ROBUSTNESS OF CHIMERA STATES WITH RESPECT TO NOISE

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Abstract
We investigate amplitude chimera states in nonlinear dynamical networks, which consist of coexisting domains of synchronized and desynchronized dynamics. In particular, we analyze the role of stochastic dynamics for these synchronization-desynchronization patterns. We address the question of robustness and control of amplitude chimera states in the presence of noise.

Key words
Nonlinear systems, networks, synchronization, chimera states, noise

1 Introduction

Chimera states in dynamical networks are intriguing spatiotemporal patterns which consist of coexisting spatial domains of synchronized and desynchronized behavior. The name is derived from the Greek mythological monster which is composed of incongruous parts of a lion, a goat, and a snake. They have recently initiated a burst of activity in different fields of research [Panaggio and Abrams, 2015]. These hybrid states discovered in the early 2000s occur in a network of identical homogeneously coupled elements which spontaneously splits into two coexisting domains of coherent (synchronized) and incoherent (desynchronized) behavior [Kuramoto and Battogtokh, 2002; Abrams and Strogatz, 2004]. Until very recently, there was no experimental evidence of chimera states. However, in 2012 two independent experiments with optical [Hagerstrom et al., 2012] and chemical [Tinsley et al., 2012] oscillators presented first experimental observations. Most recent experiments have been performed with mechanical [Martens et al., 2013; Kapitaniak et al., 2014], electronic or optoelectronic [Larger et al., 2013; Larger et al., 2015] and electrochemical [Wickramasinghe and Kiss, 2013; Schmidt et al., 2014; Schmidt and Krischer, 2015] oscillators.

Chimera states have been shown to be robust with respect to various kinds of perturbations [Laing et al., 2012]. In particular, they exist in ensembles of coupled heterogeneous elements [Laing, 2010] and networks with irregular coupling topologies [Ko and Ermentrout, 2008; Shanahan, 2010; Laing et al., 2012; Yao et al., 2013; Zhu et al., 2014]. In a recent study both of these inhomogeneity types are considered [Omelchenko et al., 2015]. Chimeras also exist in systems with delay [Ma et al., 2010; Sethia et al., 2008; Omelchenko et al., 2015a]. However, the behavior of chimera states in the presence of stochastic fluctuations still remains unresolved. This problem is especially relevant in the light of experimental realizations where noise is inevitable. It is known that even at a relatively low intensity, noise can significantly influence the behavior of a nonlinear dynamical system. Under certain conditions noise can cause the increase of coherence, e.g., in coherence resonance, and on the contrary it can also induce irregular, chaotic dynamics. Stochasticity appears due to the intrinsic fluctuations in a system, or alternatively can be implemented as an external random control force. The influence of noise strongly depends on its characteristics such as intensity, for example, which can be treated as bifurcation parameters of a stochastic system. Therefore, the investigation of noisy dynamics is on the one hand significant for the understanding of the processes occurring in nature and on the other hand relevant from the point of view of control. We are interested in understanding how robust chimera patterns are with respect to noise and in developing control mechanisms by means of noise.

It has been shown that chimera states are stable for infinitely large networks and have transient nature for finite networks. In the latter case chimeras represent a chaotic saddle state. Moreover, the lifetime of a chimera state - the time till it transforms into the completely coherent state - grows exponentially with the system size [Wolfrum and Omel'chenko, 2011; Rosin et al., 2014]. This has been found for networks of phase oscillators, where the phase dynamics in a chimera pattern is chaotic in time. Most recently discovered *amplitude chimeras* appear in systems which capture not only phase but also amplitude dynamics and are characterized by time periodic behavior of both amplitude and phase [Zakharova et al., 2014; Zakharova et al., 2015]. Interestingly, amplitude chimeras also collapse into an in-phase synchronized state and can therefore also be treated as transients towards a completely coherent regime.

Here we investigate the transient mechanism for amplitude chimeras and the impact of stochasticity on their lifetime. Especially we focus on the question of amplitude chimera control by noise. Additionally, we explain our results by proposing that an amplitude chimera pattern is a saddle state.

