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CONTROLLED SYNCHRONIZATION IN TWO FITZHUGH-NAGUMO SYSTEMS WITH SLOWLY-VARYING DELAYS

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Abstract
We study synchronization in two delay-coupled FitzHugh-Nagumo systems with slowly-varying delays, which are the simplest model of neural network. It is well known that time-varying delay in propagation between the nodes hinders synchronization. We show that external stimulus can be used to control synchronization. We develop the algorithm for synchronization of FitzHugh-Nagumo systems and find the conditions of its applicability. The simulation results confirm the efficiency of suggested algorithm.

Key words
Synchronization, FitzHugh-Nagumo system, slowly-varying delay.

1 Introduction

The ability to control nonlinear dynamical systems has brought up a wide interdisciplinary area of research that has evolved rapidly in the past decades [Schöll and Schuster, 2008]. This field has various aspects comprising stabilization of unstable fixed points (steady states) [Pyragas, 1995; Yanchuk et al., 2006], control of oscillation in systems with several degrees of freedom [Shiriaev, Freidovich and Gusev, 2010; Plotnikov and Andrievsky, 2013], or control of network dynamics [Zhou, Lu and Lü, 2008; Proskurnikov, 2013; Selivanov, Fradkov and Fridman, 2015]. One of the important areas to consider is the synchronization in neural networks.

As any other kind of physical, chemical, or biological oscillators, such neurons could synchronize and exhibit collective behavior that is not intrinsic to any individual neuron. For example, partial synchrony in cortical systems is believed to generate various brain oscillations, such as the alpha and gamma EEG rhythms. Increased synchrony may result in pathological types of activity, such as epilepsy. Coordinated synchrony is needed for locomotion and swim pattern generation in fish [Izhikevich, 2005]. On the one hand, synchronization can be good, while, on the other hand, it can be harmful. Therefore, it is important to study synchronization in neural networks. The control of synchronization has so far focused on networks of identical nodes with constant parameters [Zhou, Lu and Lü, 2008; Lu and Qin, 2009; Lu et al., 2012; Selivanov et al., 2012; Guzenko, Lehnert and Schöll, 2013; Lehnert et al., 2014]. However, in realistic networks the nodes are always characterized by some diversity meaning that the parameters of the different nodes are not identical and may vary over the time. These variations and heterogeneities in the nodes can hinder or prevent synchronization.

In order to grasp the complicated interaction of neurons in large neural networks, those are often lumped into groups of neural populations each of which can be represented as en effective excitable element that is mutually coupled to other elements [Rosenblum and Pikovsky, 2004; Popovych, Hauptmann and Tass, 2004]. In this sense the simplest model which may reveal features of interacting neurons consists of two coupled neural oscillators. Each of this can be represented by a simplified FitzHugh-Nagumo system [FitzHugh, 1961; Nagumo, Arimoto and Yoshizawa, 1962].

Note that in Refs. [Cakan, Lehnert and Schöll, 2014; Hu, Yang and Liu, 2014] FitzHugh-Nagumo systems with heterogeneities were considered. Specifically, in Ref. [Cakan, Lehnert and Schöll, 2014] authors consider a behavior of FitzHugh-Nagumo network with heterogeneous constant delays, while in Ref. [Hu, Yang and Liu, 2014] the phenomenon of vibrational resonance in a time-varying FitzHugh-Nagumo system is studied. Here we propose the algorithm to control synchronization in two delay-coupled systems with slowly-varying delays and show that it can be used to counteract the variations in connections time between the neurons.

The paper is organized as follows. After this introduction we describe the system model in Sec.2, define the synchronization problem for two FitzHugh-Nagumo systems and develop the control algorithm in Sec. 3. Section 4 shows the results of the numerical simulation of developed algorithm performance. Finally, we conclude with Sec. 5.

