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STABILIZATION OF THE DESIRED UNIFORM ROTATION IN UNDERACTUATED SYSTEMS

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ABSTRACT

In some cases the desired uniform motion may be described by a pair of first integrals of the system with zero control input. The linear-quadratic combination of these two integrals is used to construct Lyapunov function. The control is designed from the condition of decreasing Lyapunov function on the trajectories of the closed loop system. This control may be chosen a priori bounded. This method is applied to stabilize circular motion of a satellite around gravitational center, for stabilization inertia wheel pendulum and for swinging a pendubot.

1. GENERAL IDEA

Consider the following controlled Hamiltonian system

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}\ , \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}\ ,$$
$$\dot{p}_1 = \tau\ , \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}\ , \tag{1}$$

where $q_1 \in R^1, \quad q_2 \in R^n$ are generalized coordinates, $p_1 \in R^1, \quad p_2 \in R^n$ are momenta (generalized impulses), $H = H(q_2, p_1, p_2)$ is Hamiltonian, and τ is a controlling generalized force (torque), q_1 is a cyclic variable. This system under the condition $\tau = 0$ has two classical first integrals H and p_1.

In some cases the desired motion can be described by the variables $q_1 = \omega_d t + q_{10}, q_2 = 0, ..., q_n = 0, \dot{q}_{1d} = \omega_d, \dot{q}_2 = 0, ..., \dot{q}_n = 0$ Let us consider the following Lyapunov function proposed by Chetaev (Rouche e.a. 1977)

$$V = H + \lambda p_1 + \frac{\kappa}{2} p_1^2 \tag{2}$$

where constants λ and κ are chosen so that Lyapunov function would be positive definite. If the following expression

$$\tau = -F(\dot{q}_1 + \lambda + \kappa p_1) \tag{3}$$

where F is a continuous, strictly increasing, possibly bounded function with the condition $F(0) = 0$, f.e. $F(s) = s$ or $F(s) = \arctan(s)$ is chosen as control, then $\dot{V} \leq 0$ and in some cases it can be applied Barbashin and Krasovsky theorem to prove asymptotic stability. The proposed method of stabilization develops the one considered earlier (Burkov, 2004).

In the following sections the examples of application of this general idea are considered.

A SATELLITE AROUND GRAVITATIONAL CENTER

If the gravitational center coincides with the origin of the polar coordinate system r, φ, then the dynamics of the system can be described as follows

$$m\ddot{r} = mr\dot{\varphi}^2 - \frac{\mu m}{r^2} \tag{4}$$
$$m\ddot{\varphi} = -\frac{2m\dot{r}\dot{\varphi}}{r} + \frac{u_\varphi}{r}$$

where m is the mass of the satellite, μ is gravitational constant and u_φ is tangential force. We suppose that the satellite is controlled by only one input $u = u_\varphi / r$. Chang e.a.,2002 considered the more simple case of two control inputs (radial and tangential). The system has the kinetic energy $K = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2)$, the potential energy $\Pi = -\frac{\mu m}{r}$ and the moment $S = mr^2\dot{\varphi}$. Let the circular motion $r = r_d, \dot{\varphi} = \omega_d$ with the well-known condition $\omega_d^2 r_d^3 = \mu$ be the desired motion. Our system under the condition $u = 0$ has classical first integrals $I_1 = \frac{2}{m}(K + \Pi)$ and $I_2 = \frac{S}{m}$. As Lyapunov function consider the linear-quadratic combination of first integrals

$$V = I_1 + \lambda I_2 + \kappa I_2^2.$$

If we choose as $\lambda = -2\omega_d - 2\kappa r_d^2 \omega_d$ and $\kappa > 3/r_d^2$, then this function will be positive definite with respect to the variables $r - r_d, \dot{r}, \dot{\varphi} - \omega_d$. We choose as control input the following expression

$$u = -F[\dot{\varphi} - \omega_d + \kappa(S - r_d^2 \omega_d)] \tag{5}$$

Proposition. The closed loop system (4), (5) has the desired circular motion under the condition $u = 0$. This motion is asymptotically stable with respect to the variables $r - r_d, \dot{r}, \dot{\varphi} - \omega_d$.

Idea of proof. Apply Barbashin-Krasovsky theorem on asymptotic stability.

Modelling. In Fig.3 there is the graph of the radius $r(t)$ for the following parameters $m = \mu = r_d = \omega_d = 1$, $\kappa = 5$, $F(\tau) = \arctan\tau$, and initial data $r(0) = 2.7, \dot{r}(0) = 0, \dot{\varphi}(0) = 1/\sqrt{2.7^3}$.

