the basis of shape-functions deduced from a 3D modeling. This methodology to the case of geometric nonlinearities in [Mehner et al., 2000]. In [Hung and Senturia, 1999; Hung et al., 1997; Liang et al., 2002; Lin et al., 2003], the authors study the efficiency of reduction methods based on the basis of modes obtained from a proper orthogonal decomposition (POD) of simulation data generated by a full 3D model. In [Chen et al., 2004], an Arnoldi type method is presented, which is based on the projection of the system in Krylov subspaces. In [Lienemann et al., 2006], the authors emphasize the necessity of developing compact modeling of MEMS and specially micro-beams submitted to electrostatic forces. In [Del Tin, 2007], the nonlinear electromechanical forces are represented by equivalent lumped forces applied to a certain number of retained modes in the reduced-order model.

The objective of the present work is to obtain a simplified model based on a minimal number of degrees-of-freedom. Constructed from an initial finite element model, the reduced model should involve only one or two degrees-of-freedom, and still be able to represent

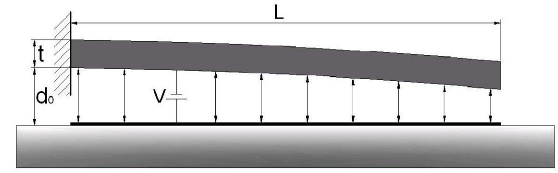

Figure 1. Clamped micro-beam loaded uniformly

the nonlinear dynamics of the system. The paper focuses on the case of an electrostatically-actuated micro-beam, and two linear reduction techniques are considered, i.e. the Guyan method and a reduction method based on linear eigenmodes. An important issue concerns the representation of the nonlinear electromechanical forces in the reduced model. In order to avoid resorting to the initial finite element model for their computation, we show that a simplified analytical expression can be used. In particular, a comparison is realized between Taylor series expansions and Padé approximations.

2 Initial Model

For the purpose of understanding the physical phenomena of electro-mechanical coupling, we restrict the analysis in this paper to the example of the cantilever micro-beam represented in Figure 1, with the geometric parameters: beam length $L = 300\ \mu m$, beam thickness $t = 3\ \mu m$, initial gap distance $d_0 = 1\ \mu m$. The Young modulus is $E = 77\ GPa$ and the volumic mass is $\rho = 2648\ kg$.

The 2D mechanical structure is modeled according to the Euler-Bernoulli beam assumptions, and it is discretized in space using the finite element method. The electrostatic pressure on the beam is computed according to

$$p_e = \frac{1}{2}\frac{\varepsilon_0 V^2}{d^2(x)} \quad (1)$$

where V is the voltage difference between the structure and the fixed electrode, ε_0 is the void permittivity, and $d(x)$ is the gap distance at point x.

After spatial discretization, the equation of motion can be written in matrix form as

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{g}(\mathbf{q}, V) \quad (2)$$

where $\mathbf{q} = \{\, q_1 \ldots q_N \,\}$ is the vector of mechanical generalized coordinates, composed of nodal displacements and rotations. $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the inertia, damping and stiffness matrices respectively. The vector of electromechanical forces $\mathbf{g}(\mathbf{q}, V)$ is given by its components

$$[\mathbf{g}]_i = \frac{\varepsilon_0 S V^2}{2}\frac{\varphi_i}{(d_0 - q_i)^2} \quad (3)$$

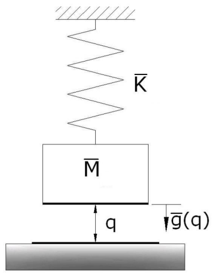

Figure 2. Single DOF system

where S is the surface of one finite element, and φ_i is a shape factor ($\varphi_i = 1$ if the coordinate corresponds to a displacement, $\varphi_i = 1/2$ if the coordinate corresponds to the displacement of the tip and $\varphi_i = 0$ if the coordinate corresponds to a rotational degree of freedom). Let us note that this model could be generalized to account for nonlinear geometric or pre-stressing effects in other types of microstructures.

