A NEW LINEAR PROGRAMMING ALGORITHM FOR OPTIMAL ESTIMATION AND CORRECTION OF AN AIRCRAFT TRAJECTORY

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1. Introduction.

Problems of optimal estimation under uncertainty and scalar optimal design of experiment may be reduced to a problem of linear programming [1]. Problems of ideal linear ideal trajectory correction and many problems of optimal design of experiment may be solved with the help of multiparametrical (so-called generalized) linear programming [3, 4, 5]. Traditional methods for solving these problems are the simplex method and the the column-generate method respectively. While using these methods two main difficulties may appear. Current basic matrix often turns out to be be ill conditioned, and the solution is usually accompanied by a large number of almost degenerate iterations thus accumulating large computational errors [2]. Besides this convergence of the column-generate method is not proved. An new efficient method for solving problems of this class the so-called *skeleton algorithm* is proposed. This algorithm helps to avoid problems mentioned above [6].

2. Idea of the skeleton algorithm.

Consider an ordinary linear programming problem:

$$\min_{\overset{m}{x}_i} \left\{ \sum_{i=1}^{n} \overset{m}{c}_i \overset{m}{x}_i : \sum_{i=1}^{n} \overset{m}{x}_i \overset{m}{a}_i = \overset{m}{b},\ \overset{m}{x}_i \geq 0,\ i = 1, ..., n \right\}. \tag{1}$$

Here $\overset{m}{x}_i$ are scalar optimization variables; other parameters are fixed: $\overset{m}{c}_i$ are objective function's coefficients, $\overset{m}{a}_i \in R^m$ are columns of the condition matrix, and $\overset{m}{b} \in R^m$ is a right side vector, $\overset{m}{b} \neq 0$. Let us call the number of linear conditions of a problem a dimension of the problem. Here and below problems' dimensions are shown with an upper index.

Let I_B be a set of indices of a basis $\{\overset{m}{a}_i : i \in I_B\}$. Let us construct a new linear programming problem that has one additional column $\overset{m}{a}_0$ and an additional variable $\overset{m}{x}_0$:

$$\min_{\overset{m}{a}_i} \left\{ \overset{m}{c}_0 \overset{m}{x}_0 + \sum_{i=1}^{n} \overset{m}{c}_i \overset{m}{x}_i : \overset{m}{x}_0 \overset{m}{a}_0 + \sum_{i=1}^{n} \overset{m}{x}_i \overset{m}{a}_i = \overset{m}{b},\ x_i \geq 0,\ i = 0, 1, ..., n \right\}, \tag{2}$$

where

$$\overset{m}{a}_0 = \sum_{i \in I_B} \overset{m}{x}_i \overset{m}{a}_i, \quad \overset{m}{c}_0 = \sum_{i \in I_B} \overset{m}{c}_i \overset{m}{x}_i.$$

Solutions of (1) and (2) are equal, besides this the basic feasible solution $\overset{m}{x}_0 = 1, \overset{m}{x}_1 = 0, ..., \overset{m}{x}_n = 0$ of (1) gives an objective function's value equal to that given by the $\left\{\overset{m}{a}_i,\, i \in I_B\right\}$ basis in (2) [2].

Consider an auxiliary linear programming problem of dimension $m-1$:

$$\min_{\overset{m-1}{x}_i} \left\{ \sum_{i=1}^{n} \overset{m-1}{c}_i \overset{m-1}{x}_i : \quad \sum_{i=1}^{n} \overset{m-1}{x}_i \overset{m-1}{a}_i = \overset{m-1}{b}, \quad \overset{m-1}{x}_i \geq 0 \,\forall\, i \right\}, \tag{3}$$

$$\overset{m-1}{c}_i \doteq \overset{m}{\Delta}_i = \overset{m}{c}_i - \overset{m}{\pi}{}' \overset{m}{a}_i \tag{4}$$

is the so-called reduced cost coefficients, $\overset{m}{\pi}$ is any solution of $\overset{m}{c}_0 - \overset{m}{\pi}{}' \overset{m}{a}_0 = 0$, $\overset{m-1}{b}$ is a vector such that conditions are simultaneous, and $\overset{m-1}{a}_i$ are $(m-1)$-vectors obtained from

$$\overset{m}{a}_i = B \begin{pmatrix} \overset{m-1}{g}_i \\ \overset{m-1}{a}_i \end{pmatrix}, i = 0, 1, ..., n, \tag{5}$$

where $B = (\overset{m}{a}_0,\, V)$, V is a $m \times (m-1)$-matrix such that matrix B is nondegenerate. Vector $\overset{m-1}{a}_i \in R^{m-1}$ is an *image* of $\overset{m}{a}_i$, and $\overset{m}{a}_i$ is a *prototype* of $\overset{m-1}{a}_i$.

