I. Introduction.

The understanding of the dynamic behavior in real physical or industrial system is of almost importance, for analysis, synthesis, prediction, etc.

It's sensible to consider that the behavior of many physical systems like phytoplankton ,solar activity, oscillation of waves is a combination between chaotic or stochastic processes, which can be successfully used for prediction of health applications, meteorological phenomena etc.

Many physical/ chemical or sometimes financial phenomena are considered as being only chaotic ((ex. Belousov–Zhabotinsky_reaction 1) or purely stochastic (stock model price, integral Ito, Black-Scholes model), but in fact they are both deterministic and stochastic (1-2).

So it is of utmost interest to find new models taking into account both behaviors, stochastic and chaotic, to understand and predict better the real physical phenomena, but also to model data for biomedical applications like (ECG, IRM, …. To be completed) The original idea in this paper is to juxtapose methods from stochastic signal analysis (nonstationary Gaussian processes, statistics from limit theorems by Nordin, Hurst exponent), and nonlinear (chaotic) dynamical system analysis (phase portrait, phase delayed plot, Lyapunov exponents), to develop a common methodology to analyze complex time series. Assuming that these two behaviors are inherently correlated, we are analyzing if there exists a correlation exists between the stochastic quantifiers (Hurst exponent, Garch method,ARMA) and chaotic quantifiers (Lyapunov exponents). To do that, different kind of stochastic-chaotic mixed processes shall be modeled and analyzed from different points of view to be developed.

Proposed methodology. As classical approach, we assume a priory stochastic nature of time series model and construct a mathematical model as a random process. Hurst exponent is defined like the estimate $\hat{H}$ of approximated fractional Brownian motion for these time series. On the other hand, for some deterministic systems, where the state is the solution of nonlinear differential or difference equation like $x_n = X(t_n)$, the behavior can be highly irregular and extremely complex. In some cases the behavior is estimated like chaotic. In the first approximation, we can determine the chaocity by the property of the system to construct i trajectories in a bounded domain of the phase space. Properties of dynamical systems which generate chaotic solutions, has been widely discussed (results and references in the monographs The simplest example is an one-dimensional dynamical system

$$x_{n+1} = f(x_n, \mu)$$

which generates chaotic solution for some functions f and values of parameter μ. In particular, for logistic function f

$$x_{n+1} = \mu x_n(1 - x_n), 0 < \mathrm{x}_n < 1, \mu > 0$$

the plot of solution looks like white noise with some values $\mu > 3,6$

So, the problem statement the nature of time series analysis nature is do the observed data have stochastic nature, or deterministic.

A lot of papers have been devoted to this problem by 90s. The essence of these results is as follows. Let's construct some statistics of observed time series, the values of which will be different from random or deterministic chaotic sequences.

There are a lot of criteria of difference between chaotic and stochastic nature of time series developed in the last years.

One of the main characteristics of the a priori deterministic series is the Lyapunov exponent λ. It's using a presence of dynamical system, which is generating research data by estimation of Lyapunov exponent, so it doesn't work for the algorithm of random process for calculation λ. The criterion of chaotic for a deterministic time series is a positive Lyapunov exponent. It's equal $\lambda = \ln 2$ for logistic sequence $x_{n+1} = 4x_n(1 - x_n)$.

Note that the above results have been proved only for a certain class of dynamical systems which generated deterministic chaos. As usual, the situation of mixture "chaotic-randomness" is a normal for the natural observed data (one of the main task is to determine their correlation in the time series). It's normal to expect that the quality of the approximation of this mixture depends on the specified ratio in the proposed model (approximation of a random process fbm and the quality is defined by the specified statistics A_n, B_n, D_n).

fBm	H=0.2	H=0.5	H=0.6
Tent map	0.5766	0.0078	1.0569
Mixture (α=0.2)	0.9591	0.8721	2.6039
Lorenz	1.8544	1.6360	1.9381
Mixture (α=0.5)	0.9244	1.0678	2.8903

Reminder.

Let's assume that the observed data can be successfully modeled by non-stationary Gaussian process (fractional Brownian motion). Statistical hypothesis T explains that the investigated time series $z_1,..,z_n$ is an implementation of fBm.