INERTIA WHEEL PENDULUM

The inertia wheel pendulum is shown schematically in Fig.2. It is a physical pendulum with a symmetric disk attached to the end which is free to spin about an axis parallel to the axis of rotation of the pendulum. The disk is actuated by a motor and the coupling torque generated by the angular acceleration of the disk can be used to actively control the system. The equations of motion are as follows

$$(J + J_r)\ddot{\theta} + J_r\ddot{\varphi} + mgl\sin\theta = 0 \qquad (6)$$

$$J_r\ddot{\theta} + J_r\ddot{\varphi} = u,$$

where θ is the pendulum angle, φ is the disk angle, u is the motor torque input and J, J_r, m, g, l are positive parameters. Let $\theta = 0, \dot{\varphi} = \omega_d$ be the desired motion. Our system has the kinetic energy

$$K = \frac{1}{2}((J + J_r)\dot{\theta}^2 + J_r\dot{\varphi}^2 + 2J_r\dot{\theta}\dot{\varphi})$$

and potential energy $\Pi = mgl(1 - \cos\theta)$. We consider the following simple control input

$$u = -F(\dot{\varphi} - \omega_d). \qquad (7)$$

Kolesnichenko e.a., 2002 proposed another control input which needs information on $\dot{\theta}$.

Proposition. The closed loop system (6), (7) has the desired down position which is asymptotically stable with respect to the variables $\theta, \dot{\theta}, \dot{\varphi} - \omega_d$ and unstable upper position $\theta = \pi, \dot{\theta} = 0, \dot{\varphi} = \omega_d$.

Idea of proof. Consider Lyapunov function

$$K + \Pi - \omega_d\frac{\partial(K + \Pi)}{\partial\dot{\varphi}}$$

Modelling. In Fig.3 there is the graph of the angle $\theta(t)$ for the following parameters $J = J_r = m = l = g = \omega_d = 1$, $F(\tau) = \arctan\tau$, and initial data $\theta(0) = 1, \dot{\theta}(0) = 0, \dot{\varphi}(0) = 1$.

PENDUBOT

Consider the two link underactuated planar robot, called pendubot, see Fig.4. We will consider the following notations: m_1 is the mass of the first link, m_2 the mass of the second link, q_1 the angle that the first link makes with the axis OX, q_2 the angle that the second link makes with the first link, l_1 and l_2 the lengths of links, l_{c1} and l_{c2} the distances to center of masses, I_1 and I_2 the moments of inertia of he links about their centroids. We introduce the following parameters: $n_1 = m_1l_{c1}^2 + m_2l_c^2 + I_1$, $n_2 = m_2l_{c2}^2 + I_2$, $n_3 = m_2l_1l_{c2}$, $n_4 = m_1l_{c1} + m_2l_1$, $n_5 = m_2l_{c2}$. The kinetic energy is expressed as follows $K = \frac{1}{2}(v_1, v_2)D(q_2)(v_1, v_2)^t$, where $v_i = \dot{q}_i$ and

$$D = \begin{pmatrix} n_1 + n_2 + 2n_3\cos q_2 & n_2 + n_3\cos q_2 \\ n_2 + n_3\cos q_2 & n_2 \end{pmatrix}$$

Using Lagrange procedure the equations of dynamics are obtained (Fantoni e.a. 2000)

$$d_{11}\dot{v}_1 + d_{12}\dot{v}_2 + n_3\sin q_2(-2v_1v_2 - v_2^2) = u, \qquad (8)$$

$$d_{21}\dot{v}_1 + d_{22}\dot{v}_2 + n_3\sin q_2 v_1^2 = 0,$$

where u is controlling torque acting on the first link. Let $v_1 = \omega_d$, $q_2 = 0, v_2 = 0$ be the desired motion. We consider the following simple control input

$$u = -F(\dot{q}_1 - \omega_d). \qquad (9)$$

Proposition. The closed loop system (8), (9) has the desired rotation which is asymptotically stable with respect to the variables $v_1 - \omega_d, q_2, v_2$ and unstable rotation $v_1 = \omega_d, q_2 = \pi, v_2 = 0$.

Idea of proof. Consider Lyapunov function

$$K - \omega_d\frac{\partial K}{\partial\dot{q}_2}$$

Modelling. In Fig.5 there is the graph of the velocity $v_1(t)$ for the following parameters $m_i = l_i = \omega_d = 1$, $F(\tau) = \tau$, and initial data $q_1(0) = 0, q_2(0) = 0.3, v_1(0) = 1.5, v_2(0) = 0.5$.

CONCLUSION

The Chetaev method for constructing Lyapunov functions is effective to solve difficult problems of nonlinear stabilization. Some questions are still open in the considered problems; for example, robustness of stabilization with respect to noise in measurements and external deterministic or stochastic disturbances. It is necessary to investigate the domain of attractions. This is the theme for future research.

