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SYNTHESIS OF A MULTIFUNCTIONAL TRACKING SYSTEM FOR ELECTROMECHANICAL CONTROL PLANTS

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Abstract
A class of affine nonlinear SISO systems with relative order of the equivalent input-output form independent from external unmatched disturbances is formalized. For this class of systems, the methods for the synthesis of a multifunctional tracking system under the conditions of parametric uncertainty of the control plant and incomplete measurements have been developed. For information support of discontinuous control, an original method of reduced observer synthesis has been developed. The observer evaluates mixed variables (combinations of state variables, external disturbances and their derivatives) according to measurements the tracking error only. The simulation results of the designed algorithms for an inverted pendulum under conditions of uncertainty are given.

Key words
Nonlinear affine SISO system, tracking, observer of states and disturbances, decomposition.

1 Introduction

The paper deals with nonlinear systems of automatic control with single input and single output (SISO systems) under the influence of external, unmatched disturbances and parametric uncertainty. The tracking problem of the output variable is considered under the assumption that only the tracking error is available for direct measurements. The synthesis of the invariant tracking system under these conditions is non-trivial task and requires the involvement and development of special methods. Standard methods of compensation or suppression of uncertainties are not directly applicable here, since they require the fulfillment of the matching conditions [Isidori, 1995; Nikiforov, 1998; Kvaternic and Lynch, 2011] and others. Therefore, the main role in solution of this problem is given to methods of informational support of the basic control law. In conditions of incomplete measurements, observers of state variables and disturbances are used for these purposes. In the classical formulation parametrically determined models of the control plant and external disturbances are required [Wonham, 1979] for the observers realization. However, adequate modeling of disturbances under constantly changing external factors seems to be an almost irresolvable problem.

In the nonlinear SISO system with unmatched disturbances, the problem of independent estimation the non-measurable variables of the state vector and external disturbances has no solution without expansion the state space by introducing dynamic models that generate external disturbances. Assuming that external disturbances are sufficiently smooth and is bounded by time functions, in this paper we use a method in which the mathematical model of a control plant is represented in the canonical form of input-output with respect to mixed variables (combinations of state variables, external disturbances and their derivatives) [Utkin, Krasnova and Akhobadze, 2008; Utkin V.A and Utkin A.V., 2014; Krasnova and Mysik, 2014; Krasnova and Utkin, 2016]. Such approach does not require direct and inverse changes of variables, since control and observation problems are solved with respect to the same variables of the new coordinate basis.

In Section 2 a class of nonlinear affine systems with the invariance of the canonical form of input-output is formalized. In this form, the properties of complete controllability and observability inherent in the unperturbed system are preserved with the transition to a new coordinate basis of mixed variables. Mixed variables are formed by transformation of state variables with affine appearance of external disturbances and their derivatives. The equivalent input-output system is obtained by sequent differentiating the tracking error. It is essential that in the obtained canonical form the matching conditions are satisfied. The basic law of discontinuous control is synthesized with respect to mixed variables, and ensures the stabilization of the tracking error. Stabilization of errors is invariant to external disturbances and the uncertainty of the multiplier term before the control input.

The main result is presented in Section 3, where a reduced observer of mixed variables is introduced. The procedure for synthesis of linear corrective actions with saturation is formalized, and the method of separation of movements in the space of observation errors is realized. Section 4 presents the simulation results of the developed algorithms for electromechanical control system operating under uncertainty conditions.

2 Canonical form with allowance for disturbances

The nonlinear affine SISO systems presented in the triangular form [Krasnova and Utkin, 2016] is considered

$$\begin{aligned} \dot{x}_i &= f_i(x_1,x_2,\dots,x_{i+1}) + \\ &+ q_i^{\mathrm{T}}(x_1,x_2,\dots,x_i)\eta,\ i=\overline{1,n-1}; \\ \dot{x}_n &= f_n(x) + q_n^{\mathrm{T}}(x)\eta + b_n(x)u, \end{aligned} \tag{1}$$

where $x=(x_1,\dots,x_n)^{\mathrm{T}} \in X \subset R^n$ is the state space vector, $u \in R$ is the control, $x_1(t) \in X_1 \subset R$ is the output (measured and controlled variable), $\eta(t)=(\eta_1(t),\dots,\eta_s(t))^{\mathrm{T}} \in R^s$ is the vector of external disturbances, the components of $\eta(t)$ are assumed to be unknown smooth bounded functions in time with bounded derivatives of order up to $(n-1)$:

$$\begin{aligned} &\left|\eta_j^{(i)}(t)\right| \le N_{ij} = \text{const} > 0,\ t \ge 0, \\ &i=\overline{0,n-1},\ j=\overline{1,s}. \end{aligned} \tag{2}$$

The functions $f_i(x_1,\dots,x_{i+1})$ and the elements of vector-lines $q_i^{\mathrm{T}}(x_1,\dots,x_i) \in R^{1\times s}$ are continuously differentiable with respect to all their arguments at least than $n-i$ times and

$$\frac{\partial f_i(x_1,\dots,x_{i+1})}{\partial x_{i+1}} \ne 0,\ i=\overline{1,n-1}, \tag{3}$$

$$b(x) \ne 0. \tag{4}$$

The conditions (3), (4) and their analogues below are of a local nature and are satisfied in an open, bounded working area of variables variation $x(t) \in X \subset R^n\ \forall t \ge 0$, which is determined by the process technology. At the same time, it is assumed that in system (1) all functions f_i, q_i^{T}, $i=\overline{1,n}$, $b(x)$ have undefined parameters, but at the same time, for all admissible ranges of parameters variations the structural properties (3) – (4) remain unchanged.

