EFFECT OF LINEARLY VARYING NORMAL FORCE UPON THE NONLINEAR MODAL ANALYSIS OF SLENDER BEAMS

Carlos E.N. Mazzilli
Department of Structural and Geotechnical Engineering
University of São Paulo
Brazil
cenmazzi@usp.br

Abstract
An improved solution to the nonlinear normal modes of a slender beam, subjected to a linearly varying normal force and end thrust, is determined, by means of an extended method of multiple time and space scales; The nonlinear free vibrations of a straight offshore riser is considered as an example. The vibration frequency of a nonlinear mode is not constant along the riser: it is seen to depend on the cross-section depth and on the vibration amplitude, thus explaining the onset of longitudinal travelling waves.

Key words
Nonlinear normal modes, nonlinear multi-modes, slender beams, axially loaded beams, vertical risers.

1 Introduction

Reliable structural performance evaluation under environmental and equipment borne loads are becoming ever more important with greater demands on the design of modern engineering systems. The linearisation assumption for light and flexible structures is questionable and whenever possible nonlinearities should be accounted for, thus leading to the use of nonlinear normal modes, whenever vibration is a concern.

Pak and Rosenberg [1968] were amongst the first to propose the extension from linear to nonlinear normal modes. Shaw and Pierre [1991] offered a rigorous and general definition for nonlinear modes and devised a technique for their evaluation in both discrete [1993] and continuous [1994] systems where modal motion is limited to a surface in the system's phase space, the invariant manifold. Their definition of nonlinear multimodes encompasses invariant manifolds of systems with internal resonance [Lacarbonara, Rega and Nayfeh, 2003; Lacarbonara and Rega, 2003; Srinil, Rega and Chucheepsakul, 2007; Srinil and Rega, 2007a; Srinil and Rega, 2007b].

Mazzilli and co-workers implemented nonlinear normal modes in the finite element analysis of beam-like structures oscillating in a purely single mode [Baracho Neto and Mazzilli, 2002; Mazzilli *et al*, 2008; Soares and Mazzilli, 2000] as well as those with internal resonance [Baracho Neto and Mazzilli, 2005; Baracho Neto and Mazzilli, 2008; Mazzilli *et al*, 2008]. Other relevant references to the literature on nonlinear modal analysis should also be acknowledged [Gendelman *et al*, 2003; King and Vakakis, 1996; Manevitch, 2001; Mikhlin, 1995; Mikhlin and Zhupiev, 1997; Mikhlin and Morgunov, 2001; Vakakis and Rand, 1992a; Vakakis and Rand, 1992b].

In a previous investigation [Mazzilli *et al*, 2008], the author studied the nonlinear normal modes of a beam subjected to an end thrust and uniformly distributed axial load, under the assumption that the primary dynamic behaviour would be determined by the average normal force along the beam length. Comparison with finite-element results showed that, in spite of overall qualitative agreement, the actual normal force variation along the beam length should be taken into account for improved quantitative results, which is the purpose of this paper.

Classical simplifying assumptions, such as neglecting longitudinal inertial forces and averaging geometric stiffness effects along the beam, as already proposed by a number of authors — see Kauderer [1958] and Singh, Sharma and Rao [1990] — lead to an equation of motion for the transversal displacement, which is de-coupled from the longitudinal one.

The method of multiple scales is used to determine the nonlinear normal modes, considering both bending and geometric stiffness effects.

Nonlinear modes are expected to be a useful tool when modelling the dynamics of vertical offshore risers, since they may provide efficient projecting functions for number of degrees-of-freedom reduction — see Shaw, Pierre and Pesheck [1999] and Mazzilli, Soares and Baracho Neto [2001] —, thus allowing for a smaller computational effort to analyse fluid-dynamic instabilities, such as those caused by vortex induced vibrations (VIV's).

The paper is organised as follows. In Section 2, the nonlinear equations of an axially loaded Bernoulli-Euler beam are derived. In Section 3, nonlinear normal modes are developed, by using a combination of a perturbation method and the invariant manifold approach. The developed methodology is applied in Section 4 to the case study of a pinned-pinned riser beam.

2 Nonlinear equations of motion of an axially-loaded beam

In this section the nonlinear equations of motion of an axially loaded beam are presented, as they appear following a Hamiltonian procedure — see Pars [1965] and Meirovitch [1970].

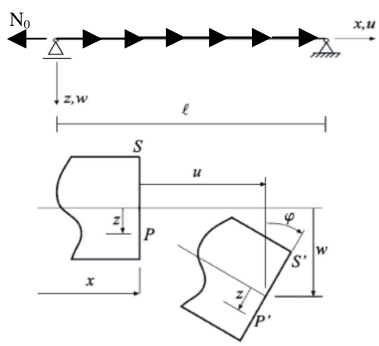

Figure 1. Schematic of an axially loaded beam with uniformly distributed load p.

Figure 1 introduces the basic notation and the kinematics of the Bernoulli-Euler beam model. Here, m and p are the mass and the axial load per unit length, and EA and EI are the axial and flexural rigidity.

