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STABILITY OF PERIODIC SOLUTIONS IN LIPSCHITZ SYSTEMS WITH A SMALL PARAMETER

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Abstract

In this paper we study the existence, uniqueness and asymptotic stability of the periodic solutions for the Lipschitz system $\dot{x} = \varepsilon g(t, x, \varepsilon)$. Classical hypotheses in the periodic case of second Bogolyubov's theorem imply our ones. By means of the results established we construct, for small ε, the curves of dependence of the amplitude of asymptotically stable 2π–periodic solutions on parameters of a forced asymmetric oscillator.

Key words

Periodic solutions, asymptotic stability, Lipschitz systems, small parameter.

1 Introduction

In the present paper we study the existence, uniqueness and asymptotic stability of the T–periodic solutions for the system

$$\dot{x} = \varepsilon g(t, x, \varepsilon), \tag{1}$$

where $\varepsilon > 0$ is a small parameter and the function $g \in C^0(\mathbb{R} \times \mathbb{R}^k \times [0, 1], \mathbb{R}^k)$ is T–periodic in the first variable and locally Lipschitz with respect to the second one. As usual a key role will be played by the averaging function

$$g_0(v) = \int_0^T g(\tau, v, 0)d\tau, \tag{2}$$

and we shall look for those periodic solutions that starts near some $v_0 \in g_0^{-1}(0)$.

In the case that g is of class C^1, we remind the periodic case of the second Bogolyubov's theorem ([Bogolyubov, 1945], Ch. 1, § 5, Theorem II) which represents a part of the averaging principle: $\det (g_0)'(v_0) \neq 0$ *assures the existence and uniqueness, for $\varepsilon > 0$ small, of a T–periodic solution of system (1) in a neighborhood of v_0, while the fact that all the eigenvalues of the matrix $(g_0)'(v_0)$ have negative real part, provides also its asymptotic stability.*

It was Mitropol'skii who noticed in [Mitropol'skii, 1959] that various applications require generalization of the second Bogolubov's theorem for Lipschitz right hand parts. Assuming that g is Lipschitz, $g_0 \in C^3(\mathbb{R}^k, \mathbb{R}^k)$ and that all the eigenvalues of the matrix $(g_0)'(v_0)$ have negative real part Mitropol'skii developed an analog of the second Bogolyubov's Theorem proving the existence and uniqueness of a T–periodic solution of system (1) in a neighborhood of v_0. The existence part of the Mitropol'skii's result has been extended to a wide class of continuous systems in [Mawhin, 2005] by means of degree theory. On the other hand stability of periodic solutions obtained via Mitropol'skii's theorem has been studied up to now for particular situations only, we refer to [Glover, Lazer and McKenna, 1989], where the Lipschitz members of the two-dimensional system (1) are represented by jumping nonlinearities and to [Buică and Daniilidis, 2007], where a certain continuity of the Clarke differential of (1) is assumed.

In the next section of the paper assuming that g is piecewise differentiable in the second variable we show in Theorem 2 that Mitropol'skii's conditions imply asymptotic stability of a T–periodic solution of system (1) in a neighborhood of v_0 without any additional assumptions. In other words we show that Bogolyubov's theorem formulated above is valid when g is not necessary C^1. Theorem 2 follows from our even more general Theorem 1 whose hypotheses do not use any differentiability neither of g nor of g_0.

2 Main results

Throughout the paper $\Omega \subset \mathbb{R}^k$ is some open set and for any $\delta > 0$ the δ-neighborhood of $v \in \mathbb{R}^k$ is denoted by $B_\delta(v_0) = \left\{v \in \mathbb{R}^k \ : \ \|v - v_0\| \leq \delta\right\}$. We have the following main result.

Theorem 1. *Let $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ and $v_0 \in \Omega$. Assume the following four conditions.*

(i) For some $L > 0$ we have that $\|g(t, v_1, \varepsilon) - g(t, v_2, \varepsilon)\| \le L\|v_1 - v_2\|$ for any $t \in [0,T],\ v_1, v_2 \in \Omega, \varepsilon \in [0,1]$.

(ii) For any $\gamma > 0$ there exists $\delta > 0$ such that

$$\Big\| \int_0^T g(\tau, v_1 + u(\tau), \varepsilon) d\tau - \int_0^T g(\tau, v_2 + u(\tau), \varepsilon) d\tau - \int_0^T g(\tau, v_1, 0) d\tau + \int_0^T g(\tau, v_2, 0) d\tau \Big\| \le \gamma \|v_1 - v_2\|$$

for any $u \in C^0([0,T], \mathbb{R}^k)$, $\|u\| \le \delta$, $v_1, v_2 \in B_\delta(v_0)$ and $\varepsilon \in [0, \delta]$.