The problem statement is to synthesize feedback on the basis of the state and external disturbances, ensuring the tracking of the output variable $x_1(t)$ for a reference signal $g(t) \in X_1 \subset R$. The reference signal is assumed to be smooth, bounded time function with bounded derivatives:

$$\left|g^{(i)}(t)\right| \le G_i = \text{const} > 0,\ t \ge 0,\ i=\overline{0,n}. \tag{5}$$

In most papers on tracking systems, the control plant is considered to be systems (1) with matched disturbances

$$q_i^{\mathrm{T}}(x_1,\dots,x_i) \equiv 0,\ i=\overline{1,n-1}, \tag{6}$$

for which the expressions (3) are conditions for the local observability of the state vector variables with respect to the output, and the expressions (3) – (4) are the conditions of controllability [Isidori, 1995; Fradkov, Miroshnik and Nikiforov, 1999]. Due to the triangular composition of the arguments of functions $f_i(x_1,\dots,x_{i+1}), i=\overline{1,n-1}$, such system is a carrier of the input-output structure with a relative degree n. Thus, for a given system there is the coordinate transformation of local variables to a canonical form in which the output x_1 and the input u with a non-zero gain is separated by integrators:

$$\begin{aligned} &\dot{x}_1 = f_1(x_1,x_2) = \bar{x}_2,\ \partial f_1/\partial x_2 \ne 0; \\ &\dot{\bar{x}}_2 = \frac{\partial f_1}{\partial x_1} f_1(x_1,x_2) + \frac{\partial f_1}{\partial x_2} f_2(x_1,x_2,x_3) = \\ &= h_2(x_1,x_2,x_3) = \bar{x}_3,\ \frac{\partial f_1}{\partial x_2} \ne 0,\ \frac{\partial f_2}{\partial x_3} \ne 0 \Rightarrow \frac{\partial h_2}{\partial x_3} \ne 0; \\ &\dot{\bar{x}}_i = \sum_{j=1}^{i} \frac{\partial h_{i-1}}{\partial x_j} f_j(x_1,\dots,x_{j+1}) = h_i(x_1,\dots,x_{i+1}) = \bar{x}_{i+1}, \\ &\partial h_i/\partial x_{i+1} \ne 0,\ i=\overline{3,n-1}; \\ &\dot{\bar{x}}_n = \sum_{j=1}^{n} \frac{\partial h_{n-1}}{\partial x_j} f_j + \frac{\partial h_{n-1}}{\partial x_n}[(q_n^{\mathrm{T}}(x)\eta + b_n(x)u] = \\ &= h(x) + q^{\mathrm{T}}(x)\eta + b(x)u, \\ &q^{\mathrm{T}}(x) = \frac{\partial h_{n-1}}{\partial x_n} q_n^{\mathrm{T}},\ b(x) = \frac{\partial h_{n-1}}{\partial x_n} b_n \ne 0. \end{aligned} \tag{7}$$

The canonical form (7) is the standard basis for the synthesis of tracking and observation systems, where the problems of ensuring invariance with respect to matched disturbances are solved depending on the type of uncertainty by suppressing or compensating them.

To synthesize the basic control law for the output variable of system (1) with unmatched disturbances, it is expedient to use the canonical representation with respect to mixed variables (combinations of state variables, external disturbances and their derivatives) [Utkin, Krasnova and Akhobadze, 2008; Utkin V.A and Utkin A.V., 2014; Krasnova and Mysik, 2014;

Krasnova and Utkin, 2016]. This approach to the synthesis problem determines the requirements for the class of admissible systems (1), which are given in the following lemma.

Lemma 1. If the conditions (3), (4) in system (1) are fulfilled and the elements of vector-lines $q_i^{\mathrm{T}} \in R^{1\times s}$, $i=\overline{1,n-1}$) are the constants (including zero) and / or do not contain any arguments other than

$$q_i^{\mathrm{T}}(x_1,...,x_i),\ i=\overline{1,n-1}, \tag{8}$$

then the system (1), (3), (4), (8) is the carrier of the input-output structure in the coordinate basis of mixed variables with relative degree n.

Proof. To obtain the canonical form, it is necessary to differentiate the first equation of the system (1) ($n-1$) times. In contrast to the system with matched disturbances (6), this process generates the derivatives of external disturbances up to the ($n-1$) order. Taking into account the notation (7), we have:

$$\dot{x}_1 = f_1(x_1,x_2)+q_1^{\mathrm{T}}(x_1)\eta = y_2,\ \partial f_1/\partial x_2 \neq 0;$$

$$\dot{y}_2 = h_2(x_1,x_2,x_3)+\overline{q}_2(x_1,x_2,\eta,\dot{\eta}) = y_3, \partial h_2/\partial x_3 \neq 0,$$

$$\overline{q}_2(x_1,x_2,\eta,\dot{\eta}) = \frac{\partial f_1}{\partial x_1}q_1^{\mathrm{T}}(x_1)\eta + \frac{\partial f_1}{\partial x_2}q_2^{\mathrm{T}}(x_1,x_2)\eta + \frac{d}{dt}q_1^{\mathrm{T}}(x_1)\eta;$$