An important property of this system is the so-called static pull-in voltage $V_{pi,s}$. It is defined as the maximal value of V for which a stable static equilibrium exists. Beyond the pull-in voltage, the system becomes unstable and the beam gets in contact with the electrode. In this work, Eq. (2) is solved numerically in the time domain using the classical Newmark algorithm. In particular, the dynamic response is computed when a step voltage is applied at the initial time. In this case, the dynamic pull-in voltage $V_{pi,d}$ is defined as the voltage at which the dynamic response is no more periodic but becomes unstable. Let us note that the following inequality holds: $V_{pi,d} < V_{pi,s}$.

3 Model Reduction

The aim of compact modeling is to extract from the rather complex model in Eq. (2) a simpler analytical expression. For example, one could represent the structure illustrated in Figure 1 by a simple spring-mass system shown in Figure 2. This simplified one-dimensional system consists in a capacitor made of two parallel plates between which a voltage is applied. The upper plate is supported by a spring and the lower plate is grounded. This spring-mass model is representative of the mode of operation of the micro-beam shown in Figure 1 but also of many other electrostatically-actuated MEMS devices. The dynamic equilibrium equation of the simplified undamped system is

$$\overline{M}\ddot{q} + \overline{K}q = \overline{g}(q, V) \quad (4)$$

where q is the displacement of the mass, as shown in Figure 2; $\overline{M}$, $\overline{K}$ are the equivalent mass and stiffness coefficients respectively and $\overline{g}(q, V)$ is the equivalent nonlinear electrostatic force. The objective of the present work is to define a systematic procedure in order to get suitable expressions for $\overline{M}$, $\overline{K}$ and $\overline{g}$.

3.1 Principle of a Subspace Reduction Method

Subspace reduction methods allow to extract the reduced model represented by $\overline{M}$, $\overline{K}$ and $\overline{g}$ from the initial model. We propose to construct the subspace from the linearized, undamped and unactuated system. Note that for pre-stressed micro-structures, the geometric stiffness matrix should be included in the linearized form. Let us consider the eigenvalue problem of dimension $N \times N$ defined from the linearized equations of motion

$$\mathbf{K}\,\mathbf{x} = \omega^2\,\mathbf{M}\,\mathbf{x}. \tag{5}$$

The aim of a reduction method is to build a model of dimension $m < N$ (ideally $m = 1$ in our case) which exhibits a dynamic behavior as close as possible to the original system. To do this, a transformation matrix $\mathbf{R}$ of dimension $N \times m$ is sought such that the degrees-of-freedom of the original model and the m generalized coordinates $\boldsymbol{\eta}$ of the reduced model are related by

$$\mathbf{q} = \mathbf{R}\,\boldsymbol{\eta}. \tag{6}$$

According to Hamilton's principle, the reduced mass and stiffness matrices are given by

$$\overline{\mathbf{M}} = \mathbf{R}^T\,\mathbf{M}\,\mathbf{R} \quad \text{and} \quad \overline{\mathbf{K}} = \mathbf{R}^T\,\mathbf{K}\,\mathbf{R}. \tag{7}$$

According to the principle of virtual work, the expression of the nonlinear force vector is given by

$$\overline{\mathbf{g}} = \mathbf{R}^T\,\mathbf{g}(\mathbf{q}, V). \tag{8}$$

Hence, the whole procedure relies on the definition of the subspace matrix $\mathbf{R}$. In the following, two different approaches are considered for the definition of $\mathbf{R}$: Guyan's method and a method on linear eigenmodes.