Let's $B(t), 0 \le t \le 1$ is a fractional Brownian motion with Hurst exponent H ,f is a twice differentiable function, where

$$\mathrm{E}(|\, f^{(k)}(B(t)\,|^p) < \infty,\ k = 1,2;$$

Let's note

$$\xi_k = n^H \left(B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right) \right) \sim \aleph(0;1),\ n \to \infty$$

where N is normal distribution μ=0 and standard deviation σ=1.

The following limit relations have to be verified papers of Nourdin $[16-19]$:

1. $\displaystyle \frac{n^H}{n} \sum_{k=1}^{n} f\left(B\left(\frac{k}{n}\right)\right) \xi_k^3 \to -\frac{3}{2} \int_0^1 f'(B(s))ds,\ H \in \left(0; \frac{1}{2}\right);$

2. $\displaystyle \frac{n^{2H}}{n} \sum_{k=1}^{n} f\left(B\left(\frac{k}{n}\right)\right) (\xi_k^2 - 1) \to \frac{1}{4} \int_0^1 f''(B(s))ds,\ H \in \left(0; \frac{1}{4}\right);$

3. $\displaystyle \frac{1}{n^H} \sum f\left(B\left(\frac{k}{n}\right)\right) \xi_k^3 \to 3 \int_0^{B(1)} f(x)\,dx,\ H \in \left(\frac{1}{2};1\right).$

Denote

$$\alpha_k = n^H B\left(\frac{k}{n}\right) = \sum_{j=1}^{k-1} \xi_j$$

Assume in the first formula $f(x) = x, f(x) = x^2$, so let's get:

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_k \xi_k^3 \to -\frac{3}{2},\ H \in \left(0; \frac{1}{2}\right), \qquad (1.1)$$

$$\frac{1}{n^{1+H}} \sum \alpha_k^2 \xi_k^3 \to 3\eta,\ H \in \left(0; \frac{1}{2}\right), \qquad (1.2)$$

where $\eta \sim \aleph\left(0; \frac{1}{2H+2}\right)$

Let's put $f(x) = x^2$ in the second formula

$$\frac{1}{n}\sum_{k=1}^{n} \alpha_k^2 (\xi_k^2 - 1) \to \frac{1}{2},\ H \in \left(0; \frac{1}{4}\right). \tag{1.3}$$

For the third formula in $f(x) = x, f(x) = 1$

$$\frac{1}{n^{2H}}\sum_{k=1}^{n} \alpha_k \xi_k^3 \to \frac{3}{2} B^2(1),\ H \in \left(\frac{1}{2}; 1\right), \tag{1.4}$$

$$\frac{1}{n^H}\sum_{k=1}^{n} \xi_k^3 \to 3B(1),\ H \in \left(\frac{1}{2}; 1\right).$$

The relations (1.3) and (1.4) enable us to test the hypothesis $T = \{the\ statistics\ z_1,...,z_n$ *, which obtained by transformation of real data, are modeled by increments of fractional Brownian motion* $\}$.

The algorithm of checking (with known *H*) is following:

Denote

$$c = \frac{1}{n}\sum_{k=1}^{n} z_k^2$$

we assume that hypothesis *T* is done or:

$$z_k = \sqrt{c}\ \xi_k = \sqrt{c}\ n^H \left(B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right)$$

Let's put

$$v_k = \sum_{j=1}^{k-1} z_j$$

and calculate the statistics

$$A_n = \frac{1}{n}\sum v_k z_k^3, \text{if } H \in \left(0;\frac{1}{2}\right);$$

$$B_n = \frac{1}{n^{1+H}}\sum v_k^2 z_k^3, \text{ if } H \in \left(0;\frac{1}{2}\right); \qquad (1.5)$$

$$D_n = \frac{1}{n^{2H}}\sum v_k z_k^3, \text{ if } H \in \left(\frac{1}{2};1\right);$$

If the hypothesis *T* is true, then according to (1.1-1.4) and the following theoretical leads to:

$$A_n \to -\frac{3}{2}c^2; \quad B_n \to 3c^{\frac{5}{2}}\eta; \quad D_n \to \frac{3}{2}c^2 B^2(1); \quad \text{where } \eta \sim \aleph\left(0;\frac{1}{2H+2}\right)$$

The hypothesis T accepted by comparing the experimental values of statistics A_n, B_n, D_n with their limiting theoretical values. Let's define deviation from the limit value like $\sigma = \left|\frac{A_n - A}{A}\right|$ for statistics A_n.