$$\dot{y}_i = h_i(x_1,...,x_{i+1}) + \overline{q}_i(x_1,...,x_i,\eta,\eta^{(1)},...,\eta^{(i-1)}) = y_{i+1},\ \frac{\partial h_i}{\partial x_{i+1}} \neq 0,$$

$$\overline{q}_i = \sum_{j=1}^{i}\frac{\partial h_{i-1}}{\partial x_j}q_j^{\mathrm{T}}(x_1,...,x_j)\eta + \frac{d}{dt}\overline{q}_{i-1}, i=\overline{3,n-1};$$

$$\dot{y}_n = \sum_{j=1}^{n}\frac{\partial h_{n-1}}{\partial x_j}f_j + \sum_{j=1}^{n}\frac{\partial h_{n-1}}{\partial x_j}q_j^{\mathrm{T}}\eta + \tfrac{d}{dt}\overline{q}_{n-1} + \frac{\partial h_{n-1}}{\partial x_n}b_n u,$$

and the system (1), (3), (4), (8) can be represented in the canonical form

$$\begin{aligned}\dot{x}_1 &= y_2,\\ \dot{y}_i &= y_{i+1}, i=\overline{2,n-1};\\ \dot{y}_n &= h(x)+\overline{q}(x,\overline{\eta})+b(x)u,\end{aligned} \tag{9}$$

where $h(x)=\sum_{j=1}^{n}\frac{\partial h_{n-1}}{\partial x_j}f_j$, $b(x)=\frac{\partial h_{n-1}}{\partial x_n}b_n \neq 0$,

$$\overline{q} = \sum_{j=1}^{n}\frac{\partial h_{n-1}}{\partial x_j}q_j^{\mathrm{T}}\eta + \tfrac{d}{dt}\overline{q}_{n-1},\ \overline{\eta} = \mathrm{col}(\eta,\eta^{(1)},...,\eta^{(n-1)}).$$

The presence of external, unmatched disturbances with input channels (8) does not affect the fulfillment of the conditions $\partial h_i/\partial x_{i+1} \neq 0$, $i=\overline{2,n-1}$. Consequently, the transformation to mixed variables is diffeomorphism.

In the canonical system (9), the control input appears only in the last, n-th equation, the coefficient before control $b(x)$ is not equal to zero and does not depend on external disturbances. Thus, the relative degree of system (1) with unmatched disturbances is n. Lemma 1 has been proved.

Thus, the class of non-linear affine systems (1), (3) (4) was single out in which the relative degree inherent in a system with matched disturbances (6) does not change when unmatched disturbances appear, if condition (8) is satisfied.

Note that if conditions (8) are not satisfied, then in the process of obtaining of the canonical form, an early appearance of control in the i - th ($i=\overline{2,n-1}$) equation with a multiplier depending on the disturbance is possible. Then the relative degree changes, and the question of controllability remains open.

If conditions (8) are satisfied in the new coordinate basis of the system (9), the mixed variables

$$y_{i+1} = h_i(x_1,...,x_{i+1}) + \overline{q}_i(x_1,...,x_i,\eta,\eta^{(1)},...,\eta^{(i-1)}),\ i=\overline{1,n-1}$$

are observable with respect to the output x_1. The system is controllable, and external disturbances and their derivatives belong to the control space. This allows us to apply to the system (9) well known methods of ensuring invariance for solving the control problems of the output variable.

To solve the tracking problem, one can obtain a canonical form with respect to the tracking error $e_1 = x_1 - g$ and its derivatives

$$e_{i+1} = h_i(x_1,...,x_{i+1}) + \overline{q}_i - g^{(i)}, i=\overline{1,n-1}. \tag{10}$$

By virtue of (10), we obtain an input u - output e_1 system in which the structural properties of observability and controllability are preserved and which is the basis for further constructions:

$$\dot{e}_i = e_{i+1}, i=\overline{1,n-1};\ \dot{e}_n = \psi(x,t)+b(x)u, \tag{11}$$

where $\psi(x,t) = h(x)+\overline{q}(x,\overline{\eta})-g^{(n)}$. In system (11) $e=(e_1,...,e_n)^T \in R^n$ is the state vector, $\overline{q}(x,\overline{\eta})-g^{(n)}$ is the vector of external disturbances. Taking into account (2)–(5), we assume that for admissible variations of the parameters and $\forall x(t) \in X \subset R^n$, $t \geq 0$ we have the estimates

$$|\psi(x,t)| \leq F,\ 0 < \overline{b} \leq |b(x)| \leq \overline{\overline{b}}, \tag{12}$$

$$|e_i(t)| \leq E_i,\ i=\overline{1,n},$$

where E_i, F, $\overline{b}$, $\overline{\overline{b}}$ are known constants, obtained from technological limitations for the "worst" case. The sign $b(x)$ is a constant and known.

The basic law of combined control that compensates the external disturbances [Krasnova and Utkin, 2016], in this case is not realized due to the uncertainty of the multiplier $b(x)$.

To ensure the invariance to the uncertainties, we will use the "force" methods of suppressing them.