3.2 Guyan's Method of Reduction

The formulation of the model reduction problem due to Guyan is based on the partition of the coordinates $\mathbf{q}$ into significant (master or retained) coordinates $\mathbf{q}_R$ and insignificant (slave or condensed) coordinates $\mathbf{q}_C$ so that the equation of motion takes the form

$$\begin{bmatrix} \mathbf{K}_{RR} & \mathbf{K}_{RC} \\ \mathbf{K}_{CR} & \mathbf{K}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_R \\ \mathbf{q}_C \end{Bmatrix} = \omega^2 \begin{bmatrix} \mathbf{M}_{RR} & \mathbf{M}_{RC} \\ \mathbf{M}_{CR} & \mathbf{M}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_R \\ \mathbf{q}_C \end{Bmatrix}. \tag{9}$$

The transformation matrix is defined by [Géradin and Rixen, 1997]

$$\mathbf{R} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{CC}^{-1}\,\mathbf{K}_{CR} \end{bmatrix}. \tag{10}$$

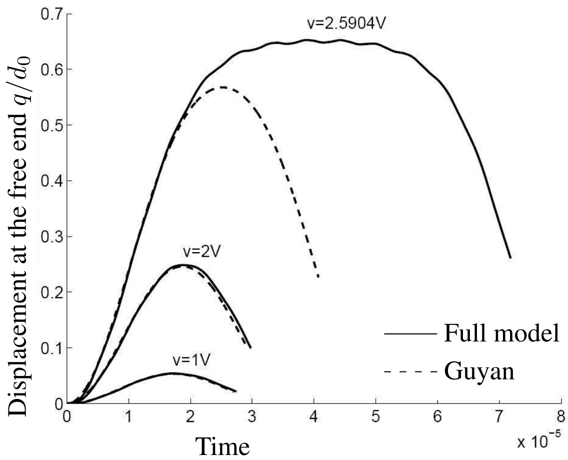

Figure 3. Dynamic behavior of the clamped micro-beam loaded uniformly (Guyan)

Compact modeling of MEMS implies to retain a single coordinate $q_R = \eta$ ($m = 1$). In the example of Figure 1, the master coordinate is chosen at the free extremity of the beam. In Figure 3, one observes that the dynamic behavior of the beam is not well captured when the voltage comes close to the dynamic pull-in voltage. Let us note that this figure represents only a portion of the first oscillation in each case.

A possible explanation is that Guyan's method is well-suited if the loads are only applied to the master coordinates, but that it is not appropriate in the present situation since the loads are distributed over the whole set of coordinates.

3.3 Reduction Based on Linear Eigenmodes

The reduction matrix can be defined as the truncated modal matrix

$$\mathbf{R} = \begin{bmatrix} \mathbf{x}_1 \ldots \mathbf{x}_m \end{bmatrix}$$

whose columns are the eigenmodes of Eq. (5). In this case, the transformation (6) leads to the normal equations

$$\ddot{\eta}_r + \omega_r^2\,\eta_r = \frac{\mathbf{x}_r^T\,\mathbf{g}(\mathbf{q}, V)}{\mu_r} \qquad r = 1, \ldots, m \tag{11}$$

where μ_r denotes the modal mass. Results are shown in Figures 4 and 5 for a basis of 1 mode and 3 modes respectively.

Using only one eigenmode, a remarkably good approximation of the dynamic behavior of the system can be obtained. Using three eigenmodes allows to improve the fidelity of the reduced model near the dynamic pull-in instability. In this example, one concludes that the eigenmodes computed from the linearized model form a sufficiently good basis for model reduction, and that it is not really necessary to consider more sophisticated approaches.