(iii) Let g_0 be the averaging function given by (2) and consider that $g_0(v_0) = 0$.

(iv) There exist $q \in [0,1)$, $\alpha, \delta_0 > 0$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^k$ such that $\|v_1 + \alpha g_0(v_1) - v_2 - \alpha g_0(v_2)\|_0 \le q\|v_1 - v_2\|_0$ for any $v_1, v_2 \in B_{\delta_0}(v_0)$.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$ system (1) has exactly one T–periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

When solution $x(\cdot, v, \varepsilon)$ of system (1) with initial condition $x(0, v, \varepsilon) = v$ is well defined on $[0,T]$ for any $v \in B_{\delta_0}(v_0)$, the map $v \mapsto x(T, v, \varepsilon)$ is well defined and it is said to be the *Poincaré map* of system (1). The proof of existence, uniqueness and stability of the T–periodic solutions of system (1) in Theorem 1 reduces to the study of corresponding properties of the fixed points of this map.

In general it is not easy to check assumptions (ii) and (iv) in the applications of Theorem 1. Thus we give also the following theorem based on Theorem 1 which assumes certain type of piecewise differentiability instead of (ii) and deals with properties of the matrix $(g_0)'(v_0)$ instead of the Lipschitz constant of g_0. For any set $M \subset [0,T]$ measurable in the sense of Lebesgue we denote by $\mathrm{mes}(M)$ the Lebesgue measure of M.

Theorem 2. *Let $g \in C^0(\mathbb{R} \times \Omega \times [0,1], \mathbb{R}^k)$ satisfy (i). Let g_0 be the averaging function given by (2) and consider $v_0 \in \Omega$ such that $g_0(v_0) = 0$. Assume that*

(v) given any $\widetilde{\gamma} > 0$ there exist $\widetilde{\delta} > 0$ and $M \subset [0,T]$ measurable in the sense of Lebesgue with $\mathrm{mes}(M) < \widetilde{\gamma}$ such that for every $v \in B_{\widetilde{\delta}}(v_0)$, $t \in [0,T] \setminus M$ and $\varepsilon \in [0, \widetilde{\delta}]$ we have that $g(t, \cdot, \varepsilon)$ is differentiable at v and $\|g'_v(t, v, \varepsilon) - g'_v(t, v_0, 0)\| \le \widetilde{\gamma}$.

Finally assume that

(vi) g_0 is continuously differentiable in a neighborhood of v_0 and the real parts of all the eigenvalues of $(g_0)'(v_0)$ are negative.

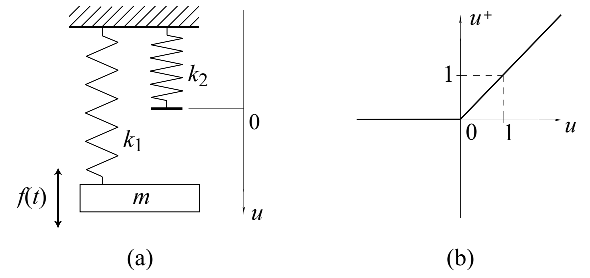

Figure 1. (a) A driven mass attached to an immovable beam via a spring with piecewise linear stiffness, see e.g. [Kryukov, 1967], (b) the jumping nonlinearity $u \mapsto u^+$.

Then there exists $\delta_1 > 0$ such that for every $\varepsilon \in (0, \delta_1]$, system (1) has exactly one T–periodic solution x_ε with $x_\varepsilon(0) \in B_{\delta_1}(v_0)$. Moreover the solution x_ε is asymptotically stable and $x_\varepsilon(0) \to v_0$ as $\varepsilon \to 0$.

For proving Theorem 2 we observe that the property (v) implies (ii), while the property (vi) implies (iv).

3 An example

The differential equation for the coordinate u of the mass attached via nonlinear spring to an immovable beam drawn at Fig. 1 is written down as follows (see [Kryukov, 1967])

$$m\ddot{u} + c\dot{u} + k_1 u + k_2 u^+ = f(t), \tag{3}$$

where f is a force applied to the mass in the vertical direction.