Focusing on a practically significant class of electromechanical systems, in which the control action has a known discontinuous character, the following basic control law is used

$$u = -M \operatorname{sgn} b(x) \cdot \operatorname{sgn} s, \quad (13)$$

$$s = c^{\mathrm{T}} e = c_1 e_1 + ... + c_{n-1} e_{n-1} + e_n, \quad (14)$$

where $c_i = \text{const} > 0$, the roots λ_i of the equation $\lambda^{n-1} + c_{n-1}\lambda^{n-2} + ... + c_2\lambda + c_1 = 0$ have negative real parts $\operatorname{Re}\lambda_i < 0 \quad \forall i = \overline{1, n-1}$, $M = \text{const} > 0$, $\operatorname{sgn}(\cdot)$ is a sign function [Utkin, 2009].

We find the lower bound for choosing the amplitude of discontinuous control from a sufficient condition $s\dot{s} < 0$. [Utkin, Guldner and Shi, 2009]. By virtue of (11)–(14), we have:

$$s\dot{s} = s\left(\sum_{i=1}^{n-1} c_i e_{i+1} + \psi(x,t) - bM \operatorname{sgn} b \cdot \operatorname{sgn} s\right) \le$$
$$\le |s|\left(\sum_{i=1}^{n-1} c_i E_{i+1} + F - M\bar{b}\right) < 0 \Rightarrow \quad (15)$$
$$\Rightarrow M > \left(\sum_{i=1}^{n-1} c_i E_{i+1} + F\right) / \bar{b}.$$

With the amplitude chosen on the basis of (15) for a finite time

$$0 < t_s \le \sum_{i=1}^{n} c_i E_i \Big/ \left(M\bar{b} - \sum_{i=1}^{n-1} c_i E_{i+1} - F\right), \; c_n = 1$$

on the surface $s = 0$ in space R^n there will be appear a sliding mode. When $t > t_s$ the dynamic order of the system (11) equal to n decreases to $(n-1)$. Expressing the variable $e_n = -c_1 e_1 - ... - c_{n-1} e_{n-1}$ from equality $s = 0$ (14) and substituting it in (11), we have a stable system.

$$\dot{e}_i = e_{i+1}, \; i = \overline{1, n-2};$$
$$\dot{e}_{n-1} = -c_1 e_1 - ... - c_{n-1} e_{n-1}, \quad (16)$$
$$s(t) = 0,$$

where the choice c_i, $i = \overline{1, n-1}$ provides the desired rate of convergence of the tracking error $\lim_{t\to\infty} e_1(t) = 0$.

For realization of the basic law of combined control (13), current estimates of mixed variables (10) are required. To solve this problem, the next section presents an original method for synthesizing a reduced observer for mixed variables, in which the method of separation of motions is also realized.

3 Synthesis of the observer of mixed variables

Our goal is to create a multifunctional tracking system that can support various modes of operation of the control plant without reconfiguring the feedback parameters. The analytical form of the control action is not introduced. It is assumed that only its current values $g(t)$ are observed. Derivatives of the reference signal are unknown, but are limited. In inequalities (5), the maximum admissible estimates for all possible modes of operation are laid. Only the tracking error $e_1(t) = x_1(t) - g(t)$ is measured directly. To realize (13) - (14) and estimate the mixed variables (10), we construct a reduced order $(n-1)$ observer on the basis of system (11)

$$\dot{z}_i = z_{i+1} + v_i, \; i = \overline{1, n-2}; \; \dot{z}_{n-1} = v_{n-1}, \quad (17)$$

where $z = (z_1, ..., z_{n-1})^T \in R^{n-1}$ is the state vector, v_i ($i = \overline{1, n-1}$) are corrective actions of observer. Corrective actions are formed on the basis of measurements $e_1(t)$ so as to ensure the stabilization of the system with respect to observation errors $\varepsilon_i = e_i - z_i$, $i = \overline{1, n-1}$. By virtue of (11), (17), the system takes the form

$$\dot{\varepsilon}_i = \varepsilon_{i+1} - v_i, \; i = \overline{1, n-2}; \; \dot{\varepsilon}_{n-1} = e_n - v_{n-1}, \quad (18)$$

where $e_n(t)$ is supposed to be limited external influence (12)

The effective method for estimating immeasurable state variables and external influences without introducing their dynamic models is to construct a state observer (17) with discontinuous corrective actions [Krasnova and Kuznetsov, 2005; Utkin, Krasnova and Akhobadze, 2008; Utkin V.A and Utkin A.V., 2014; Krasnova and Utkin, 2016]. The organization of sliding modes is carried out in the space of observation errors. But in the multidimensional case, the realization of such algorithms on the on-board computer can lead to the emergence of non-ideal sliding modes. This leads to inadequate quality (non-smoothness) of the reconstructed signals and undesirable effects in the realization of discontinuous control (13). For this reason, in systems with discontinuous control it is recommended to use observers with continuous corrective actions. Advantages of observers on sliding modes can be ensured in the pre-limit situation by using of so-called S-shaped continuous corrective influences [Teel, 1996; Krasnova and Mysik, 2014].