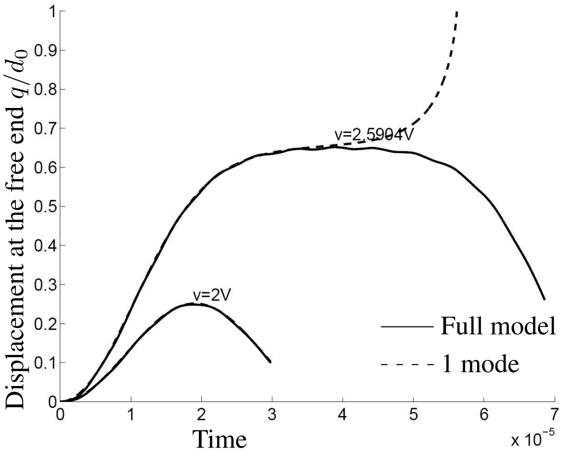

Figure 4. Dynamic behavior of the clamped micro-beam loaded uniformly (1 Linear Mode)

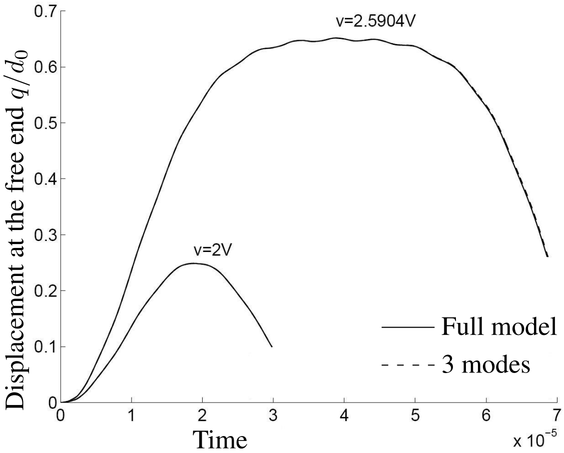

Figure 5. Dynamic behavior of the clamped micro-beam loaded uniformly (3 Linear Modes)

However, even though the model can be reduced to one single mode, the expression of the nonlinear electrostatic force requires a significant number of operations. The next section discusses several possible simplifications of the force.

4 Reduction of the Force Term

Let us consider a model reduced to a single modal dof. In this case, the modal amplitude η can be normalized so that the tip displacement is given by $q_{tip} = \eta d_0$. The electrostatic force in terms of the generalized coordinate η is obtained from (8)

$$\overline{g}(\eta, V) = \mathbf{x}_1^T \mathbf{g} = \frac{\varepsilon_0 S V^2}{2} \sum_{i=1}^{N} \frac{x_i \varphi_i}{(d_0 - \eta\, x_i)^2} \qquad (12)$$

where $x_i = [\mathbf{x}_1]_i$ is the coordinate i in the first mode-shape vector.

The evolution of the electrostatic force shown in Figure 6 exhibits a series of vertical asymptotes corre-

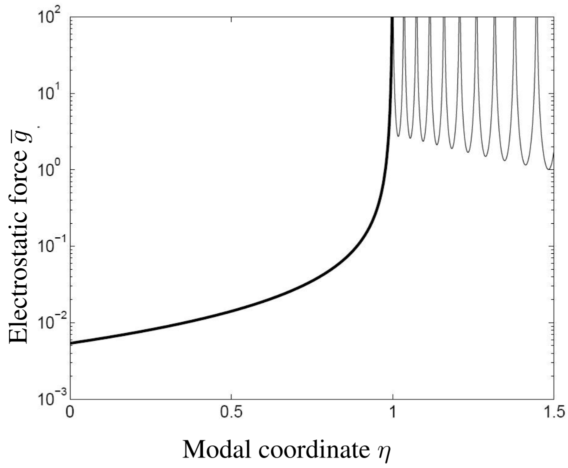

Figure 6. Evolution of the electrostatic force $\overline{g}$ vs. the modal coordinate η

sponding to the poles

$$\eta_i = \frac{d_0}{x_i} \qquad (i = 1, \ldots, N). \qquad (13)$$

However, only the left part of the curve is physical since contact occurs for $\eta = 1$. In the following, several approximated functions $\widetilde{g}$ are proposed for the function $\overline{g}$ defined in Eq. (12).

