DESIGN OF OPTIMAL BOUNDARY CONTROL FOR ELASTIC BEAM MOTIONS BASED ON AN INTEGRODIFFERENTIAL APPROACH

Georgy Kostin
Laboratory of Mechanics
of Controlled Systems
Institute for Problems in Mechanics
of the Russian Academy of Sciences
Russia
kostin@ipmnet.ru

Vasily Saurin
Laboratory of Mechanics
and Optimization of Structures
Institute for Problems in Mechanics
of the Russian Academy of Sciences
Russia
saurin@ipmnet.ru

Abstract

A projective approach is considered to modelling and optimization of controlled motions for elastic beams. The approach is based on an integrodifferential formulation of the original PDE system and FEM technique. A discretization scheme is worked out and an explicit energy criterion of solution quality is proposed. The result ODE system is used to design the optimal control that minimizes the terminal beam energy. The numerical results obtained are analyzed and discussed.

Key words

Elastic Beam, Optimal Control, FEM.

1 Introduction

The design of control strategies for dynamic systems with distributed parameters has been actively studied in recent years. Processes such as oscillations, heat transfer, diffusion, and convection are part of a large variety of applications in science and engineering. The theoretical foundation for optimal control problems with linear partial differential equations (PDEs) and convex functionals was established by [Lions, 1971], [Lions, 1988]. Linear hyperbolic equations are treated, besides in Lions' book, in [Ahmed and Teo, 1981], [Butkovsky, 1969]. An introduction to the control of vibrations can be found in [Krabs, 1995]. Oscillating elastic networks are investigated in [Gugat, 2005], [Lagnese et al., 1984], [Leugering, 2000].

Different approaches to discretization of dynamical models with distributed parameters are developed to reduce the original initial-boundary value problem to an ODE system. It is worth noting the variational and projection methods using to solve control problems for elastic structure motions. The method of integrodifferential relations (MIDR) was proposed in [Kostin and Saurin, 2006] for the optimal control design of elastic beam motions. Variational principles on the basis of the MIDR and their relations with the conventional variational formulations for linear elastic systems are studied in [Kostin and Saurin, 2009]. A projective approach is developed as a modification of the Galerkin method in the frame of the MIDR for dynamical systems described by linear parabolic PDEs in [Rauh et al, 2010]. In the paper this approach combined with the finite element method is extended to modeling and optimization of controlled dynamical systems with distributed elastic and inertial parameters.

2 Statement of the optimal control problem

Consider controlled motions of a rectilinear elastic beam described by the following PDE system with boundary and initial conditions:

$$
\begin{aligned}
&\dot{p} + m'' = 0,\\
&\eta := \dot{w} - p/\rho = 0,\\
&\xi := w'' - m/\kappa = 0,\\
&y \in (0, L), \quad t \in (0, T);\\
&m(t, L) = m'(t, L) = w'(t, 0) = 0,\\
&w(t, 0) = u(t); \quad w(0, x) = w_0(x),\\
&p(0, x) = p_0(x), \quad u(0) = w_0(0).
\end{aligned}
\tag{1}
$$

Here p is the linear momentum density; m is the bending moment in the beam cross section; w are the lateral displacements; L and ρ are the length and linear density of the beam, respectively; κ is its flexural rigidity; w_0 and p_0 are known functions of the spatial coordinate x; u is the control input (displacement of the beam end) and T is the terminal instant of the control process. The dotted symbols denote the partial derivatives with

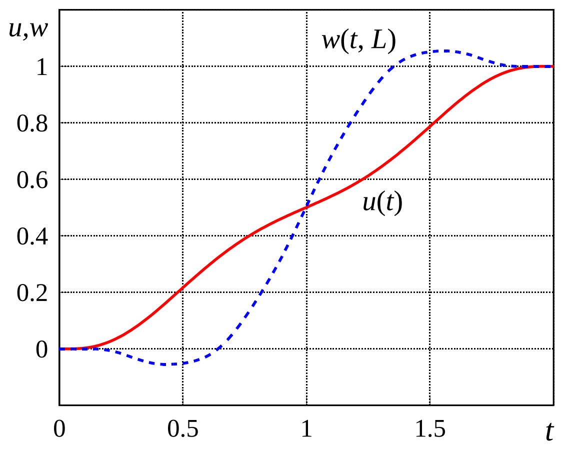

Figure 1. Optimal control $u(t)$ and displacement $w(t, L)$.