The limit distribution functions for B_n and D_n

$$F_1(x) = P\left\{3c^{2,5}\,\eta < x\right\} = \Phi\left(\frac{x}{3\sigma}c^{=2,5}\right),\ F_2(x) = 2\Phi\left(\frac{1}{c}\sqrt{\frac{2}{3}x}\ \right) - 1, x > 0,$$

Where Φ is an Laplace function $\sigma = (2H+2)^{-0,5}$

Hypothesis *T* is accepted, if

$$\delta < \beta_0, |B_n| < \beta_1 \text{ for } H<0,5\ ;\ 0 < D_n < \beta_2 \text{ for } H>0,5 \qquad (1.6)$$

where β_1, β_2 are quintiles of distributions F_1 and F_2, corresponding to the selected significance level $\alpha = 0,1$. Then

$$\beta_1 = \frac{4,95c^{2,5}}{\sqrt{2H+2}}; \beta_2 = 4.08c^2; c = z^2$$

1. Generate 3000 values fBm for *H=0,2 0,4 0,6 0,8* $x_1,...,x_n$, $x_0 = 0, n = 3000$ and to normalize these values (such as $u_k = \frac{x_k}{\sqrt{\frac{1}{n}\sum_1^n x_k^2}}$).

Fractional Brownian motion is defined as a Gaussian random process with characteristics:

$$\mathrm{E}\,B_H(t) = 0, B_H(0) = 0, \mathrm{E}\,B_H(t)B_H(s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

Where *H* is the Holder exponent which measures the regularity and smoothness of trajectories of process , and t and *s* are periods of time.

Note that with *H=0,5* we get a standard Wiener process or white noise. With $H < \frac{1}{2}$ the increments are form the sequence with short, $H > \frac{1}{2}$ with a long memory.

Smoothness of the trajectories of the process $B_H(t)$ is defined by the parameter *H:* almost all the trajectories satisfy the Holder condition:

$$|X(t) - \mathrm{X(s)}| \le c|t-s|^{\alpha}, \alpha < H$$

which generalizes known Levy's result for the Wiener process (reference).

The increments of fBm $B_H(t_2) - B_H(t_1),\ B_H(t_4) - B_H(t_3), \mathrm{t}_1 < t_2 < t_3 < t_4$

are form a Gaussian random vector with a correlation between the coordinates:

$$\frac{1}{2}((t_4 - t_1)^{2H} + (t_3 - t_2)^{2H} - (t_4 - t_2)^{2H} - (t_3 - t_1)^{2H})$$

For discrete time:

$$\xi_k = B_H(\frac{k}{n}) - B_H(\frac{k-1}{n}),$$

we obtain the correlation coefficient:

$$\rho(\xi_j, \xi_k) = \frac{1}{2}(|k-j+1|^{2H} + |k-j-1|^{2H} - 2|k-j|^{2H}),$$

it means that increments are forming stationary (in the narrow sense) sequence.

2. Take 3000 from the table (solution of differential equation in Lorenz attractor) in application (5001-8000) and cutting the trend. Trend is consider as a mean in our case. It's

possible to approximate time series by linear function. Usually this is done using a logarithmic, exponential, or (not so often) polynomial transformation of data.

3. Denote the obtained array as $c_1,...,c_{3000}$ and normalize it : $v_k = \frac{c_k}{\sqrt{\frac{1}{n}\sum_1^n c_k^2}}$ Let's break the interval of observations $[1;n]$ for m time windows $\Delta_1 = [1;T_1]$, $\Delta_2 = [T_1+1;T_2]$, $\Delta_m = [T_{m-1}+1;T_m]$.

Denote:

$$\hat{\mu}_r = \frac{Z(T_r) - Z(T_{r-1}+1)}{T_r - T_{r-1} - 1} = \frac{b(T_r) - b(T_{r-1}+1)}{T_r - T_{r-1} - 1}$$

If we approximate the trend by piecewise linear function:

$$b(t) = Z(1) + \hat{\mu}_1 (t-1),\ t \in \Delta_1,$$

$$b(t) = b(T_{r-1}) + \hat{\mu}_r (t - T_{r-1}),\ t \in \Delta_r,\ r \le m.$$

Let's consider the sequence $w_k = u_k + \alpha v_k, \alpha = 0; 0,2; 0,5; 1; 2$ and study the behavior of statistics for this mixture upon calculation A_n, B_n, D_n which are the follows from limit theorems.