4.1 Simplified Approximation

A first possible approximation $\widetilde{g}$ of the electrostatic force $\overline{g}$ may be based on the form

$$\widetilde{g}(\eta, V) = \frac{K\, V^2}{(d_0 - \eta\, x_{tip})^2} \qquad (14)$$

with x_{tip} is the amplitude of the mode-shape at the tip node.

The free parameter K can be identified by imposing that $\widetilde{g} = \overline{g}$ at a particular value η_0 of the modal coordinate η. However, it is found that the resulting approximation can only be made accurate around the particular value η_0. In other words, this approximation does not allow to represent the force accurately in the whole displacement range.

4.2 Polynomial Approximation

The electrostatic force may be approximated by a polynomial of degree M, i.e.

$$\widetilde{g}(\eta, V) = V^2 \sum_{j=0}^{M} a_j\, \eta^j. \qquad (15)$$

4.2.1 Taylor Series Expansion The Taylor series expansion of $\overline{g}(\eta, V)$ in the neighborhood of $\eta = 0$ allows to identify the coefficients a_j in Eq. (15). Figure 7

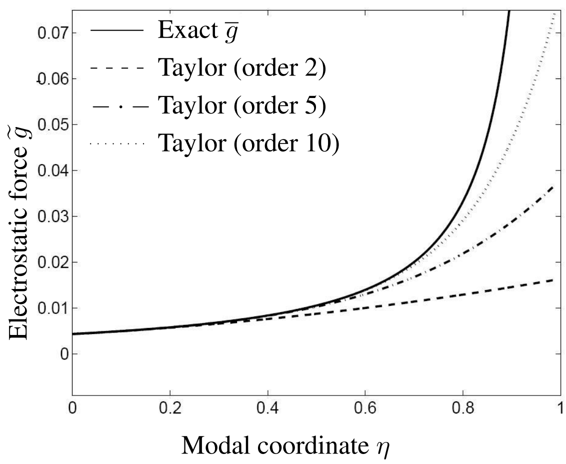

Figure 7. Evolution of the electrostatic force $\widetilde{g}$ vs. the modal coordinate η

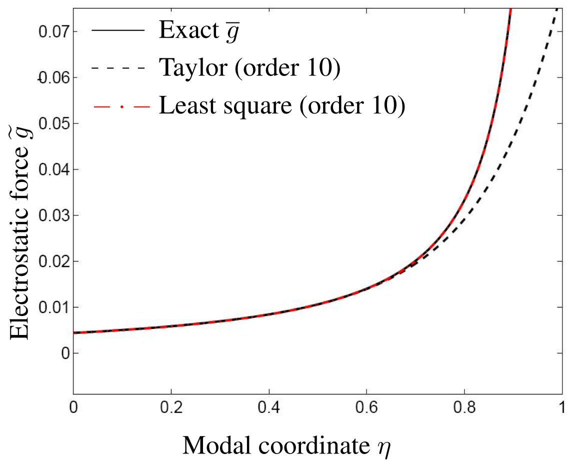

Figure 8. Evolution of the electrostatic force $\widetilde{g}$ vs. the modal coordinate η

shows the voltage-displacement curve for different degrees of Taylor series expansion. Due to the vertical asymptote for $\eta = 1$, a rather high order of approximation is required.

4.2.2 Least Squares Approximation Another way to determine the coefficients of the polynomial approximation (15) is to minimize the error with the reduced electrostatic force by solving in a least squares sense the overdetermined nonlinear system

$$V^2 \sum_{j=0}^{M} a_j \, (\eta_i)^j = \widetilde{g}(\eta_i, V) \qquad \forall \, i = 1, \ldots, P \quad (16)$$

for a set of P values of the coordinate η in the interval $\left[0 \ldots 0.9\right]$. For a polynomial of order $M = 10$, the results are illustrated in Figure 8, and a better agreement is observed than for a Taylor series approximation.