2 Model Equation

We consider two delay-coupled FitzHugh-Nagumo (FHN) systems [FitzHugh, 1961; Nagumo, Arimoto and Yoshizawa, 1962], which are the simplest model of neural network. The FHN model is paradigmatic for excitable dynamics close to a Hopf bifurcation [Lindner et al., 2004], which is not only characteristic for neurons but also occurs in the context of other systems ranging from electronic circuits [Heinrich et al., 2010] to cardiovascular tissues and the climate system [Murray, 1993; Izhikevich, 2000]. The plant is described as follows:

$$\begin{aligned} \varepsilon \dot{u_1} &= u_1 - \frac{u_1^3}{3} - v_1 + C[u_2(t-\tau) - u_1(t)] + I, \\ \dot{v_1} &= u_1 + a, \\ \varepsilon \dot{u_2} &= u_2 - \frac{u_2^3}{3} - v_2 + C[u_1(t-\tau) - u_2(t)], \\ \dot{v_2} &= u_2 + a, \end{aligned} \tag{1}$$

where u_i and v_i denote the membrane potential and recovery variable of the nodes $i = 1, 2$ respectively, ε is a time-scale parameter and typically small, meaning that u_i is a fast variable, while v_i changes slowly. τ is the delay, i.e., the time the signal needs to propagate between two nodes. Let assume that the delay τ is a differentiable function with $\dot{\tau} \leqslant d < 1$ (this is the case of *slowly-varying delays*). I is an external stimulus and will be considered as a control. The coupling strength is given by C. In the uncoupled system ($C = 0$), a is the threshold parameter: for $a > 1$ the system is excitable while for $a < 1$ it exhibits self-sustained periodic firing. This is due to a supercritical Hopf bifurcation at $a = 1$ with a locally stable fixed point for $a > 1$ and a stable limit cycle for $a < 1$.

3 Synchronization of Two FitzHugh-Nagumo Systems

Let state the problem of variable value synchronization in two coupled FHN systems. We subtract the third equation from the first one, and the fourth one from the second one (1) making the following substitution

$$\delta_1 = u_1 - u_2, \qquad \delta_2 = v_1 - v_2, \tag{2}$$

and get

$$\begin{aligned} \varepsilon \dot{\delta}_1(t) &= (1 - C - \phi(t))\delta_1(t) \\ &\quad - C\delta_1(t - \tau(t)) - \delta_2(t) + I(t), \\ \dot{\delta}_2(t) &= \delta_1(t), \end{aligned} \tag{3}$$

$\phi = 1/3(u_1^2 + u_1 u_2 + u_2^2)$, $\phi(t) \geqslant 0\ \forall t$ is nonnegative function. Then the control goal can be described as follows

$$\delta_1(t) \to 0, \quad \delta_2(t) \to c, \quad \text{while } t \to \infty, \tag{4}$$

where c is some constant.

To ensure the control goal (4) let choose the control $I(t)$ in form

$$I(t) = -\theta_1 \delta_1(t) + \theta_2 \delta_1(t - \tau), \tag{5}$$

where $\theta_1 \geqslant 0$, θ_2 are control parameters. Substitute the chosen control to the system (3) and get the equation of closed-loop system

$$\begin{aligned} \varepsilon \dot{\delta}_1(t) &= (1 - C - \phi(t) - \theta_1)\delta_1(t) \\ &\quad + (\theta_2 - C)\delta_1(t - \tau(t)) - \delta_2(t), \\ \dot{\delta}_2(t) &= \delta_1(t), \end{aligned} \tag{6}$$

For some control parameters θ_1, θ_2 the control goal (4) can be achieved. To find these appropriate parameters introduce the following Lyapunov-Krasovskii functional

$$V(t, \mathbf{\Delta}(t)) = \varepsilon \delta_1^2 + \delta_2^2 + \theta_0 \int\limits_{t-\tau(t)}^{t} \delta_1^2(s) \mathrm{d}s, \tag{7}$$

where $\mathbf{\Delta} = (\delta_1, \delta_2)$, while $\theta_0 > 0$ is some positive parameter. Find its derivative according to the system (6)

$$\begin{aligned} \dot{V}(t, \mathbf{\Delta}(t)) &= 2(1 - C - \phi(t) - \theta_1 + 0.5\theta_0)\delta_1^2(t) \\ &\quad + 2(\theta_2 - C)\delta_1(t)\delta_1(t - \tau) \\ &\quad - \theta_0(1 - \dot{\tau})\delta_1^2(t - \tau). \end{aligned} \tag{8}$$