Calculate the increments $y_k = w_k - w_{k-1}$,the statistics $R_{1n}(y) = \frac{1}{n}\sum_{k=1}^{n} |y_k|$, and normalize the increments, if

$$z_k = \frac{0,8}{R_{1n}} y_k$$

Denote $v_k = \sum_{j=1}^{k-1} z_j$.

And construct the statistics:

$$A_n = \frac{1}{n}\sum v_k z_k^3, \text{if } H \in \left(0; \frac{1}{2}\right);$$

$$B_n = \frac{1}{n^{1+H}}\sum v_k^2 z_k^3, \text{ if } H \in \left(0; \frac{1}{2}\right);$$

$$D_n = \frac{1}{n^{2H}}\sum v_k z_k^3, \text{ if } H \in \left(\frac{1}{2}; 1\right)$$

The limit theorems for statistics from increments of fractional Brownian motion have been proved in works of I.Nourdin and others [16-19] .

$$\xi_k = n^H \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right) \sim \aleph(0;1)$$

Let's denote

$$\alpha_k = n^H B\left(\frac{k}{n}\right) = \sum_{j=1}^{k-1} \xi_j .$$

There is a Mean-square convergence:

$$\frac{1}{n}\sum_{k=1}^{n} \alpha_k \xi_k^3 \to -\frac{3}{2}, \; H \in \left(0; \frac{1}{2}\right),$$

$$\frac{1}{n^{1+H}}\sum \alpha_k^2 \xi_k^3 \to 3\eta, \; H \in \left(0; \frac{1}{2}\right),$$

where

$$\eta \sim \aleph\left(0; \frac{1}{2H+2}\right);$$

$$\frac{1}{n^{2H}}\sum_{k=1}^{n} \alpha_k \xi_k^3 \to \frac{3}{2} B^2(1), \; H \in \left(\frac{1}{2}; 1\right),$$

These results allow us to estimate the adequacy of model with the basic process-fractional Brownian motion.

These statistics allow to define the contribution of chaotic component in modified fBm $v_k = \sigma\, B_H(\frac{k}{n})$ (where H is Hurst exponent, σ is an amplitude) to be compared with ideal (a=0)).

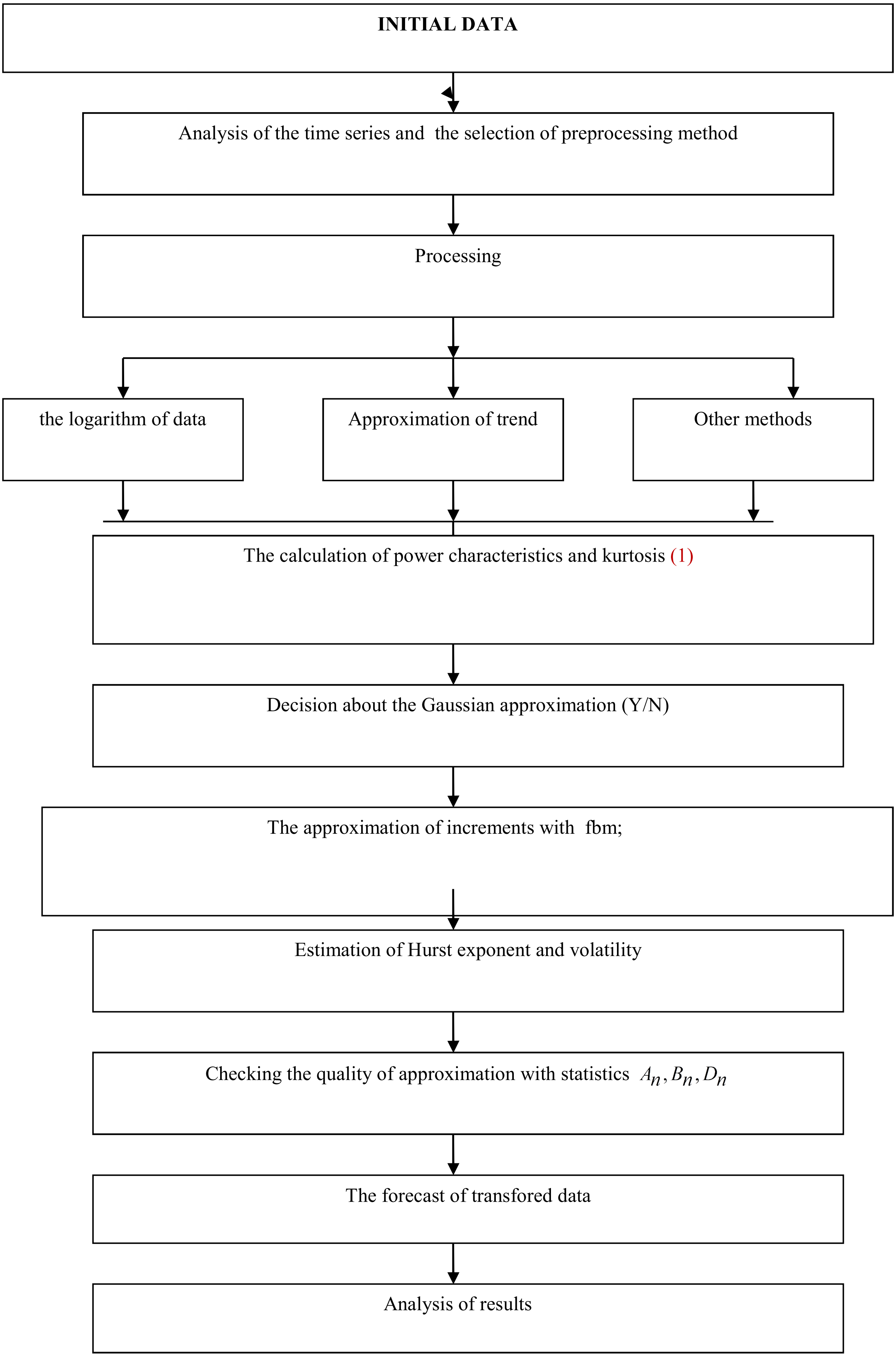
We've got the results as 4 tables for every H with 5 columns ($a = 0; 0{,}2; 0{,}5; 1; 2$) and the values of control statistics A_n, B_n, D_n and the quintiles.