Proposition 1. [7, 8] If problem (3) has a solution, then current degenerate basic solution of (2) is optimal. If the objective function's value in (3) is unbounded it is possible either to diminish the objective function's value in (2), or to find out unsolvability of problem (2).

According to [8] if the objective function of (3) is unbounded on the set of feasible solutions, there is a set of indices S ($|S| \leq m$) and values λ_j, such that for a_j, $j \in S$ in (2) we have

$$\sum_{j \in S} \lambda_j \overset{m-1}{a}_j = 0 \Leftrightarrow \sum_{j \in S} \lambda_j \overset{m}{a}_j = \overset{m}{\alpha} \overset{m}{a}_0; \quad \sum_{j \in S} \lambda_j \overset{m}{\Delta}_j < 0; \quad \lambda_j > 0,\, j \in S. \tag{6}$$

Using (5), (6) we get $\overset{m}{\alpha} = \sum\limits_{j \in S} \lambda_j \overset{m-1}{g}_i$, where values $\overset{m-1}{g}_i$ are as in (5).

While using the skeleton algorithm we get S and vectors $\overset{m-1}{a}_j$, $j \in S$ analytically for a one-dimensional problem of linear programming (i.e., problem with one linear condition).

Proposition 2. If $\overset{m}{\alpha} \leq 0$, then the objective function in (2) is unbounded on the set of feasible solutions. If $\overset{m}{\alpha} > 0$ the objective function in (2) decreases by $\left| \sum\limits_{j \in S} \lambda_j \overset{m}{\Delta}_j \right| / \overset{m}{\alpha}$ when the basis $\overset{m}{a}_0$ is replaced with $\{\overset{m}{a}_i : i \in S\}$ ([8]).

We can use the same method to find out solvability or unsolvability of (3), i.e., we can construct an auxiliary problem of dimension $m-2$, then a problem of dimension $m-3$, etc. Finally, we obtain a one-dimensional problem, i.e., a linear programming problem, that has only one row. It may be solved analytically [6]. For an k-dimensional auxiliary problem of the form (3) ($k = m-1, ..., 1$) it is convenient to assume right side vector in (3) to be equal to one of the condition columns. Then extended problem for (3) can be obtained without matrix inversion. If we do so, we can put $\overset{k}{a}_0 = \overset{k}{b}$, $\overset{k}{c}_0 = 1$.

So, the skeleton algorithm results in solving a one-dimensional problem. If it has a solution, then current basis in (1) is optimal. Vice versa a new basis $\overset{2}{a}_1, \overset{2}{a}_2$ of a two-dimensional problem is found. According to (6), we have

$$\overset{2}{\lambda}_1 \overset{2}{a}_1 + \overset{2}{\lambda}_2 \overset{2}{a}_2 = \overset{2}{\alpha} \overset{2}{a}_0. \tag{7}$$

Now, there are two variants.

1. $\overset{2}{\alpha} \leq 0$ in (7). Then two-dimensional problem has no solution and we go over to a three-dimensional problem rewriting (7) as

$$\overset{2}{\lambda}_1 \overset{3}{a}_1 + \overset{3}{\lambda}_2 \overset{2}{a}_2 - \overset{2}{\alpha} pt(\overset{2}{a}_0) = \overset{3}{\alpha} \overset{3}{a}_0, \tag{8}$$

where $pt(a)$ is a prototype of a. Going on with this we will reach an initial problem (2) and replace a_0 with a set of vectors.

2. $\overset{2}{\alpha} > 0$ in (7). Then the objective function's value decreases according to proposition 2. We then go over to an extended two-dimensional problem of the form (2), find $\overset{2}{a}_0$ from (7), and consider a new one-dimensional problem. If $\overset{j}{\alpha} > 0$, $j = 2, 3, ...$, we diminish the objective function and check the solvability of a j-dimensional problem using $(j-1)$-dimensional auxiliary problem.