We should choose control parameters such that to make the Lyapunov function derivative be negative for all $\delta_1(t)$, $\delta_1(t - \tau)$ except zero, i.e., the following inequality should be fulfilled

$$\begin{aligned} &2(1 - C - \phi(t) - \theta_1 + 0.5\theta_0)\delta_1^2(t) + 2(\theta_2 - C) \\ &\quad \times \delta_1(t)\delta_1(t - \tau) - \theta_0(1 - \dot{\tau})\delta_1^2(t - \tau) < 0, \end{aligned} \tag{9}$$

that can be presented in form

$$\begin{bmatrix} \delta_1(t) & \delta_1(t - \tau) \end{bmatrix} W \begin{bmatrix} \delta_1(t) \\ \delta_1(t - \tau) \end{bmatrix} < 0, \tag{10}$$

where

$$W = \begin{bmatrix} 2(1 - C - \phi - \theta_1) + \theta_0 & \theta_2 - C \\ \theta_2 - C & -\theta_0(1 - \dot{\tau}) \end{bmatrix} < 0. \tag{11}$$

Thus, we should make matrix W be negative-definite (11) by tuning the control parameters θ_0, θ_1, θ_2. We use Sylvester's criterion to determine whether the matrix W is negative-definite. Since $\theta > 0$ and $\dot{\tau} \leqslant d < 1$ then the lower principal minor of matrix W is negative. The matrix W is negative-definite if and only if its determinant is positive. Since $\phi(t) \geqslant 0$ and $\dot{\tau} \leqslant d < 1$, then control parameters must satisfy the inequality

$$-2(1-C-\theta_1+0.5\theta_0)\theta_0(1-d)-(\theta_2-C)^2 > 0, \quad (12)$$

that can be rewritten as

$$-(1-d)\theta_0^2+2(1-d)(\theta_1+C-1)\theta_0-(\theta_2-C)^2 > 0. \quad (13)$$

This inequality is fulfilled for some positive θ_0, when the following quadratic equation for θ_0 has real roots

$$-(1-d)\theta_0^2+2(1-d)(\theta_1+C-1)\theta_0-(\theta_2-C)^2 = 0, \quad (14)$$

and the following inequality is fulfilled by Vieta's formulas

$$\theta_1 + C - 1 > 0. \quad (15)$$

Thus, the discriminant of the equation (14) must be positive

$$4(1-d)^2(\theta_1+C-1)^2-4(1-d)(\theta_2-C)^2 > 0, \quad (16)$$

that, considering inequality (15), can be presented as

$$\theta_1 > \frac{|\theta_2 - C|}{\sqrt{1-d}} - C + 1. \quad (17)$$

If the control parameters θ_1, θ_2 satisfy the obtained inequality (17), then Lyapunov function derivative (8) is nonpositive. When $\delta_1(t) = 0$ and $\delta_1(t-\tau) = 0$, Lyapunov function derivative (8) equals zero and $\dot{\delta}_2(t) = 0$ from the second equation of the system (6). This means the achievement of control goal (4).

Thus, the following theorem takes place

Theorem 1. *Let the delay τ be slowly-varying differential function in the plant* (1)*, i.e., $\dot{\tau} \leqslant d < 1$. Then the control $I(t)$ in form* (5)*, where parameters $\theta_1 \geqslant 0$ and θ_2 satisfy the inequality* (17)*, ensures the control goal* (4)*, meaning the substitutions* (2).

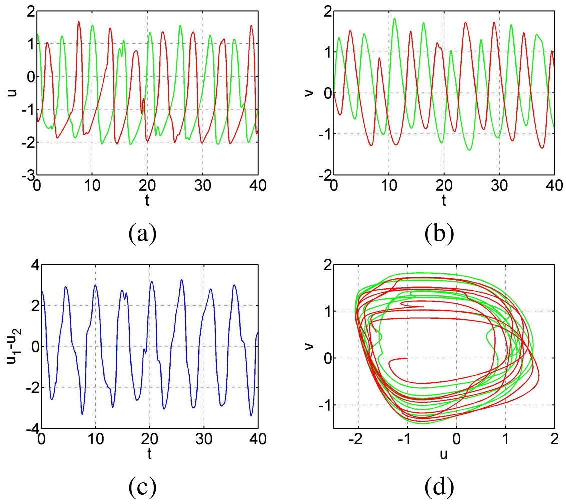

Figure 1. Dynamics of two coupled FitzHugh-Nagumo systems according to Eq. (1) without control. Green line marks system one, while red line marks system two. (a) and (b): time series of the membrane potential and the recovery variable, respectively; (c): time series of synchronization errors of the membrane potentials; and (d): phase space. The system parameters: $a = 0.7$, $C = 1$, $\varepsilon = 0.1$, $\tau(t) = 3 + 1/2\cos(t)$. The initial conditions: $u_1(t) = \cos(t)$, $u_2(t) = -\cos(t)$, $v_1(t) = \sin(t)$, $v_2(t) = -\sin(t)$ for $t \in [-\tau, 0]$.