Let's random value *X* has a distribution function *F(X).* α-quintile is an solution of equation $x_a = F^{-1}(a)$

where α is probability, that random value *X* will take a value less or equal to

$x_a, P(x <= x_a) = a$.

The block-diagram, which is describe approximation method


INITIAL DATA
Analysis of the time series and the selection of preprocessing method
Processing
the logarithm of data
Approximation of trend
Other methods
The calculation of power characteristics and kurtosis (1)
Decision about the Gaussian approximation (Y/N)
The approximation of increments with fbm;
Estimation of Hurst exponent and volatility
Checking the quality of approximation with statistics A_n, B_n, D_n
The forecast of transfored data
Analysis of results

Appendix

Table 1: Control statistics of mixture between fBm and Lorenz attractor *(α = 1;2;10; n=3000)*

H_{fbm}		$\hat{H}$	A_n	B_n	D_n	$A = \lim A_n$	β_1	β_2
0,2	α =1	0,15	10,0	−83	2046	−0,84	1,57	3,05
	α =2	0,15	6,83	−39,3	1397	−1,26	2,6	3,7
	α =10	0,2	−48,8	−182	−4663	−45,5	227	22,5
0,4	α =1	0,15	11,8	−113	2413	−0,75	1,37	2,88
	α =2	0,15	11,9	−111	2430	−0,81	1,52	3,0
	α =10	0,3	1,37	−4,11	28,6	−3,14	7,73	5,90
0,6	α =1	0,15	11,0	−101	2265	−0,72	1,31	2,83
	α=2	0,15	10,7	−93	2180	−0,72	1,31	2,83
	α =10	0,15	7,27	−38,0	1487	−0,78	1,43	2,93
0,8	α =1	0,15	10,95	−101	2239	−0,72	1,31	2,83
	α =2	0,15	10,4	−92,5	2132	−0,72	1,31	2,83
	α =10	0,15	6,32	−38,0	1293	−0,73	1,32	2,84

The table data shows about "aggressiveness" of the chaotic component in relation to stochastic for $H_{fBm} \geq 0{,}2$ $(for\ a = 1, a = 2)$.
The character of combination of generated *fBm* (for specified values *H)* defines Lorenz attractor, that means that the values of control characteristics are so far from the limits), which indicates the impossibility of approximation.
For $H_{fBm} = 0{,}2$ satisfactory approximation is possible only for $a = 10$.
Persistence ($\hat{H} > 0{,}5$) of investigated time series ($D_n < \beta_2$) means it has stochastic nature: antipersistent ($A_n \approx A$, $|B_n| < \beta_1$) admits the existence of chaotic component.
Thus, satisfactory approximation in natural time series means their stochastic nature, or the presence of "resonance" (with a random component) chaotic sequence.