However, a detailed inspection would reveal higher errors in the neighborhood of $\eta = 0$, and it turns out that errors in this region strongly penalize the quality of the model. For example, Figure 9 represents the stable part of the static voltage-displacement curve. The vertical asymptotes represent the static pull-in voltage, at which the stable equilibrium position disappears. One observe that the results given by the Taylor series expansion are more accurate in this case.

4.3 Padé Approximation

The Padé approximation is based on the representation of the electrostatic force in terms of a rational fraction of the form

$$\widetilde{g}(\eta, V) = V^2 \frac{P_L(\eta)}{Q_M(\eta)} \quad (17)$$

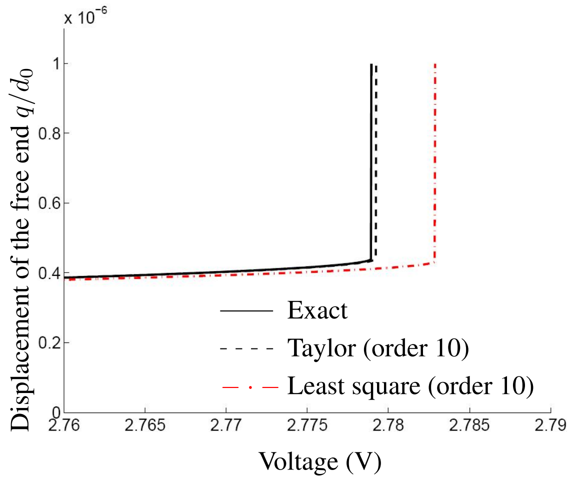

Figure 9. Voltage-displacement curve

with

$$P_L(\eta) = \sum_{i=0}^{L} a_i \, \eta^i,$$
$$Q_M(\eta) = \sum_{i=0}^{M} b_i \, \eta^i. \quad (18)$$

It consists in solving the nonlinear system

$$\widetilde{g}^{(k)} = \overline{g}^{(k)} \qquad \text{for } k = 1, \ldots, M \quad (19)$$

where the superscript (k) denotes the derivative of order (k) at $\eta = 0$. The results of the voltage-displacement curve using a Padé approximation of order 1 and 2 respectively are shown in Figure 10. The quality of the approximation is remarkable in both cases. As in Figure 9, the voltage displacement curve could be plotted to assess the quality of the resulting model. For the sake of conciseness, this plot is not given here, but we could verify that the order 2 Padé

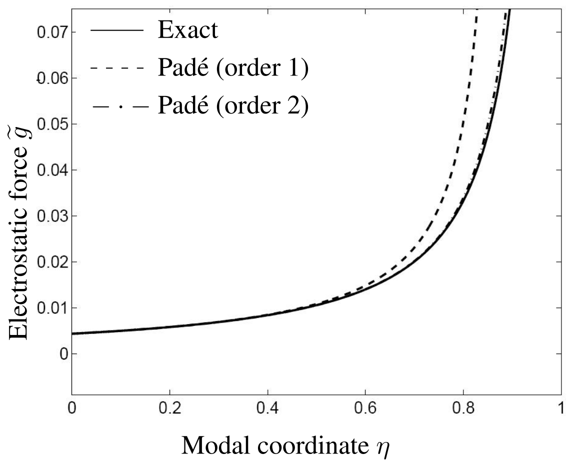

Figure 10. Electrostatic force vs Modal Coordinate

approximation leads to a similar level of accuracy than the order 10 Taylor series expansion.

5 Conclusion

This paper concerns the compact modeling of MEMS devices with nonlinear electromechanical forces. Ideally, the reduced-order model should only involve one generalized coordinate. In the considered example, a projection method based on the first eigenmode of the linearized unactuated structure appears to be a good choice in terms of simplicity and accuracy. To take into account the influence of distributed electrostatic forces, it is necessary to approximate the nonlinear force in terms of the reduced coordinate. It is shown that the Padé approximation of order 2 is superior in terms of accuracy and numerical efficiency.

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