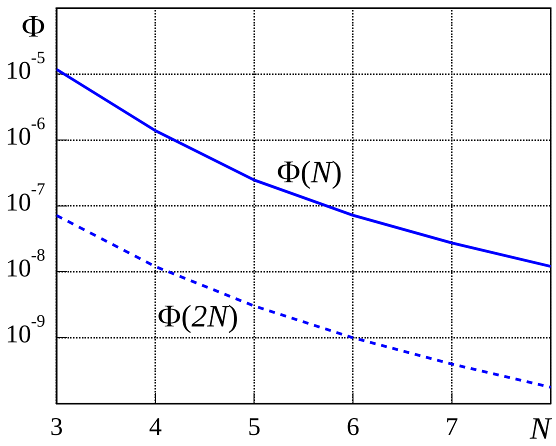

Figure 2. Integral error vs. N for the optimal motion.

respect to the time t, and the primed symbols stand for the partial derivatives with respect to the coordinate x.

The problem is to find an optimal control $u^*(t)$ in a given control set $\mathcal{U}$ that moves the beam from its initial state to the terminal position $w(T, 0) = w_T$ in the fixed time T and minimizes an objective function $J[u]$ (terminal mechanical energy of the beam):

$$
\begin{aligned}
&J[u] = E(T) \to \min_{u \in \mathcal{U}}, \\
&E(t) = \int_0^L \psi(t, x) dx, \quad \psi = \frac{\rho p^2}{2} + \frac{\kappa (w'')^2}{2}.
\end{aligned} \tag{2}
$$

To solve the initial-boundary value problem (1), we apply the MIDR, in which the local equalities $\eta = \xi = 0$ and initial conditions are replaced by integral relations, whereas the first equation in (1) and boundary conditions are satisfied exactly.

3 Discretization algorithm

Let us firstly eliminate the function m taking into account the first equation in (1) and boundary conditions at $x = L$ as follows

$$
m = -\int_0^{L-x} \int_0^{y} \dot{p}(t, L - y_1) dy_1 dy \tag{3}
$$

and then define a space mesh with the nodes:

$$
\begin{aligned}
&x_0 = 0, x_M = L, 0 \leq x_{j-1} < x_j, \\
&I_j = (x_{j-1}, x_j), j = 1, \ldots, M.
\end{aligned}
$$

To find an approximate solution of the optimization problem (2), the functions p and w are approximated by piece-wise polynomial space splines

$$
\begin{aligned}
&w \in S_w^{(N+2)} = \\
&\quad \Big\{ w(t,x) : w = \sum_{j=0}^{N+2} w_{ij}(t) \, (x/L)^j \,, \\
&\qquad x \in I_i, \; i = 1, \ldots, M; \\
&\qquad w \in C^1, \; x \in [0, L], \\
&\qquad w_{10}(t) = u, \; w_{11}(t) = 0 \Big\}, \\
&p \in S_w^{(N)} = \\
&\quad \Big\{ p(t,x) : p = \sum_{j=0}^{N} p_{ij}(t) \, (x/L)^j \,, \\
&\qquad x \in I_i, \; i = 1, \ldots, M \Big\}, \\
&\mathbf{w}(t) = \{ \mathbf{w}_1, \ldots, \mathbf{w}_M \} \,, \\
&\mathbf{w}_i = \{ w_{i2}, \ldots, w_{i,N+2} \} \,, \\
&\mathbf{p}(t) = \{ \mathbf{p}_1, \ldots, \mathbf{p}_M \} \,, \\
&\mathbf{p}_i = \{ p_{i0}, \ldots, p_{iN} \} \,, \quad i = 1, \ldots, M \,,
\end{aligned} \tag{4}
$$

where $\mathbf{w}(t)$ and $\mathbf{p}(t)$ are vector-functions defining the unknown displacements w and linear momentum density p.

A projective approach is used to reduce the original PDE system to a system of ODEs with initial conditions in the form

$$\int_0^L \eta(t,x,\mathbf{w},\mathbf{p},u)\chi(x)dx = 0,$$
$$\int_0^L \xi(t,x,\mathbf{w},\mathbf{p},u)\chi(x)dx = 0,$$
$$\forall\chi \in S_\chi^{(N)} = \Big\{\chi(x): \chi = \sum_{j=0}^{N} \chi_{ij}x^j/L^j,\ x\in I_i,\ i=1,\ldots,M\Big\}, \tag{5}$$
$$\int_0^L [w(x,\mathbf{w}(0),u(0)) - w_0(x)]\,\chi(x)dx = 0,$$
$$\int_0^L [p(x,\mathbf{p}(0)) - p_0(x)]\,\chi(x)dx = 0.$$