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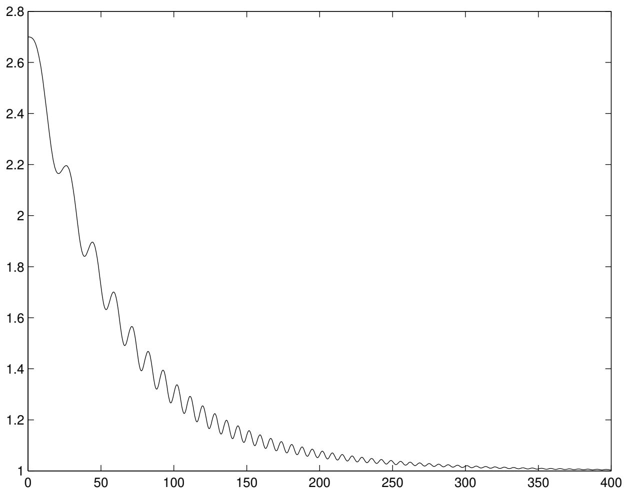

Figure 1: Radius $r(t)$

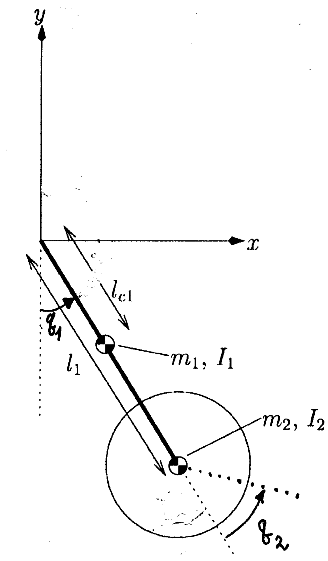

Figure 2: Inertia wheel pendulum

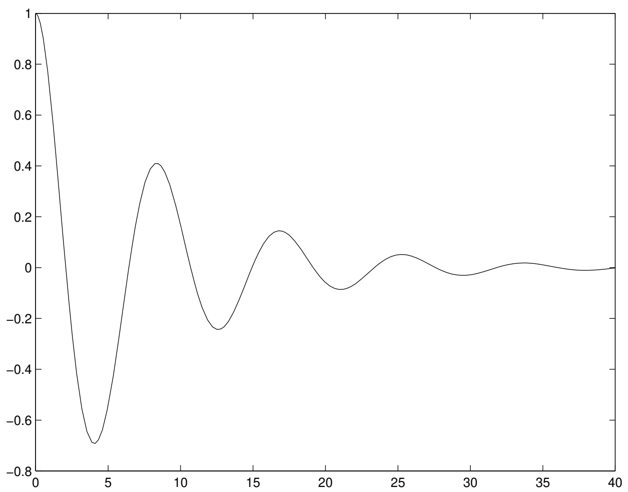

Figure 3: Angle $\theta(t)$

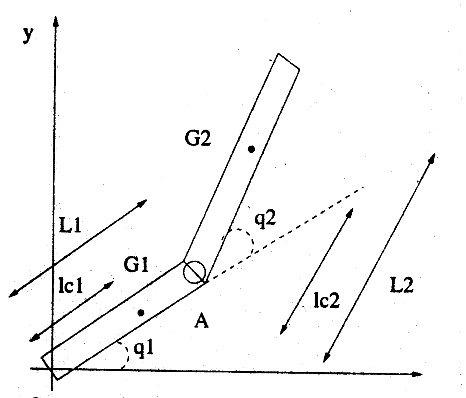

Figure 4: Pendubot

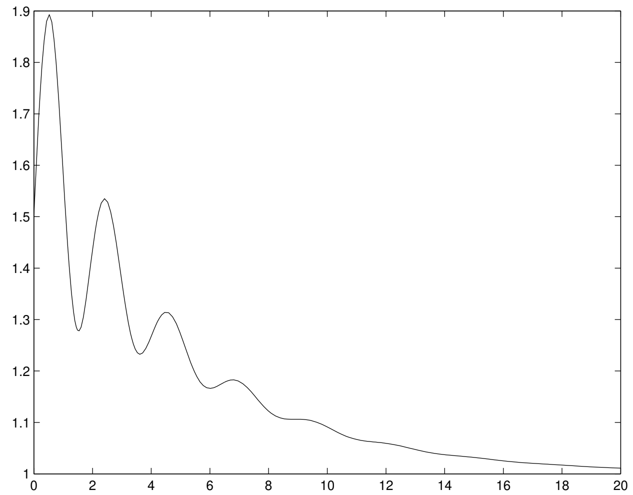

Figure 5: Velocity $v_1(t)$