2 Model

We consider a network of N supercritical Stuart-Landau (SL) oscillators. The uncoupled dynamics of each node $j \in \{1, ..., N\}$ is given by $\dot{z}_j = f(z_j)$, with

$$f(z_j) = (\lambda + i\omega - |z_j|^2)z_j, \qquad (1)$$

where $z_j = x_j + i\,y_j = r_j e^{i\phi_j} \in \mathbb{C}$, with $x_j, y_j, r_j, \phi_j \in \mathbb{R}$, and $\lambda, \omega > 0$. In the uncoupled case, the system undergoes a Hopf bifurcation at $\lambda = 0$, so that for $\lambda > 0$ the single SL oscillator exhibits self-sustained oscillations with frequency ω and radius $r_j = \sqrt{\lambda}$. The only steady state $x_j = 0$ and $y_j = 0$ is unstable.

2.1 Deterministic dynamics on a network

We investigate a ring of N non-locally coupled SL oscillators, where each node is coupled to its P nearest neighbors in both directions with the strength σ:

$$\dot{z}_j = f(z_j) + \frac{\sigma}{2P}\sum_{k=j-P}^{j+P}(Rez_k - Rez_j) \qquad (2)$$

where $j = 1, 2, ..., N$, and all indices are modulo N. The coupling strength σ, and the coupling range P/N are identical for all connections. By definition, the values of P are restricted to positive integer numbers. We further set the values of σ to be positive real numbers. As the coupling term only involves the real parts, the rotational S^1 symmetry of the system is broken. This is a necessary condition to obtain nontrivial steady states $z_j \neq 0$ [Zakharova et al., 2013].

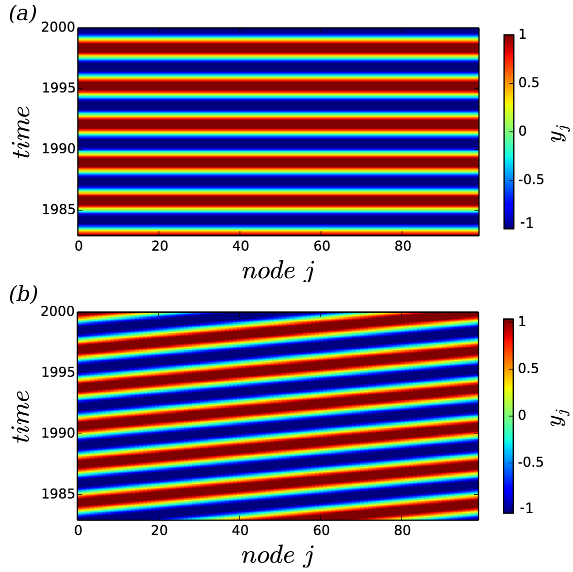

Figure 1. Space-time plots of the synchronized regimes: (a) in-phase synchronization; (b) coherent travelling wave. The colors indicate the value of y_j for each node j. Different amplitude-chimera-like initial conditions were used in panels (a) and (b). Parameters: $N = 100, \sigma = 14, P/N = 0.04, \omega = 2, \lambda = 1.$

2.2 Stochastic dynamics on a network

In order to study the impact of noise we supplement Eq. (2) with a noise term leading to stochastic differential equations:

$$\dot{z}_j = f(z_j) + \frac{\sigma}{2P}\sum_{k=j-P}^{j+P}(Rez_k - Rez_j) + \sqrt{2D}\xi_j(t) \qquad (3)$$

where $\xi_j(t)$ is the normalized source of Gaussian white noise: $\langle \xi_j(t)\xi_j(t+\tau)\rangle = \delta(\tau)$, $\langle \xi_j(t)\rangle = 0$, and D is the noise intensity. The noise terms are spatially uncorrelated: $\langle \xi_i(t)\xi_j(t)\rangle = \delta_{ij}$, i.e. every node j gets a different input $\xi_j(t)$.