For reducing computational load of control algorithms in this paper the method of synthesis of correction actions on the base of functions with saturation is proposed. This approach provides solution of observation problem with prescribed accuracy. In comparison with linear observer with high-gain coefficients [Khalil and Praly, 2014], the procedure of state space expansion of the observer for estimation external disturbances is omitted. The main idea is to provide in system (18) stabilization of estimation errors and its derivatives with given accuracy in finite time $T>0$. Then for $T>0$, the observer variables converge to small vicinity of unmeasured interconnected variables $z_i(t)\approx e_i(t)$, $i=\overline{2,n-1}$, and the estimation $v_{n-1}(t)\approx e_n(t)$ can be calculated from steady state equation $\dot{\varepsilon}_{n-1}=e_n-v_{n-1}\approx 0$.

Lemma 2. If in system (18) with correction actions in the form of sat-functions

$$v_1=M_1\text{sat}(l_1\varepsilon_1)=\begin{bmatrix} M_1\text{sgn}\varepsilon_1, |\varepsilon_1|>1/l_1, \\ M_1 l_1\varepsilon_1, |\varepsilon_1|\le 1/l_1; \end{bmatrix} \quad (19)$$

$$v_i=M_i\text{sat}(l_i v_{i-1})=\begin{bmatrix} M_i\text{sgn}v_{i-1}, |v_{i-1}|>1/l_i, \\ M_i l_i v_{i-1}, |v_{i-1}|\le 1/l_i,\ i=\overline{2,n-1} \end{bmatrix}$$

The initial conditions and variable $e_n(t)$ are bounded by known constants

$$|\varepsilon_i(0)|\le E_i,\ i=\overline{1,n-1},\ |e_n(t)|\le F_n=E_n, \quad (20)$$

then for all an arbitrary small $\delta, T>0$ there are positive real constants M_i^*, l_i^* such, that $\forall M_i, l_i$: $M_i>M_i^*$, $l_i>l_i^*$, $i=\overline{1,n-1}$ the following inequalities are fulfilled

$$|\varepsilon_i(t)|\le\delta,\ i=\overline{1,n-1}, \quad (21)$$
$$|e_n(t)-v_{n-1}(t)|\le\delta\ \forall t\ge T.$$

Proof. Let us split the time period $[0;T]$ into $2(n-1)$ segments with points

$0<t_1<t_2<...<t_{2n-3}<t_{2n-2}=T$.

Assuming that $\delta<<\min\{E_1,...,E_{n-1}\}$, the amplitudes $M_i>0$ ($i=\overline{1,n-1}$) of correction actions (19) are chosen in such a way to provide convergence to linear zone of correction actions in finite time from top to bottom:

$$|\varepsilon_1(t)|\le 1/l_1, t>t_1,\ |v_i(t)|\le 1/l_{i+1}, t>t_{2i-1}. \quad (22)$$

The parameters $l_i>0$ ($i=\overline{1,n-1}$) are used like high-gain coefficients. Its values are chosen to provide in period of time $[t_{2i-1};t_{2i}]$ the relations (21), and the following inequalities

$$|\varepsilon_{i+1}(t)-v_i(t)|=|\alpha_{i+1}(t)|\le\Delta_{i+1}<\delta, t>t_{2i}, \quad (23)$$
$$i=\overline{1,n-1};\ \varepsilon_n:=e_n.$$

The solutions of the systems (18)–(19) are bounded in each finite time interval. The parameters of correction actions are chosen with the goal of state space variables stabilization, and the next bounds can be introduced

$$|\varepsilon_i(t)|\le F_i=\text{const}\ \forall t\ge 0,\ i=\overline{1,n-1}.$$

In the closed loop system (18)–(19) $\text{sgn}v_1(t)=\text{sgn}\varepsilon_1(t)\ \forall t\ge 0$ according to procedure, and the equalities $\text{sgn}v_i(t)=\text{sgn}\varepsilon_i(t)$, $i=\overline{2,n-1}$ are guaranteed only $\forall t>t_{2i-2}$ out of zones $|\varepsilon_i|\le\Delta_i$ (23).

If $\text{sgn}v_i(0)=\text{sgn}\varepsilon_i(0)$, $i=\overline{2,n-1}$, then the closed loop system (18)–(19) at initial time moment can be expressed in the form

$$\dot{\varepsilon}_i=\varepsilon_{i+1}-M_i\,\text{sgn}\,\varepsilon_i,\ i=\overline{1,n-1},\ \varepsilon_n:=e_n.$$

The variables of this system converge monotonically to some vicinity of the origin if the amplitudes of correction actions are chosen on the base of sufficient conditions

$$\begin{aligned}&\varepsilon_i\dot{\varepsilon}_i<0\Rightarrow\varepsilon_i(\varepsilon_{i+1}-M_i\text{sgn}\varepsilon_i)\le\\&\le|\varepsilon_i|(F_{i+1}-M_i)<0\Rightarrow M_i>F_{i+1}, i=\overline{1,n-1},\end{aligned} \quad (24)$$

where $F_i=E_i$, $i=\overline{1,n-1}$. In the "worst" case the range of errors deviations can be estimated in the following way:

$$F_1=|\varepsilon_1(0)|\le E_1 \quad (25)$$
$$F_2=|\varepsilon_2(t_2)|\le E_2+(F_3+M_2)t_2,$$
$$F_i=|\varepsilon_i(t_{2i-2})|\le E_i+(F_{i+1}+M_i)t_{2i-2},\ i=\overline{2,n-1},$$
$$F_n=E_n.$$