In this section we apply Theorem 2 to find amplitudes of asymptotically stable periodic solutions of the following equation

$$\ddot{u} + \varepsilon c\dot{u} + u + \varepsilon a u^+ = \varepsilon\lambda \cos t\,, \tag{4}$$

which is a form of equation (3), where the relevant coefficients are assumed to be small (see [Bovsunovskii, 2007]). The situation when the coefficient in front of $\dot{u}$ is small and that in front of u^+ is not has been studied in [Glover, Lazer and McKenna, 1989]). The stability of large amplitude periodic solutions in (4) was addressed in [Fabry, 2007].

Some function u is a solution of (4) if and only if $(z_1, z_2) = (u, \dot{u})$ is a solution of the system

$$\begin{aligned} \dot{z}_1 &= z_2\,, \\ \dot{z}_2 &= -z_1 + \varepsilon[-a z_1^+ - c z_2 + \lambda \cos t]\,. \end{aligned} \tag{5}$$

After the change of variables

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

system (5) takes the form

$$\begin{aligned}\dot{x}_1 &= \varepsilon \sin t[a(x_1 \cos t + x_2 \sin t)^+ + \\ &\quad +c(-x_1 \sin t + x_2 \cos t) - \lambda \cos t], \\ \dot{x}_2 &= \varepsilon \cos t[-a(x_1 \cos t + x_2 \sin t)^+ + \\ &\quad +c(x_1 \sin t - x_2 \cos t) + \lambda \cos t].\end{aligned} \quad (6)$$

The corresponding averaged function g_0, calculated according to the formula (2), is

$$g_0(x_1, x_2) = \begin{pmatrix} -\pi c & \pi a/2 \\ -\pi a/2 & -\pi c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \pi\lambda \end{pmatrix}.$$

It can be easily checked that the unique zero of g_0 is

$$\left(\frac{2a\lambda}{a^2 + 4c^2}, \frac{4c\lambda}{a^2 + 4c^2} \right)$$

and the eigenvalues of $(g_0)'$ are $-\pi c \pm i\pi a$. The amplitude of this zero is

$$A = \frac{2|\lambda|}{\sqrt{a^2 + 4c^2}}.$$

To apply Theorem 1 it remains to observe the following proposition.

Proposition 1. *Let $v_0 \in \mathbb{R}^2 \setminus \{0\}$. Then the right hand side of (6) satisfies (ii) for any $c, a, \lambda \in \mathbb{R}$.*

The main result of this section can be now summarized as follows.

Proposition 2. *Assume that $c > 0$ and $A = 2|\lambda|/\sqrt{a^2 + 4c^2} \neq 0$ and take an arbitrary $R > 0$. Then for each $\varepsilon > 0$ sufficiently small, equation (4) has an asymptotically stable 2π-periodic solution whose amplitude goes to A as $\varepsilon \to 0$. Moreover, there are no other 2π-periodic solutions with amplitudes in the interval $[1/R, R]$.*

By Theorem 2 the curves of dependence of the amplitude of asymptotically stable 2π-periodic oscillations in (4) upon the parameters are drawn in Fig. 2. Particularly one can see that this amplitude tends to $+\infty$ as $\sqrt{a^2 + 4c^2} \to 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$ is fixed.

The same considerations like in Propositions 1 and 2 are applied in [Buică, Llibre and Makarenkov, 2008] for detecting amplitudes of asymptotically stable periodic solutions in the forced nonsmooth van der Pol oscillator (see [Hogan, 2003]).

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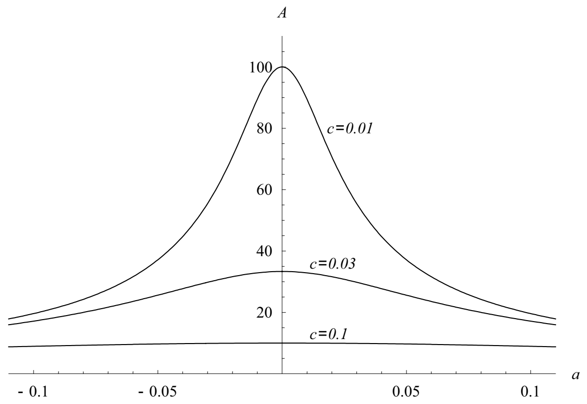

Figure 2. The curves of dependence of the amplitude of asymptotically stable 2π-periodic oscillations in (4) upon the parameter $a \in \mathbb{R}$ constructed for $\lambda = 1$ and distinct c's.

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