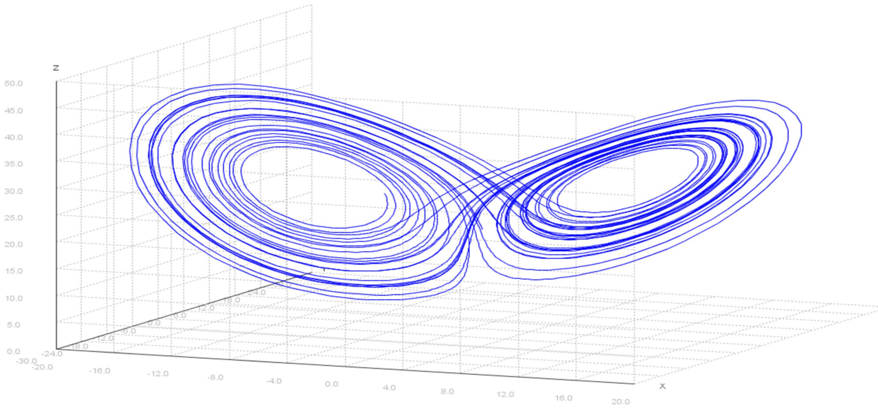


Figure 1. Phase portrait of Lorenz attractor.

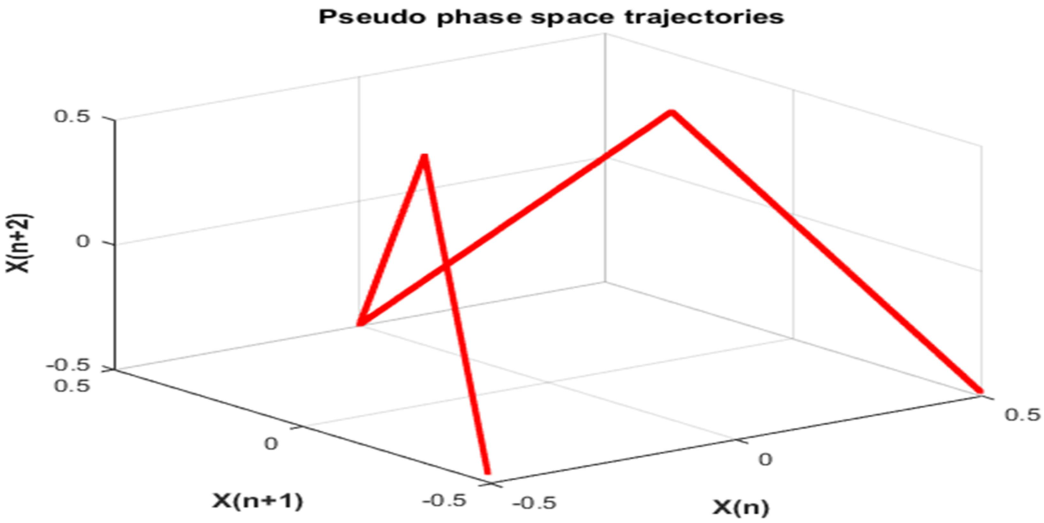


Figure 2. Phase portrait of Tent map.

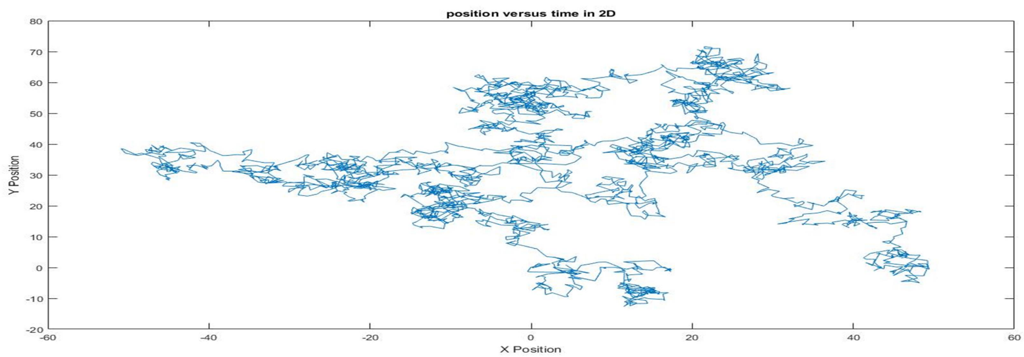


Figure 3. Phase portrait of fractional Brownian motion.

