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CONDITIONS OF YAKUBOVICH OSCILLATORITY FOR NONLINEAR SYSTEMS UNDER DISTURBANCES

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Abstract

This paper explores the concept of Yakubovich oscillatory behavior. The existing result on this topic has been extended, and sufficient conditions for Yakubovich oscillatory behavior in nonlinear systems with bounded delay have been established and proven. Estimates of the oscillation range for nonlinear dynamic systems are provided. To illustrate the theory, the oscillatory nonlinear system with cubic nonlinearity and bounded delay has been considered.

Key words

Yakubovich oscillatority, nonlinear systems, Lyapunov function, disturbance.

1 Introduction

The concept of oscillation as a process with varying degrees of repeatability has undergone significant changes throughout its existence. At the turn of the 19th and 20th centuries, it became clear that linear models of oscillations were not sufficient to describe new phenomena and processes in physics and technology. This required the development of the corresponding mathematical apparatus, namely the theory of nonlinear oscillations, the foundations of which were laid in the works of A. Poincaré, B. Van der Pol, A. A. Andronov, N. M. Krylov and N. N. Bogolyubov [Bogoliubov and Mitropolsky, 1961; Andronov et al., 1966; Leonov et al., 1996].

Based on these works many definitions of the concept of "oscillation" were introduced [Fradkov and Pogromsky, 1998; Leonov et al., 1996]. Oscillation can be understood as the behavior of a function that does not tend towards zero or diverge to $-\infty$ or $+\infty$. In other words, this function does not have a limit. The definition of oscillation should be applicable to a wide range of systems and provide a practically useful approach to study it. The most general definition of oscillation was proposed by V. V. Nemytskyy [Nemytskii, 1961], but effective criteria for determining whether a function oscillates in this sense have not yet been established. V. A. Yakubovich developed an effective and useful concept for studying oscillatory behavior [Yakubovich, 1973; Tomberg and Yakubovich, 1989]. This criterion is applicable to Lurie-type systems, which can be divided into linear and nonlinear parts. However, it does not provide estimates of the oscillation amplitude. D. V. Efimov and A. L. Fradkov extended the analysis of oscillations and developed a method for estimating their amplitude [Efimov and Fradkov, 2006; Efimov and Fradkov, 2009]. Since disturbance is an integral part of the functioning of any objects in the modern world, in this work, the effective results obtained by D. V. Efimov and A. L. Fradkov have been generalized to the case of nonlinear systems with disturbance. Oscillation is one of the most widespread regimes of system behavior, therefore there are many works devoted to studying and controlling it [Fradkov and Pogromsky, 1998; Andrievsky and Guzenko, 2014; Plotnikov and Andrievsky, 2013; Blekhman, 2023].

The rest of the paper is organized as follows. Section 2 recalls the concept of oscillation. In Sec. 3 the conditions for oscillatory behavior in a nonlinear system with a bounded disturbance are obtained. Section 4 demonstrates the application of the obtained results to a nonlinear system, and Sec. 5 presents the simulation results. Finally, Sec. 6 concludes the paper.

Notation. Throughout the paper $\mathbb{R}^n$ denotes the n dimensional real Euclidean space with vector norm $|\cdot|$; notation $z = \mathrm{col}(x, y)$ means that z is a vector of two components $x,\ y$.

2 Oscillatority

Consider a general model of a nonlinear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a state vector, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function.

Remind definition about oscillatority for nonlinear dynamical system introduced by V. A. Yakubovich [Yakubovich, 1973; Tomberg and Yakubovich, 1989] and modified by D. V. Efimov and A. L. Fradkov [Efimov and Fradkov, 2006; Efimov and Fradkov, 2009].

Definition 1. *The solution $\mathbf{x}(\mathbf{x_0}, t)$ of the system (1) with initial condition $\mathbf{x_0} \in \mathbb{R}^n$ is called a $[\alpha, \beta]$-oscillation with respect to the output $y = \eta(\mathbf{x})$, where $\eta : \mathbb{R}^n \to \mathbb{R}$ is a continuous monotone function, if the solution is defined for all $t \geq 0$ and*

$$\varliminf_{t\to+\infty} y(t) = \alpha; \quad \varlimsup_{t\to+\infty} y(t) = \beta;$$
$$-\infty < \alpha < \beta < +\infty.$$

The solution $\mathbf{x}(\mathbf{x_0}, t)$ of the system (1) with initial condition $\mathbf{x_0} \in \mathbb{R}^n$ is called oscillating if there exists an output y and constants α, β such that it is a $[\alpha, \beta]$-oscillation with respect to the output y.

The system (1) is called oscillatory if for almost all initial conditions $\mathbf{x_0} \in \mathbb{R}^n$ its solutions $\mathbf{x}(\mathbf{x_0}, t)$ are oscillating.