Important remark. If $\overset{2}{\alpha} > 0$ in (7) we are to calculate a prototype of $\overset{2}{a}_0$. According to (7) this vector is a linear combination of two vectors. That is why while manipulating auxiliary problems of dimension 2 or more we may obtain a *nonbasic* optimal solution. It may be replaced with a basic solution using the ideas of [9].

An idea of the skeleton algorithm was described here for an ordinary linear programming problem. Everything is also just for the generalized linear programming with the one exception: in some cases one-dimensional problem cannot be solved analytically. However it is usually still easy to solve. The skeleton algorithm has no degenerate iterations that is why it is convergent even for the generalized problem.

3. The algorithm.

Step 1. Construct a series of auxiliary problems of dimension $m-1$, $m-2$, ..., 1. If all coefficients of the objective function are nonnegative for one of the constructed problems, go to step 4. Vice versa after constructing one-dimensional problem go to step 2.

Step 2. Solve one-dimensional problem. If it is solvable, then go to step 4. If it is not solvable (i.e., $\overset{1}{\alpha} \leq 0$), then calculate $\overset{k}{\alpha}, k = 2, 3, 4...$ for 2, 3, 4...-dimensional auxiliary problems until some $\overset{j}{\alpha}$ will be positive. Then go to step 3.

Step 3. We now know that $\overset{j}{\alpha} > 0$, and $\overset{k}{\alpha} \leq 0$ for all $k < j$. Calculate new value of coefficient $\overset{j-1}{c_0}$ and vector $\overset{j-1}{a_0}$. Then using (4) calculate $\overset{k}{c}_i$ for $k = j-1, j-2, ..., 1$, and go to step 2.

Step 4. The end. Current solution in the initial m-dimensional problem is optimal.

4. Numerical experiments.

4.1. **Minimax estimation problem.** Consider an estimation problem with the following model of measurements:

$$y(t) = H(t)\theta + \xi(t), t \in T,$$

where θ is an m-vector of unknown parameters, t is time, $y(t)$, $\xi(t)$ are measurement results and their errors, $H(t)$ are known vectors. Let $\tau = \{t_1, ..., t_n\}$ be a set of moments of measurements, $y = (y(t_1), ..., y(t_n))'$ respective vector of measurements, $l = b'\theta$ the estimated parameter (b is a known vector). Let $\widehat{l}(y) = x'y$ be a linear algorithm of estimation, where x satisfies the condition of the algorithm unbiasedness $\widehat{l}(H'\theta) = l = b'\theta$, which is equivalent to the equality $H_i \doteq H(t_i)$: $\sum\limits_{i=1}^{n} x_i H_i = b$. Assuming $|\xi_i| \leq 1$, we formulate the minimax problem:

$$L^* = \min_{\tau, x_i} \max_{|\xi_i| \leq 1} \left\{ \widehat{l} - l : \tau \in T, \sum_{i=1}^{n} x_i H_i = b \right\}.$$

This may be reduced to a linear programming problem [1]

$$L^* = \min_{\tau, x_{i1}, x_{i2}} \left\{ \sum_{i=1}^{n} (x_{1i} + x_{2i}) : \sum_{i=1}^{n} (x_{1i} - x_{2i}) H_i = b;\ x_{1i}, x_{2i} \geq 0,\ i = 1, ..., n \right\}.$$

As opposed to an ordinary linear programming here the number of variables is infinitely. While using simplex-method we search for a basic solution, i.e., consider only such τ that consist of m moments. Then we introduce to the basis a vector from a set $\{H(t) : t \in T\}$ such that $\Delta(t) = 1 \mp \pi' H(t)$ reaches its minimum on this vector. This is the idea of the column-generate method.

Consider the following measurement model:

$$y(t) = \theta_1 + \theta_2 \cdot \sin t + \theta_3 \cdot \cos t + \xi(t), t \in [0, 1].$$

Let us take θ_2 as a controlled parameter l, and vectors corresponding to the moments 0.88, 0.888, 0.8888 as an initial basis. This case is unfavorable for the simplex-method because the basis matrix is ill-conditioned. Numerical results are shown in table 1. (One iteration of the skeleton algorithm is all calculations that decrease the objective function in the initial problem).