Here η and ξ are obtained by substituting relations (3) and (4) in (1). The following relative integral error is proposed to estimate the quality of these approximations

$$\Delta = \Phi/\Psi, \quad \Phi = \int_0^T\int_0^L \varphi(t,x)dxdt,$$
$$\Psi = \int_0^T\int_0^L \psi(t,x)dxdt, \tag{6}$$
$$\varphi = \frac{\rho\eta^2}{2} + \frac{\kappa\xi^2}{2}.$$

The consistent polynomial control input

$$u \in U = \Big\{u: u = w_0(0) + \frac{w_T t}{T} + \frac{t-T}{T}\sum_{i=1}^{K}\frac{u_i t^i}{T^i}\Big\} \tag{7}$$

is considered. After solving the initial value problem (1) for the undefined vector $\mathbf{u} = \{u_1,\ldots,u_K\}$ of control parameters, the result vector-functions $\mathbf{w}(t,\mathbf{u})$ and $\mathbf{p}(t,\mathbf{u})$ is used to minimized a modified objective function

$$J_1[\mathbf{u}] = J[\mathbf{u}] + \gamma\frac{\Phi}{T} \to \min_{\mathbf{u}},$$

where the last term is added with some dimensionless weight coefficient γ to regulate the accuracy of optimal solution.

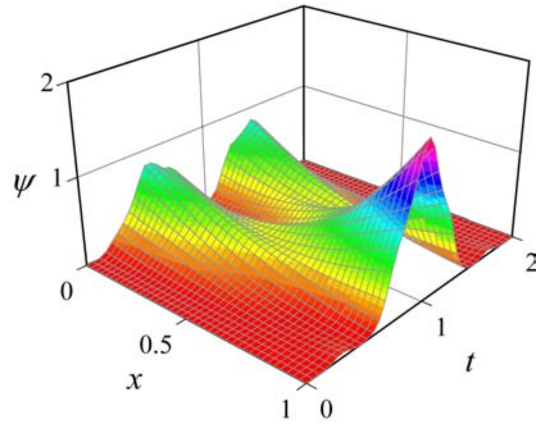

Figure 3. Energy density distribution ψ for the optimal motion.

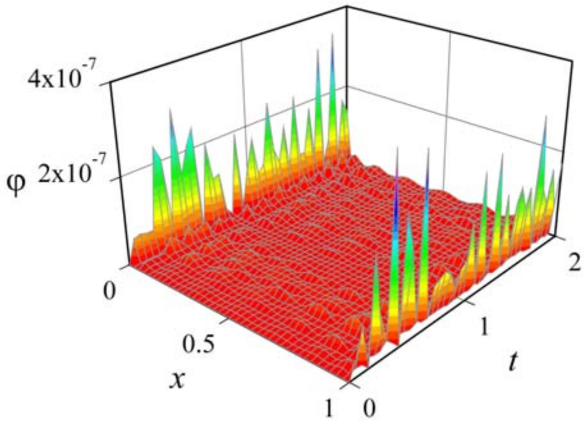

Figure 4. Local error distribution ϕ for the optimal motion.

4 Numerical results

The following dimensionless parameters of the control problem , and approximations are given:

$$\rho = \kappa = L = 1, \quad T = 2,$$
$$w_T = 1, \quad K = 4, \quad M = 1, \quad N = 8.$$

The optimal control $u^*(t)$ is shown in Fig. 1 by a solid curve. The corresponding objective function is $J_1 \approx 2.3\times10^{-5}$. The displacement of the free beam end is presented by a dash curve.

The integral error Φ is shown in Fig. 2 as a function of the approximation order N . For the order $N = 8$ the relative error of the numerical solution is equal to $\Delta|_{N=8} = 3.3\times10^{-8}$. The distribution of mechanical energy density $\psi(t,x)$ stored during the optimal motion is depicted in Fig. 3. The function of local error $\psi(t,x)$ is presented in Fig. 4.

5 Conclusion

Dynamical control problems for elastic structures are considered. A projective algorithm of numerical simulation and control optimization for these initial-boundary value problems is worked out based on the method of integrodifferential relations. The algorithms allow one to estimate explicitly the local and integral quality of numerical solutions obtained. As an example, an optimization problem of longitudinal controlled motions of a elastic beam clamped on a track is investigated.

Acknowledgements

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