If η is a vector function, then the system (1) is called oscillatory if at least one component of the output is oscillating. It should be noted that a nonlinear system has a non-empty set of equilibrium points, for which the solutions are not oscillations. Therefore, the term "almost all initial conditions" is added to the definition. The constants α and β are exact asymptotic bounds for the output y. Thus, the oscillatory property for system (1) indicates that the auxiliary output $y = \eta(\mathbf{x})$ is ultimately bounded and locally unstable. Recall the definition of ultimate boundedness [Pogromsky and Nijmeijer, 2001; Willems, 1972].

Definition 2. *The solution $\mathbf{x}(\mathbf{x_0}, t)$ of the system (1) is called ultimatelly bounded if there exist positive constants Δ_0 and Δ such that for all initial conditions $\mathbf{x}_0 \in \mathcal{B}_{x_0} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq \Delta_0\}$ the following inequality is fulfilled:*

$$|\mathbf{x}(\mathbf{x_0}, t)| \leq \Delta, \quad t \geq t^*. \tag{2}$$

In general, the property of ultimate boundedness can be established under Lyapunov-like conditions [Khalil, 2002] or under the condition of semi-passivity [Pogromsky and Nijmeijer, 2001]. The property of local instability can be established using a linearization approach [Yakubovich, 1973; Efimov and Fradkov, 2006] or Lyapunov-like conditions [Efimov and Fradkov, 2006; Efimov and Fradkov, 2009].

3 Main Result

Consider a model of a nonlinear dynamical system under bounded disturbance:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \xi, \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a state vector, $\xi = \xi(t) \in \mathbb{R}^n$ is a bounded differentiable in t disturbance, i.e. $\exists\, d > 0 : |\xi| < d$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function.

The problem is to find conditions for system (3) to be oscillatory. Before presenting the formulation of the criterion, recall the definition of a K_∞-function. A continuous function $v : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class K if it is strictly increasing and $v(0) = 0$. It belongs to K_∞ class if it belongs to class K and is radially unbounded.

Theorem 1. *Let the solutions of the system (3) with a bounded disturbance ξ, i.e. $|\xi| < d$, be ultimately bounded. Let there exists a continuous and locally Lipshitz Lyapuynov function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the inequalities*

$$v_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq v_2(|\mathbf{x}|),$$

for all $\mathbf{x} \in \mathbb{R}^n$, where $v_1, v_2 \in K_\infty$. Furthermore, let the derivative of V with respect to the system (3) satisfy the inequalities

$$\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) > 0,$$

for $X_1 < |\mathbf{x}| < X_2$, where $0 < X_1 < X_2 < \infty$. If the initial conditions of the system (3) do not belong to the set $\{\mathbf{x} : |\mathbf{x}| \leq X_1\}$, and the set

$$\Omega = \{\mathbf{x} : |\mathbf{x}| > v_2^{-1} \circ v_1(X_2)\}$$

does not contain locally stable equilibrium points of the system (3), then the system (3) is oscillatory with respect to the output $|\mathbf{x}|$, and

$$|\mathbf{x}| > v_2^{-1} \circ v_1(X_2).$$

Proof. Analyzing the properties of the system (3), consider the initial conditions of the state vector from the set $\{\mathbf{x} : |\mathbf{x}| > X_1\}$. The ultimate boundedness of the solutions of the system (1) follows from the assumption. This means that there exists a positive constant Δ such that (2) is fulfilled. Since these facts and also $\dot{V}(t) > 0$ for $X_1 < |\mathbf{x}| < X_2$, the Lyapunov function $V(t)$ has an upper bound and assymptoticaly satisifes the inequality $V(t) > v_1(X_2)$, where $|\mathbf{x}(t)| > v_2^{-1} \circ v_1(X_2)$.

Due to the initial conditions of the state vector being chosen from the set $\{\mathbf{x} : |\mathbf{x}| > X_1\}$ and the boundedness of the trajectory $\mathbf{x}(t)$, $t \geq 0$, it has a non-empty