3 Patterns in the deterministic model

It has been shown by [Zakharova et al., 2014] that the following three types of regimes can be found in the network Eq. (2) depending on the choice of coupling parameters: in-phase synchronized regime (or traveling wave), amplitude chimera (AC) and chimera death (CD). The in-phase synchronized regime and coherent traveling waves are shown as space-time plots color coded by the variable y_j in Fig. 1(a) and Fig. 1(b), respectively. It is not surprising to find synchronized solutions in a network of identical coupled elements. However, the interplay of nonlocally coupled network topology and symmetry-breaking coupling gives rise to the amplitude chimera - a striking hybrid state characterized by the spatial coexistence of two domains with

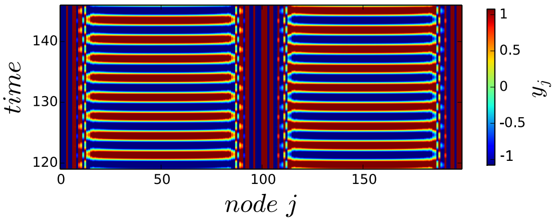

Figure 2. Space-time plot of an amplitude chimera state. Parameters: $N=200$, $\sigma=18$, $P/N=0.04$, $\omega=2$, $\lambda=1$.

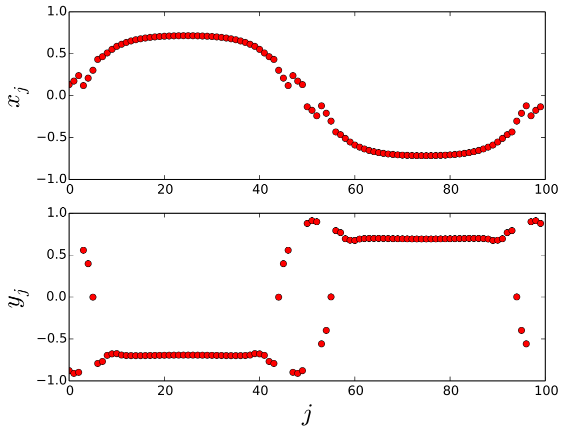

Figure 3. Snapshot of an amplitude chimera (a): Real parts x_j, (b): imaginary parts y_j. The dots mark the values of the corresponding nodes j. Parameters: $N = 100, \omega = 2, \lambda = 1, \sigma = 14, P/N = 0.04$.

different behavior: one is oscillating with spatially coherent amplitude, while the other demonstrates oscillations with spatially incoherent amplitudes; in both domains the phases remain spatially correlated, in contrast to the common phase chimeras, e.g., in the Kuramoto model, where the chimera behavior is manifested in the partially incoherent phases, and the amplitudes are not relevant. Fig. 2 shows a typical space-time plot, and Fig. 3 depicts a snapshot for the amplitude chimera. Moreover, the chimera behavior of steady states, i.e., chimera death, is also observed in the network Eq. (2) [Zakharova et al., 2014]. In this regime the network splits into two parts: spatially coherent oscillation death, where the neighboring nodes populate the same branch of the inhomogeneous steady state, and spatially incoherent oscillation death, where the sequence of populated branches of neighboring elements is random. A typical space-time plot for the chimera death with two coherent and two incoherent domains is given in Fig. 4. Additionally, chimera death patterns with different number of clusters in the coherent domain can be found while tuning the coupling parameters of the system Eq. (2) [Zakharova et al., 2014; Zakharova et al., 2015].

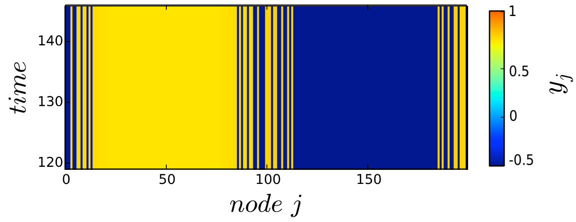

Figure 4. Space-time plot of a chimera death state. Parameters: $N = 200, \omega = 2, \lambda = 1, \sigma = 18, P/N = 0.4$.