4 Simulation

The simulation was carried out in Matlab R2009b.

Firstly, we consider the case of the system (1) behavior without control. The system parameters: $a = 0.7$, $C = 1$, $\varepsilon = 0.1$, $\tau(t) = 3 + 1/2\cos(t)$. The initial conditions: $u_1(t) = \cos(t)$, $u_2(t) = -\cos(t)$, $v_1(t) = \sin(t)$, $v_2(t) = -\sin(t)$ for $t \in [-\tau, 0]$. The two coupled FHN systems do not synchronize: Figure 1 shows in (a) and (b) the time series of the membrane potentials and the recovery variables, respectively, in (c) the synchronization errors of the membrane potentials, in (d) the phase portrait.

Now we use the control according to Eq. (5) with $\theta_1 = 5$, $\theta_2 = 1$ in order to synchronize the two systems. Figure 2 shows the results. After a transient time of approximately 20 units of time the two systems reach the desired synchronized state (see the time series of the membrane potential and the recovery variable in Fig. 2(a), (b) and their synchronization errors in Fig. 2(c), (d).) Thus, the control is successful. Note that the control I is bounded and tends to zero while $t \to \infty$ (see Fig. 2(f)).

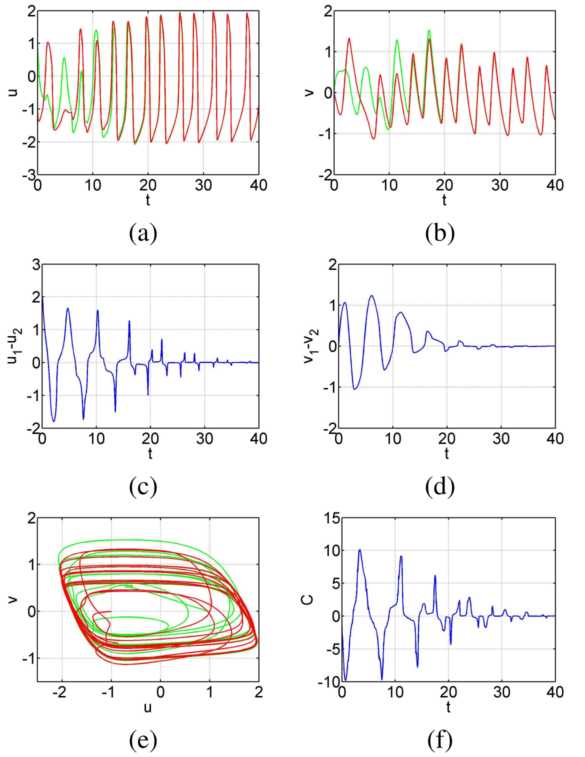

Figure 2. Control of synchronization of two coupled FitzHugh-Nagumo systems (Eq. (1)) with control algorithm in form (5). (a) and (b): time series of the membrane potential and the recovery variable, respectively; (c) and (d): time series of synchronization errors of the membrane potential and the recovery variable, respectively; (e): phase space; and (f): time series of the external stimulus adapted according to Eq. (5). $\theta_1 = 5, \theta_2 = 1$. Other parameters and initial conditions as in Fig 1.

5 Conclusion

We have proposed the method for controlling synchrony in two delay-coupled FitzHugh-Nagumo systems with slowly-varying delays, a neural model which is considered to be generic for excitable systems close to a Hopf bifurcation. We have posed the synchronization problem and introduced the Lyapunov-Krasovskii functional to find an appropriate control parameters and prove the synchronization problem. Based on this function we have derived a controller, which makes the synchrony stable despite the time-varying delay.

We have found the applicability conditions of proposed method and have formulated the theorem of control goal achievement. The simulation has shown that this method ensures the control goal. Given the paradigmatic nature of the FitzHugh-Nagumo system we expect our method to be extended to the case with several nodes of the system and be applicable in a wide range of neural system models.

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