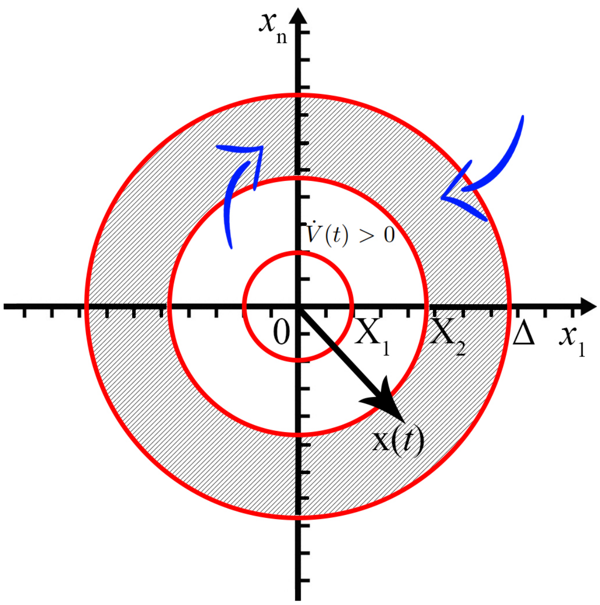

Figure 1. Ultimate boundedness and local instability of the solution $\mathbf{x}(t)$ of the system (1) with a bounded disturbance ξ.

compact and closed ω-limit set contained in the set Ω. By the assumption, the set Ω does not contain stable equilibrium points of the system (3). Therefore, the non-empty compact and closed ω-limit set of trajectory $\mathbf{x}(t)$ also does not contain such invariant subsets. Then, this solution is $[\alpha;\beta]$-oscillation with respect to the output $|\xi|$ for $v_2^{-1} \circ v_1(X_2) < \alpha < \beta < \Delta$, which means the oscillatority of the system (3). The theorem has been proven.

Corollary 1. *Since* $|x_i| \le |\mathbf{x}|\ \forall\, x_i,\ i = 1,\dots,n,$ *then* $v_2^{-1} \circ v_1(X_2)$ *is the lower bound of the oscillation amplitude of the modulus of the state coordinate* $|x_i|$.

The graphical proof of Theorem 1 is depicted in Fig. 1. Initial conditions $\mathbf{x}(0)$ should not be chosen within the ball of radius X_1 because the value of the Lyapunov function derivative $\dot{V}(t)$ can be both positive and negative.

If the initial conditions $\mathbf{x}(0)$ of the system (1) belong to the ring of radii X_1 and X_2, then the trajectory of the solution tends to the ring with radii X_2 and Δ (dashed area in Fig. 1) because of the positiveness of the Lyapunov function derivative in this area $\dot{V}(t) > 0$ for $X_1 < |\mathbf{x}| < X_2$.

The same is valid for the area $|\mathbf{x}| > \Delta$ because of the ultimate boundedness of the solution. The solution $\mathbf{x}(t)$ asymptotically oscillates in the ring of radii X_2 and Δ, because there are no stable equilibrium points.

4 Example

Let's take a look at a nonlinear system with a bounded disturbance ξ, i.e. $0 < |\xi| < d$:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 + x_2 + \xi, \\ \dot{x}_2 &= x_2 - x_2^3 - x_1. \end{aligned} \tag{4}$$

One can check that this system has the only equilibrium point $(0;0)$ for $\xi \equiv 0$. To prove the local instability of the equilibrium point, consider the following Lyapunov function:

$$V(\mathbf{x}) = \frac{1}{2}\left(x_1^2 + x_2^2\right), \tag{5}$$

where $\mathbf{x} = \mathrm{col}(x_1, x_2)$ is a state vector. Find its derivative with respect to the system (4):

$$\dot{V}(\mathbf{x}) = -x_1(x_1^3 - x_1 - \xi) - x_2(x_2^3 - x_2).$$

Express the variable x_2 in terms of x_1 as $x_2 = \gamma x_1$ for $x_1 \ge x_2$ or $x_1 = \gamma x_2$ otherwise, where $\gamma \in [-1;1]$ is a constant. Then, $\dot{V}$ can be presented as:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= -(1+\gamma^4)x_1^4 + (1+\gamma^2)x_1^2 + \xi x_1, \\ &\qquad \text{for } x_1 \ge x_2, \\ \dot{V}(\mathbf{x}) &= -(1+\gamma^4)x_2^4 + (1+\gamma^2)x_2^2 + \gamma\xi x_2, \\ &\qquad \text{for } x_1 < x_2. \end{aligned} \tag{6}$$

To find the area of positive values of $\dot{V}$, one needs to calculate the roots of the corresponding cubic equations (6):

$$\begin{aligned} x_1^3 - \frac{1+\gamma^2}{1+\gamma^4}x_1 - \frac{\xi}{1+\gamma^4} &= 0, \\ x_2^3 - \frac{1+\gamma^2}{1+\gamma^4}x_2 - \frac{\gamma\xi}{1+\gamma^4} &= 0. \end{aligned}$$

One should find the range of values for x_1 and x_2, for which $\dot{V}(t) > 0$. This is possible if and only if the cubic equations have three roots. In turn, this is fulfilled if and only if:

$$\begin{aligned} \frac{\xi^2}{4(1+\gamma^4)^2} - \frac{(1+\gamma^2)^3}{27(1+\gamma^4)^3} &< 0, \text{ for } x_1 \ge x_2, \\ \frac{\gamma^2\xi^2}{4(1+\gamma^4)^2} - \frac{(1+\gamma^2)^3}{27(1+\gamma^4)^3} &< 0, \text{ for } x_1 < x_2. \end{aligned}$$

Both of these conditions are fulfilled for

$$|\xi| < \frac{2\sqrt{3}}{9},$$

To find the narrowest range of values for x_1 and x_2, for which $\dot{V}(t) > 0$, one needs to choose $\gamma = 0$ for the first inequality and $|\gamma| = 1$ for the second one.