The inequalities for the amplitudes M_i, which provides (22) for given time are

$$M_i>\frac{|\varepsilon_i(t_{2i-2})|}{t_{2i-1}-t_{2i-2}}+F_{i+1}, i=\overline{1,n-1}. \quad (26)$$

Taking into account (25), the following relation can be introduced from (26) (from bottom to top)

$$\begin{aligned}&M_{n-1}>\frac{E_{n-1}+(F_n+M_{n-1})t_{2n-4}}{t_{2n-3}-t_{2n-4}}+F_n\Rightarrow\\&M_{n-1}^*=\frac{E_{n-1}+F_n t_{2n-3}}{t_{2n-3}-2t_{2n-4}},\ 2t_{2n-4}<t_{2n-3}<T,\\&M_i^*=\frac{E_i+F_{i+1}t_{2i-1}}{t_{2i-1}-2t_{2i-2}},\ 2t_{2i-2}<t_{2i-1},\ i=\overline{n-1,2},\\&M_1^*=\frac{E_1}{t_1}+F_2.\end{aligned} \quad (27)$$

Thus, the parameters M_i^* (27) was calculated and $\forall M_i>M_i^*$, $i=\overline{1,n-1}$ the inequalities (22) are fulfilled.

Let us suppose, for example

$\Delta t=t_{2i-1}-2t_{2i-2}$, $i=\overline{2,n-1}$

and $\Delta t=t_1=t_2-t_1=t_4-t_3=...=t_{2n-2}-t_{2n-3}>0$.

Then the upper bound for choice $\Delta t>0$, which provides convergence of estimation errors in finite time $T > 0$ is

$$0 < \Delta t \le T/(2(2^{n-1}-1)). \quad (28)$$

The amplitudes M_i are chosen sequentially from bottom to top on the base of (27), (25) for accepted value of Δt (28).

With (22)–(23) the closed loop system (18)–(19) can be rewritten in the following form

$$\dot{\varepsilon}_1 = -M_1 l_1 \varepsilon_1 + \varepsilon_2, \ |\varepsilon_1| \le 1/l_1 \ \ \forall t > t_1; \quad (29)$$

$$\dot{\varepsilon}_i = -M_i l_i v_{i-1} + \varepsilon_{i+1} = -M_i l_i (\varepsilon_i - \alpha_i) + \varepsilon_{i+1},$$

$$|v_{i-1}| \le 1/l_i \Rightarrow |\varepsilon_i| \le 1/l_i + \Delta_i \ \ \forall t > t_{2i-1}, \ \ i = \overline{2, n-1}.$$

For the variables of the system (29) in time intervals $[t_{2i-1}; t_{2i-1} + \Delta t = t_{2i}]$ the next estimations are true

$$|\varepsilon_1(t_2)| \le \frac{|\varepsilon_2(t)|}{M_1 l_1} + \left(\frac{1}{l_1} - \frac{|\varepsilon_2(t)|}{M_1 l_1}\right) e^{-M_1 l_1 \Delta t} \le \quad (30)$$

$$\frac{F_2}{M_1 l_1} + \frac{M_1 - F_2}{M_1 l_1} e^{-M_1 l_1 \Delta t},$$

$$|\varepsilon_i(t_{2i})| \le \frac{F_{i+1}}{M_i l_i} + \Delta_i + \frac{M_i - F_{i+1}}{M_i l_i} e^{-M_i l_i \Delta t}, \ \ i = \overline{2, n-1}.$$

Taking into account the relations $v_1 = M_1 l_1 \varepsilon_1 \ \forall t > t_1$, $v_i = M_i l_i (\varepsilon_i - \alpha_i) \ \forall t > t_{2i-1}$, $i = \overline{2, n-1}$, the lower bounds for coefficients $l_i > 0$ can be calculated from (30) to provide (23)

$$(M_i - F_{i+1}) e^{-M_i l_i \Delta t} \le \Delta_{i+1} \Rightarrow$$

$$l_i > \frac{1}{\Delta t M_i} \ln \frac{M_i - F_{i+1}}{\Delta_{i+1}}, \ i = \overline{1, n-1}. \quad (31)$$

According to (30)–(31) for time interval $t_{2i} < t \le T$ we have for the system (29) variables

$$|\varepsilon_1| \le \frac{|\varepsilon_2| + \Delta_2}{M_1 l_1} \quad (32)$$

$$|\varepsilon_i| \le \frac{|\varepsilon_{i+1}| + \Delta_{i+1}}{M_i l_i} + \Delta_i, \ i = \overline{2, n-1}.$$

From (31), (32) it follows that inequalities $|e_n(t) - v_{n-1}(t)| \le \delta$, $|\varepsilon_{n-1}(t)| \le \delta$ are true $\forall t \ge T$ for each $l_{n-1} > l_{n-1}^*$, if l_{n-1}^* is chosen according to

$$l_{n-1}^* = \max\left\{\frac{F_n + \delta}{M_{n-1}\delta}; \frac{1}{\Delta t M_{n-1}} \ln \frac{M_{n-1} - F_n}{\delta}\right\}. \quad (33)$$