Table 1.

iteration	simplex-method		skeleton algorithm	
	obj. func.	opt. moments	obj. func.	opt. moments
1	483039.19	{0.88, 0.888, 0.8888}	483039.19	{0.88, 0.888, 0.8888}
2	2502.10	{0, 0.888, 0.8888}	7.83	{0, 0.5, 1}
3	8.85	{0, 0.444, 0.8888}		
4	7.93	{0, 0.444, 1}		
5	7.83	{0, 0.5, 1}		

Number of elementary operations is 1,245 and 995 respectively.

4.2. **Optimal problem of linear ideal correction of an aircraft trajectory.** Let l be a k-vector of parameters of the system, and let b be deviation of l from its nominal value. Suppose that l changes by the value of $U_i u_i$ while correcting the trajectory with the impulse u_i, $i = 1, ..., n$, where U_i is a matrix with m columns and $dim(u_i)$ rows. Now assume that correction expenses are proportional to $\|u_i\|$, and formulate an optimal problem of linear ideal correction according to [1, 3]

$$\min_{u_i}\left\{\ \sum_{i=1}^{n}\|u_i\|:\ \ \sum_{i=1}^{n}U_iu_i=b\ \right\}.$$

This problem may be reduced to the following generalized linear programming problem:

$$(9)\qquad \min_{x_i,a_i}\left\{\ \sum_{i=1}^{n}x_i:\ \ \sum_{i=1}^{n}x_ia_i=b,x_i\geq 0,a_i=U_i\gamma_i,\ \|\gamma_i\|=1,i=1,...,n\ \right\}.$$

Consider a problem of correction for the single mass point in space, that follows a parabolic path on the time interval $[T,3T/2]$, where moment T corresponds to the highest point of the trajectory. Let us take deviations of the point's coordinates and velocities at the moment t as a correction impulse. Comparison of the skeleton algorithm and the simplex-method for problem (9) shows an inefficiency of the latter (see Table 2).

Table 2.

iteration	simplex-method		skeleton algorithm	
	obj. func.	opt. moments	obj. func.	opt. moments
1	1005.11	{2.2, 2.5, 2.8}	1005,11	{2.2, 2.5, 2.8}
2	232.01	{2, 2.5, 2.8}	12,03	{2, 2.5, 3}
3	12.05	{2, 2.5, 3}		
4	12.04	{2, 2.25, 3}		
5	12.03	{2, 2.13, 3}		
6	12.03	{2, 2.07, 3}		
7	12.03	{2, 2.04, 3}		
8	12.03	{2, 2.02, 3}		
9	12.03	{2, 2.01, 3}		
10	12.03	{2, 2.00, 3}		

Total time of calculations in MatLab is 8 and 1.2 seconds respectively.

4.3. **Nonbasic solution.** As it was mentioned above, while using the skeleton algorithm we may obtain a *nonbasic* optimal solution. Here is a simple example of such a situation. Let us solve a standard linear programming problem

$$\min_{x}\left\{c'x:Ax=b,x\geq 0\right\},$$

where $c=(-\frac{1}{2},-1,0,0,0,0,0,0)'$, $b=(4,7,10,14,32,24)'$,

$$A=\begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ -1 & 2 & 0 & 2 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 6 & 0 & 0\\ 2 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\ 2 & -3 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Simplex method gives us a basic optimal solution $x^{(1)}=(8,10,10,1,0,0,6,38/3)'$, while the result of the skeleton algorithm's computations is $x^{(2)}=(10,9,10,1,3,3/4,9,11)'$. This is a nonbasic solution, that gives the same value of the objective function.

5. Conclusion.

The subject of this paper is a new algorithm of linear programming, which can be used for the minimax estimation problem and optimal problem of linear ideal correction of a trajectory. Proposed algorithm helps to avoid almost degenerate iterations and large computational errors. The algorithm is easy enough because it does not use inversion of matrices. For generalized linear programming problems it is not necessary to store all arrays of columns for auxiliary problems in computer's memory due to the columns' analytic representation. That is why the algorithm is especially effective for generalized linear programming. However extra numerical experiments are needed to find out the algorithm's efficiency.

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