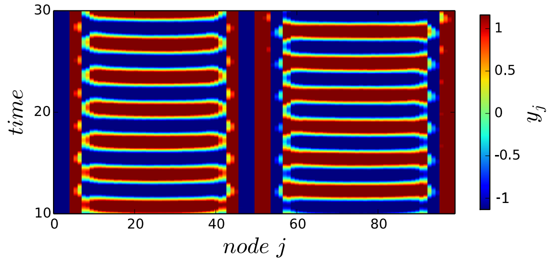

Figure 5. Space-time plot of an amplitude chimera state in presence of noise with intensity $D = 5 \cdot 10^{-3}$. Parameters: $N = 100$, $\sigma=19$, $P/N=0.04$, $\omega=2$, $\lambda=1$.

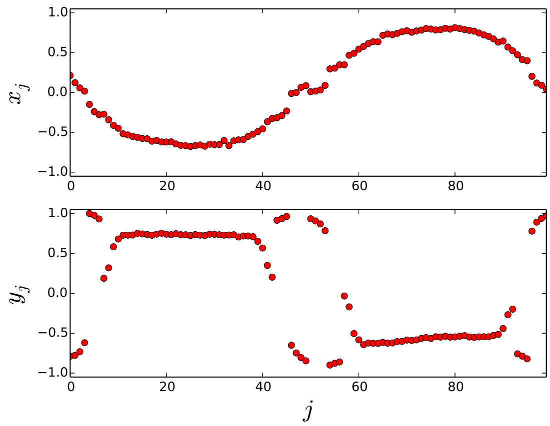

Figure 6. Snapshot of an amplitude chimera in presence of noise with intensity $D = 5 \cdot 10^{-3}$. Parameters: $N = 100$, $\sigma = 19$, $P/N=0.04$, $\omega=2$, $\lambda=1$.

4 Stochastic case

Our numerical investigations of the stochastic model Eq. (3) show that amplitude chimeras exist in the presence of random fluctuations and are robust to noise even of high intensity. A space-time plot and a snapshot of amplitude chimeras for $D=5 \cdot 10^{-3}$ are given in Figs. 5 and 6, respectively. Although chimera patterns are preserved, their lifetime is dramatically decreased by noise and strongly depends on its intensity: the larger the noise intensity the shorter the lifetime.

In order to quantitatively characterize the impact of noise on the chimera structures we analyze their transient lifetime t_{tr} for various coupling strengths σ in the presence of noise of different intensity. For the range

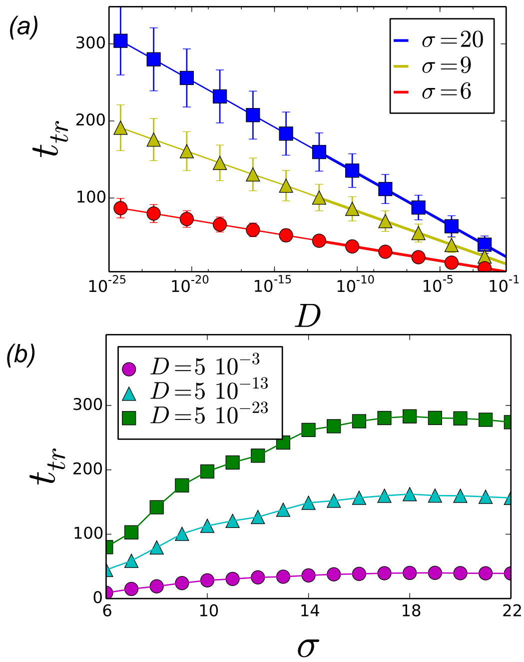

Figure 7. Transient times of amplitude chimeras (a) in dependence on noise intensity D (logarithmic scale) for different values of coupling strength $\sigma = 6, 9, 20$, (b) in dependence on σ for different values of D. The black symbols represent the average over 50 realizations of the randomized initial conditions, the colored lines correspond to a linear fit of those (following eg. (4)), the colored error bars denote the corresponding standard deviation of the resulting t_{tr} for the 50 initial conditions; dots and red bars: $\sigma = 6$, triangles and yellow bars: $\sigma = 9$, squares and blue bars: $\sigma = 20$. Other parameters: $N = 100$, $P/N = 0.04$, $\omega = 2$, $\lambda = 1$.