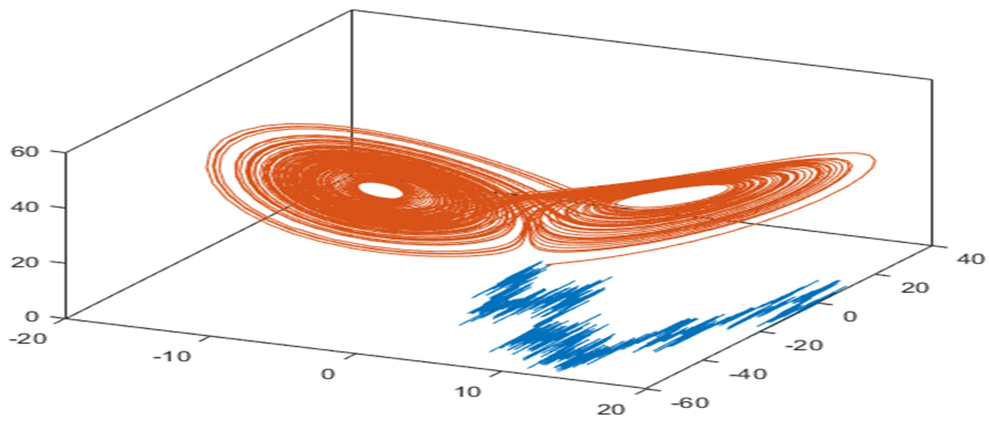


Figure 4. Phase portrait of Lorenz attractor+fractional Brownian motion (*H=0.2)*

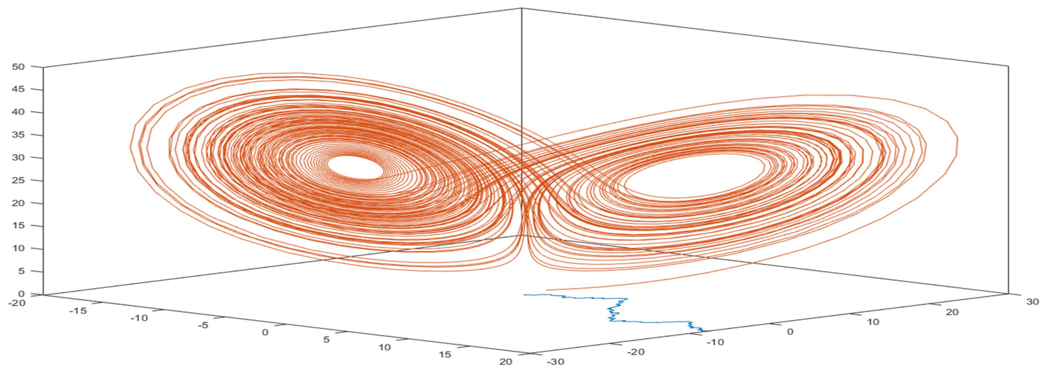


Figure 5. Phase portrait of Lorenz attractor+fractional Brownian motion (*H=0.8)*