For accepted $l_{n-1} > l_{n-1}^*$ the accuracy

$$0 < \Delta_{n-1} \le \delta - (F_n + \delta)/(M_{n-1} l_{n-1}),$$

is provided by appropriate choice of feedback coefficients l_{n-2} (31). Both inequalities

$$|\varepsilon_{n-1}(t) - v_{n-2}(t)| \le \Delta_{n-1} < \delta, t > t_{2n-4},$$

$$|\varepsilon_{n-1}| \le \delta \Rightarrow |\varepsilon_{n-2}(t)| \le \frac{\delta + \Delta_{n-1}}{M_{n-2} l_{n-2}} + \Delta_{n-2} < \delta \ \ t \ge T$$

are fulfilled $\forall l_{n-2} > l_{n-2}^*$, if

$$l_{n-2}^* = \max\left\{\frac{\delta + \Delta_{n-1}}{\delta M_{n-2}}; \frac{1}{\Delta t M_{n-2}} \ln \frac{M_{n-2} - F_{n-1}}{\Delta_{n-1}}\right\}. \quad (34)$$

For selected $l_{n-2} > l_{n-2}^*$ the size of zone Δ_{n-2} is calculated $0 < \Delta_{n-2} \le \delta - (\delta + \Delta_{n-1})/(M_{n-2} l_{n-2})$. This values is provided by appropriate choice of coefficient l_{n-2} (31), and so on further. Thus, both inequalities

$$|\varepsilon_{i+1}(t) - v_i(t)| \le \Delta_{i+1} \ \ \forall t > t_{2i},$$

$$|\varepsilon_{i+1}(t)| \le \delta \Rightarrow |\varepsilon_i| \le \frac{\delta + \Delta_{i+1}}{M_i l_i} + \Delta_i < \delta \ \ \forall t \ge T$$

are fulfilled $\forall l_i > l_i^*$, $i = \overline{n-3, 1}$. The lower bounds are chosen sequentially on the basis of inequalities

$$l_i^* > \max\left\{\frac{\delta + \Delta_{i+1}}{\delta M_i}; \frac{1}{\Delta t M_i} \ln \frac{M_i - F_{i+1}}{\Delta_{i+1}}\right\}, \quad (35)$$

where $0 < \Delta_{i+1} \le \delta - (\delta + \Delta_{i+2})/(M_{i+1} l_{i+1})$. The Lemma 2 has been proved.

Let us note that estimations (20) are true for zero initial conditions $z_i(0) = 0$ in the observer (17). With measurements of $e_1(t)$ the initial value of the variable $\varepsilon_1(t)$ can be calculated $z_1(0) = e_1(0) \Rightarrow \varepsilon_1(0)$. This value can be used to accelerate the convergence process of observation errors.

With using observer of mixed variables (17), (19) the basic discontinuous control law (13)–(14) is realized in the form

$$u = -M \operatorname{sgn} b(x) \operatorname{sgn}(c_1 e_1 + c_2 z_2 + \ldots + c_{n-1} z_{n-1} + v_{n-1}),$$

and according to (21) this control input provides convergence to $(c_2 + c_3 + \ldots + c_{n-1} + 1)\delta = \Delta$-vicinity of sliding surface $s = 0$ in finite time $t_s > T$.

Thus for $t > t_s$ in the closed loop system (1), (13)–(14), (17), (19) the real sliding mode exists

$$\dot{e}_i = e_{i+1}, \ i = \overline{1, n-2};$$

$$\dot{e}_{n-1} = -c_1 e_1 - \ldots - c_{n-1} e_{n-1} - \Delta,$$

$$|s(t)| \le \Delta.$$

According to the last relations, the tracking problem is solved with some accuracy

$$|e_1(t)| \le \bar{\delta} \ \ \forall t > t_s.$$

4 Example

The procedure of tracking system synthesis of inverted pendulum is considered as an example of designed procedure. The direct current motor (DC motor) is used as actuator of electromechanical system [Angeli, 2001]. The dynamical model of the plant can be described by third order differential equations

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = a_{21}\sin x_1 - a_{22}x_2 + a_{23}(x_3+\eta), \quad (36)$$
$$\dot{x}_3 = -a_{32}x_2 - a_{33}x_3 + bu,$$

where x_1 $[rad]$ is angular position of the pendulum, x_2 $[rad/s]$ is angular velocity, x_3 $[N\cdot m]$ is electromechanical torque applied to the pendulum suspension axis, u is control input (DC motor voltage); the variables $x_i(t)$ are bounded by the following constants $|x_i(t)| \le X_i$; $\eta(t)$ is unknown time function, which characterize the external bounded disturbances influence (2), $b_3 > 0$, $\forall a_{ij} > 0$, b_3, a_{32}, a_{33} are known coefficients, $a_{21} = g/l$, $a_{22} = \kappa/l$, $a_{23} = 1/(ml^2)$, $\bar{g} = 9{,}8\,[m/s^2]$ is gravitation force acceleration, m $[kg]$, l $[m]$ the mass and length of pendulum correspondingly, κ $[Pa\cdot s]$ is viscous friction coefficient, the parameters m, l, κ are not known exactly, but their ranges are known.

The tracking problem for desired trajectory $g(t)$ (5) with respect to output variable $x_1(t)$ is stated under assumption that only output variable $e_1(t) = x_1(t) - g(t)$ is available for measurements.