In this case, using trigonometric Vieta's formula, one can find the roots of the cubic equation:

$$\begin{aligned} x_1^* &= \frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}\xi}{2}\right) + \frac{2\pi k}{3}\right], \\ x_2^* &= \frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}\xi}{4}\right) + \frac{2\pi k}{3}\right], \end{aligned} \tag{7}$$

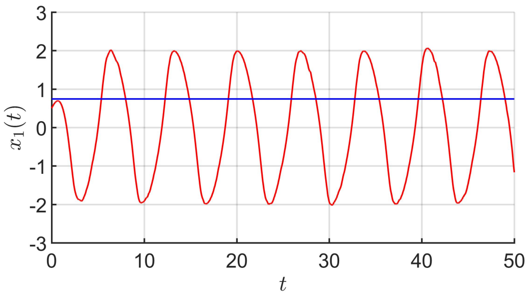

Figure 2. Dynamics of the system (4) state vector component x_1 with a bounded disturbance (9). Initial conditions: $x_1(0) = 0.5$, $x_2(0) = 0$.

for $k = 0, 1, 2$.

Analyzing the roots, and noting that $\dot{V} \to -\infty$ as $t \to \pm\infty$, one can find the area $|\mathbf{x}| \in [X_1; X_2]$ where $\dot{V}$ takes positive values:

$$X_1 = -\frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}|\xi|}{2}\right) - \frac{2\pi}{3}\right],$$

$$X_2 = -\frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}|\xi|}{2}\right) - \frac{4\pi}{3}\right].$$

To obtain conditions that are independent of ξ, one can use the assumption that $|\xi| < d$ for the boundedness of the disturbance:

$$X_1 = -\frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}d}{2}\right) - \frac{2\pi}{3}\right], \quad (8a)$$

$$X_2 = -\frac{2\sqrt{3}}{3}\cos\left[\frac{1}{3}\arccos\left(\frac{3\sqrt{3}d}{2}\right) - \frac{4\pi}{3}\right]. \quad (8b)$$

The function $\dot{V}$ is strictly positive for all $\{\mathbf{x} : |\mathbf{x}| \in (X_1; X_2)\}$.

The ultimate boundedness of the state $\mathbf{x}$ of the system (4) directly follows from the same Lyapunov function (5). Thus, using Theorem 1, the following result can be obtained.

Theorem 2. *Let the initial conditions of the system* (4) *with a bounded disturbance* ξ *, i.e.* $|\xi| \le d < 2\sqrt{3}/9$, *not belong to the set* $|\mathbf{x}| < X_1$. *Then, the system* (4) *is oscillatory with respect to the output* $|\mathbf{x}|$, *and the amplitude of oscillation of the components of the state vector* $\mathbf{x}$ *is greater than* X_2. *The values of* X_1 *and* X_2 *are defined by* (8a) *and* (8b), *respectively.*

5 Simulation

Let us consider the system (4) with the following disturbance for simulation:

$$\xi(t) = \frac{1}{3}\sin(100t), \quad (9)$$

which satisfies the conditions of Theorem 2. The upper bound of the modulus of this disturbance d is equal to $1/3$. Thus, the values of X_1 and X_2 can be calculated using formulas (8a) and (8b), respectively:

$$X_1 \sim 0.3949, \quad X_2 \sim 0.7422.$$

Initial conditions are chosen as follows: $x_1(0) = 0.5$, $x_2(0) = 0$, which also satisfy the conditions of Theorem 2.

Figure 2 presents the results of simulation. The dynamics of the state vector component $x_1(t)$ is marked by red color, while the estimate on the oscillation amplitude X_2 is marked by blue color. One can see that the solution is oscillating, which illustrates the results of Theorem 2.

6 Conclusion

This paper examines the property of Yakubovich oscillatority. The previous result on this topic was generalized, and sufficient conditions for Yakubovich oscillatority in nonlinear systems with bounded delay were formulated and proven. Estimates of the range of oscillations for nonlinear dynamic systems are provided. As an example, the oscillatory nonlinear system with bounded delay has been considered.

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