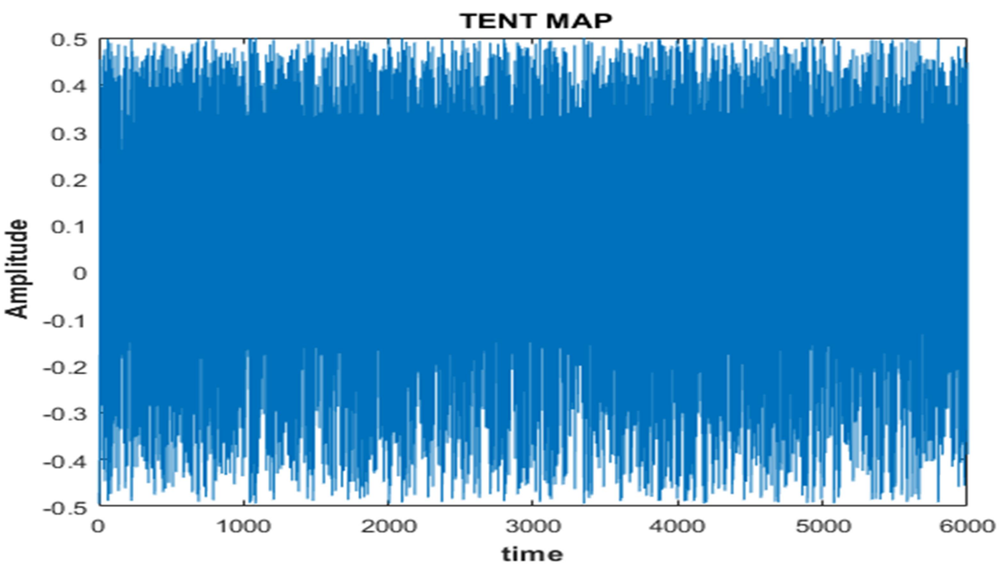


Figure 6. Tent map.

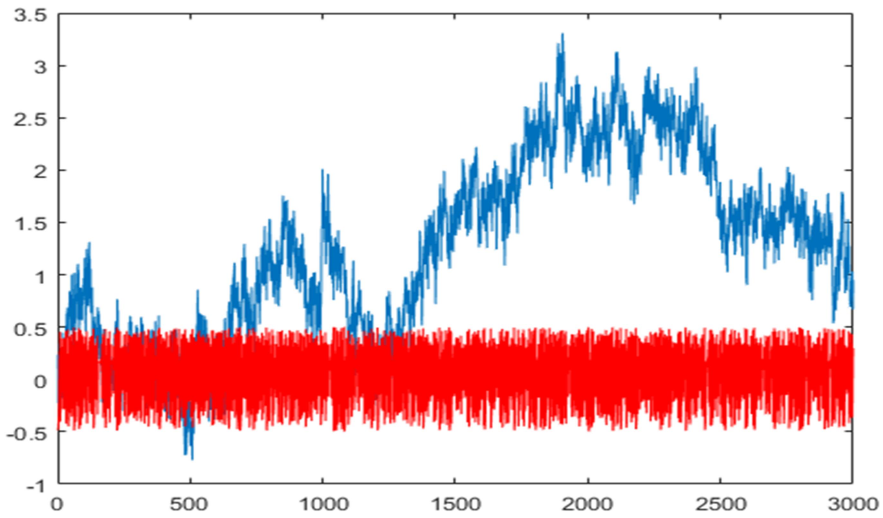


Figure 7. Tent map+ fractional Brownian motion (*H=0.2)*

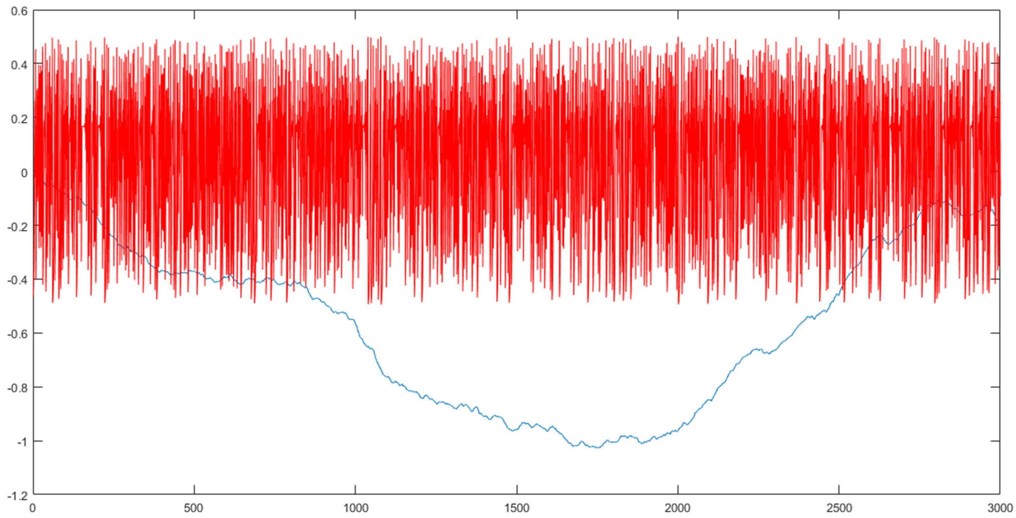


Figure 8. Tent map+ fractional Brownian motion (*H=0.8, α=2)*

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