The Bernoulli-Euler kinematical assumption leads to the following expressions for the displacements of a generic point P:

$$\begin{aligned} u_p &= u - z\sin\varphi \cong u - zw' \\ w_p &= w + z(\cos\varphi - 1) \cong w \\ \varphi &= \arctan\left(\frac{w'}{1+u'}\right) \cong w'. \end{aligned} \tag{1}$$

For small strains, both the Lagrangian and the engineering strain are identical for practical purposes. The strain at a generic point P of the riser along the longitudinal direction is:

$$\begin{aligned} \varepsilon_P &= u'_P + \frac{1}{2}(u'_P)^2 + \frac{1}{2}(w'_P)^2 \\ &\cong u' - zw'' + \frac{1}{2}(w')^2 = \varepsilon - zw'' \end{aligned} \tag{2}$$

where ε is the strain at the cross-section centroid:

$$\varepsilon = u' + \frac{1}{2}(w')^2. \tag{3}$$

The assumption $u_p = O(w_P^2)$ is implicit in the approximation introduced in (2). Equation (4) for the transversal motion, decoupled from the longitudinal motion, has been derived using Hamilton's Principle and can be followed in Mazzilli *et al* [2008].

$$m\ddot{w} + EIw^{iv} - N(x,t)w'' + pw' = 0 \tag{4}$$

where

$$\begin{aligned} N(x,t) &= N_0(t) - px = \\ &\overline{N} + p\left(\frac{\ell}{2} - x\right) + \frac{EA}{2\ell}\int_0^\ell w'^2 dx \\ \overline{N} &= -\frac{EAu_0}{\ell} \end{aligned} \tag{5}$$

It is assumed that the beam is fixed at $x = \ell$, that is, $u_\ell = 0$, and that an axial force $N_0(0)$ is applied to the originally rectilinear beam at $x = 0$ in time $t = 0$ together with the rightward-distributed axial load p, thus giving origin to a certain leftward axial displacement $-u_0$, before the onset of transversal vibration. The left end is then fixed, that is, $u_0(t) = u_0 = const$, so that once the transversal oscillation takes place, the axial force at the left end varies with time (as a matter of fact, $N_0(t) \geq N_0(0)$, due to the axial strain increase caused by bending):

$$\begin{aligned} N_0(t) &= N_0(0) + \frac{EA}{2\ell}\int_0^\ell w'^2 dx \\ N_0(0) &= -\frac{EAu_0}{\ell} + \frac{p\ell}{2} \end{aligned} \tag{6}$$

Once w is determined from (4), the axial displacements can be obtained from Mazzilli *et al* [2008] as:

$$\begin{aligned} u(x,t) = &-\frac{\overline{N}(\ell - x)}{EA} + \frac{px(\ell - x)}{2EA} \\ &-\frac{1}{2}\int_0^x \left(\frac{dw}{d\xi}\right)^2 d\xi + \frac{x}{2\ell}\int_0^\ell w'^2 dx. \end{aligned} \tag{7}$$

Therefore, (4) is conveniently rewritten as:

$$\ddot{w} + \alpha w^{IV} - \beta w'' - \gamma\left(\frac{\ell}{2} - x\right)w'' + \gamma w'$$
$$- \mu w'' \int_0^\ell w'^2 dx = 0; \quad (8)$$
$$\alpha = \frac{EI}{m}; \quad \beta = \frac{\overline{N}}{m}; \quad \gamma = \frac{p}{m}; \quad \mu = \frac{EA}{2m\ell}.$$

Equation (8) can be put in a non-dimensional form, after the following variable transformation is introduced:

$$v = \frac{w}{\ell}; \quad z = \frac{x}{\ell}; \quad \tau = \omega_1 t;$$
$$\omega_1 = \frac{\pi}{\ell}\sqrt{\beta + \frac{\pi^2}{\ell^2}\alpha} \quad (9)$$

In (9), ω_1 stands for the linear-theory first-mode frequency of a beam subjected to a constant normal force of value $\overline{N}$. Thus:

$$\frac{\partial^2 v}{\partial \tau^2} + \hat{\alpha}\frac{\partial^4 v}{\partial z^4} - \hat{\beta}\frac{\partial^2 v}{\partial z^2} - \varepsilon^2\hat{\gamma}\left(\frac{1}{2} - z\right)\frac{\partial^2 v}{\partial z^2}$$
$$+ \varepsilon^2\hat{\gamma}\frac{\partial v}{\partial z} - \hat{\mu}\frac{\partial^2 v}{\partial z^2}\int_0^1\left(\frac{\partial v}{\partial z}\right)^2 dz = 0; \quad (10)$$
$$\hat{\alpha} = \frac{\alpha}{\omega_1^2\ell^4}; \quad \hat{\beta} = \frac{\beta}{\omega_1^2\ell^2};$$
$$\varepsilon^2\hat{\gamma} = \frac{\gamma}{\omega_1^2\ell}; \quad \hat{\mu} = \frac{\mu}{\omega_1^2\ell}.$$

Note that axial loading is responsible for a non-constant coefficient for the term $\partial^2 v / \partial z^2$.