of noise intensity $10^{-25} \leq D \leq 10^{-2}$ we compare the results for t_{tr} for three choices of the coupling strength ($\sigma = 6, 9, 20$) in a semi-logarithmic plot (Fig. 7 (a)). The transient times linearly decrease with the logarithm of the applied noise intensity for all σ. Therefore, our numerical results suggest the following relation between the chimera lifetime and the noise intensity:

$$t_{tr} = -\frac{1}{\mu} \, ln(D) + \eta \Leftrightarrow D \sim e^{-\mu t_{tr}}, \qquad (4)$$

where μ and η are constants depending on the coupling strength σ: the stronger the coupling, the steeper is the slope. Fig. 7 (b) shows that t_{tr} increases monotonically with the coupling strength σ up to a maximum value.

The lifetime of amplitude chimeras in the deterministic case strongly depends on the choice of initial conditions as reported in [Zakharova et al., 2015]. In general the sensitivity of chimera states to the initial configurations is explained by the fact that classical chimera states typically coexist with the completely synchronized state, for which the basin of attraction is significantly larger. Although we do not have a precise geometric vision of the phase space or a strict analytical proof of the stability for amplitude chimeras, all our numerical results support the idea that amplitude chimera patterns can be seen as a saddle state composed of stable (solid lines in Fig. 8) and unstable (dashed lines Fig. 8) manifolds. The set of initial conditions can be represented as a volume restricted in phase space (green circle in Fig. 8). The observed amplitude chimera corresponds to trajectories starting from this set and passing the saddle-point from the stable direction towards the unstable manifold. The lifetime of an amplitude chimera, therefore, depends on the chosen trajectory: the closer to the saddle point it gets, the longer is the lifetime. In other words, the transient time is determined by the time the system spends in the vicinity of the saddle point where coherent and incoherent oscillating domains coexist before it escapes to the in-phase synchronized regime. Such a phase space scenario explains the sensitivity of transient times to initial conditions since they determine the particular path the system takes.

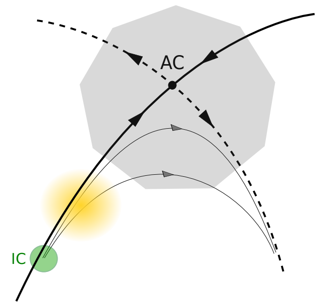

Figure 8. Schematic saddle-point representation of the amplitude chimera (AC). Solid lines denote the stable manifold and dashed lines correspond to the unstable manifold. The green circle defines the set of initial conditions and yellow shading represents the impact of noise.

Our numerical investigations of the stochastic model Eq. (3) show that Gaussian white noise dramatically reduces the impact of initial conditions on the lifetime of amplitude chimeras. In more detail, we have tested a set of realizations of initial conditions which lead to significantly different lifetimes of amplitude chimeras without random forcing. In the presence of relatively weak noise $D = 5 \cdot 10^{-13}$ all realizations result in amplitude chimeras with similar lifetime (results not shown). This again supports our view of the amplitude chimera as a saddle point, and allows for the following explanation. The stochastic force, which continuously perturbs the system, makes it randomly switch between different trajectories close to the saddle point. Therefore, the system's dynamics is not determined by a single trajectory anymore, but rather affected by a set of trajectories belonging to the N-dimensional hyper sphere. This reduces the sensitivity of the amplitude chimera lifetime to initial conditions. In Fig. 8 the impact of noise is illustrated by yellow shading, denoting the stochastic forces applied to the system.

5 Conclusion

We have shown that amplitude chimeras can be considered as transients towards in-phase synchronized states. In the deterministic case their lifetime depends on the choice of initial conditions. We have found that amplitude chimeras are robust to noise and also exist in networks with time-delayed coupling. Moreover, the lifetime of amplitude chimeras can be controlled by varying the noise intensity and the value of time delay, which, therefore, play the role of control parameters. The presence of noise allows one to decrease the lifetime of amplitude chimeras, while time delay can significantly increase it. Furthermore, we provide an intuitive picture of the stability for amplitude chimera states. The results of our numerical investigations suggest that amplitude chimeras can be represented as a saddle point in the high-dimensional phase space.

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