The external disturbances in system (36) are unmatched disturbances, the conditions (3), (4), (8) are fulfilled. Let us write the system (36) in canonical form with respect to output variable (11) by using non-singular coordinate transformations (10)

$$\dot{e}_1 = e_2,\ \dot{e}_2 = e_3,\ \dot{e}_3 = \psi(t) + bu, \quad (37)$$

where $b = a_{23}b_3$, $0 < \bar{b} \le b \le \bar{\bar{b}}$,

$$e_1 = x_1 - g,\ e_2 = x_2 - \dot{g},$$
$$e_3 = a_{21}\sin x_1 - a_{22}x_2 + a_{23}(x_3+\eta) - \ddot{g},$$
$$\psi(t) = a_{21}x_2\cos x_1 + a_{33}a_{21}\sin x_1 -$$
$$-(a_{23}a_{32} + a_{22}a_{33})x_2 - (a_{22}+a_{33})e_3 +$$
$$+ a_{33}a_{23}\eta + a_{23}\dot{\eta} - (a_{22}+a_{33})\ddot{g} - \dddot{g},$$
$$|e_i(t)| \le E_i, i = 1,2,3,\ E_{1,2} = 2X_{1,2},$$
$$|\psi(t)| \le F\ \forall t \ge 0.$$

The basic control law (13) and inequality for amplitude choice (15) are chosen in the form

$$u = -M\,\mathrm{sgn}\,s,\ s = c_1e_1 + c_2e_2 + e_3,$$
$$M > (c_1E_2 + c_2E_3 + F)/\bar{b}.$$

For informational support of selected control law we one can introduce the reduced observer of mixed variables (17) in the feedback loop. This observer is second order observer with correction actions in the form of sat-functions (19)

$$\dot{z}_1 = z_2 + v_1,\ v_1 = M_1\mathrm{sat}(l_1\varepsilon_1), \quad (38)$$
$$\dot{z}_2 = v_2,\ v_2 = M_2\mathrm{sat}(l_2v_1).$$

According to (37), (38) the equations for observation errors $\varepsilon_i = e_i - z_i$, $i = 1,2$ are

$$\dot{\varepsilon}_1 = \varepsilon_2 - v_1,\ \dot{\varepsilon}_2 = e_3 - v_2.$$

The basic control input is realized in the form

$$u = -M\,\mathrm{sgn}\,s(c_1e_1 + c_2z_2 + v_2). \quad (39)$$

The numeric experiment of closed loop system (36), (38), (39) is provided in the Matlab–Simulink environment under the following parameters values

$$c_1 = c_2 = 4,\ a_{32} = 2,\ a_{33} = 10,\ b_3 = 10,$$
$$X_1 = 2\pi,\ X_2 = 1,\ l_1 = 50,\ l_2 = 100.$$

Different operation regimes are considered under set of external disturbances functions and parameters $m \in [0.9;1.1]$, $l \in [0.9;1.1]$, $\kappa \in [7;9]$ variations. For "worst" case the following parameters of controller (38), (39) are used

$$M = 24, l_1 = 50, l_2 = 100,\ M_1 = 5,\ M_2 = 10. \quad (40)$$

The convergence of the output variable $x_1(t)$ of the system (36) to the desired constant signal is shown on the Fig. 1. Fig. 2 depicts the convergence of the output variable to harmonic reference signal $g = \sin(0.5t)$ under influence of harmonic external disturbances $\eta(t) = 0.5\sin 2t$ with the same feedback coefficients (40).

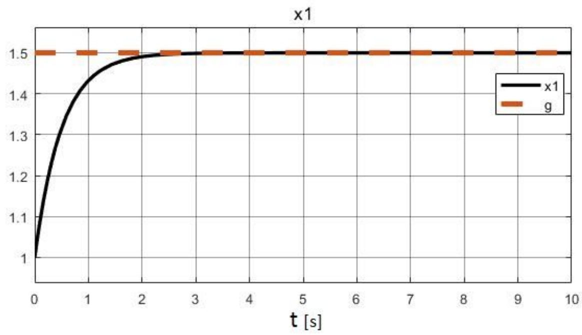

Figure 1.

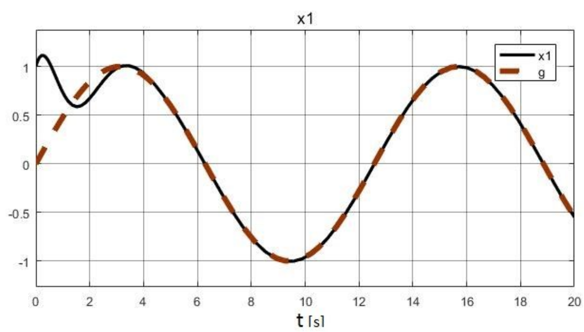

Figure 2.

The simulation results show the efficiency of proposed approach for synthesis of tracking system under uncertainty of system's parameters and external disturbances.

5 Conclusion

A major solution aspect of the problem consists in the development of estimation methods for the unmeasurable external disturbances and functional uncertainties using observation subsystems. Such an approach appreciably simplifies the structure of the resulting controller, as there is no need to introduce the autonomous dynamic models of the external disturbances, to describe in detail and perform real calculations using the nonlinear expressions (which is especially topical for controller synthesis based on the complete nonlinear model). Moreover, this approach relaxes the requirements to the amount of a priori information about the plant and its operating conditions. The result aims at creating universal and easily implementable invariant tracking systems that do not require (a) readjustment under a considerable variation of the parameters and external factors during operation and (b) a complete set of sensors in the control system.

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