Extending the method of multiple scales — see Nayfeh and Mook [1979] and Nayfeh and Nayfeh [1994] —, the solution will be sought in the form of an asymptotic expansion in terms of time and space scales, as well:

$$v(z,\tau) = \varepsilon v_1(z_0, z_1, ..., \tau_0, \tau_1 ...)$$
$$+ \varepsilon^2 v_2(z_0, z_1, ..., \tau_0, \tau_1 ...) + ... \quad (11)$$

$$\tau_j = \varepsilon^j \tau; \quad z_j = \varepsilon^j z; \quad 0 < \varepsilon < 1$$

where ε is a book-keeping parameter.

The following differential operators and relationships are introduced:

$$D_j^q = \frac{\partial^q}{\partial \tau_j^q};$$
$$\frac{\partial}{\partial \tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + ...;$$
$$\frac{\partial^2}{\partial \tau^2} = D_0^2 + \varepsilon\, 2 D_0 D_1 + \varepsilon^2\left(D_1^2 + 2 D_0 D_2\right) + ...;$$
$$\Delta_j^q = \frac{\partial^q}{\partial z_j^q}; \quad (12)$$
$$\frac{\partial}{\partial z} = \Delta_0 + \varepsilon\Delta_1 + \varepsilon^2\Delta_2 + ...;$$
$$\frac{\partial^2}{\partial z^2} = \Delta_0^2 + \varepsilon\, 2\Delta_0\Delta_1 + \varepsilon^2\left(\Delta_1^2 + 2\Delta_0\Delta_2\right) + ...;$$
$$\frac{\partial^4}{\partial z^4} = \Delta_0^4 + \varepsilon\, 4\Delta_0^3\Delta_1 + \varepsilon^2\left(6\Delta_0^2\Delta_1^2 + 4\Delta_0^3\Delta_2\right) + ...$$

Substituting (11)-(12) into (10) and collecting terms of the same order of ε, it is possible to arrive at differential equations whose solutions and solvability conditions allow for the characterization of the nonlinear normal modes.

2.1 Order ε solution

The equation of order ε leads to:

$$D_0^2 v_1 + \hat{\alpha}\Delta_0^4 v_1 - \hat{\beta}\Delta_0^2 v_1 = 0 \quad (13)$$

where v_1 must satisfy the boundary conditions $v_1(0,\tau) = v_1(\ell,\tau) = \Delta_0^2 v_1(0,\tau) = \Delta_0^2 v_1(\ell,\tau) = 0$. Thus, the solution v_1 can be written in the form:

$$v_1(z_0, z_1, ..., \tau_0, \tau_1 ...) =$$
$$\sum_k v_{1k}(z_0, z_1, ..., \tau_0, \tau_1 ...),$$

$$v_{1k}(z_0, z_1, ..., \tau_0, \tau_1 ...) = \quad (14)$$
$$r_{1k}(z_0, z_1, ..., \tau_1, \tau_2 ...) sin\,\hat{\omega}_k\tau_0 +$$
$$s_{1k}(z_0, z_1, ..., \tau_1, \tau_2 ...)\cos\hat{\omega}_k\tau_0.$$

Substituting (14) into (13), one obtains:

$$r_{1k} = R_{1k} sin k\pi z_0;$$
$$s_{1k} = S_{1k} sin k\pi z_0; \quad (15)$$
$$\hat{\omega}_k = k\pi\sqrt{\hat{\beta} + k^2\pi^2\hat{\alpha}};$$

where $\hat{\omega}_k$ is the normalised frequency of linear-mode k with respect to ω_1, taking into account both the beam bending and geometric stiffness. For long risers, the geometric stiffness prevails by large over the bending stiffness. Therefore, an almost linear relationship between $\hat{\omega}_k$ and the mode number k takes place, which favours internal resonance and a strong modal coupling, Mazzilli *et al* [2008], which would require consideration of nonlinear multimodes — see Baracho Neto and Mazzilli [2002, 2005, 2008]. Yet, this situation is beyond the scope of the present paper.

3 Nonlinear normal modes

A nonlinear normal mode is a free-vibration motion of a nonlinear system about its static equilibrium configuration, which takes place on a two-dimensional invariant manifold embedded in the phase space, so that it is tangent at the equilibrium point to the corresponding linear system eigenplane — see Shaw and Pierre [1991, 1993]. Hence, once the initial conditions have set a motion on this manifold, it will stay there. The nonlinear normal mode turns out to be the solution (14) for a certain mode *k*. It is pursued following the steps of the method of multiple scales. Therefore, from this point on, the summation implied in the solution (14) will not apply, that is, only the terms referring to the particular mode *k* under analysis will be considered.

3.1 Order ε^2 solution

The equation of order ε^2 leads to:

$$\begin{aligned}
&D_0^2 v_{2k} + \hat{\alpha}\Delta_0^4 v_{2k} - \hat{\beta}\Delta_0^2 v_{2k} = \\
&-2D_0 D_1 v_{1k} - 4\hat{\alpha}\Delta_0^3\Delta_1 v_{1k} - 2\hat{\beta}\Delta_0\Delta_1 v_{1k} \\
&= \begin{bmatrix} -2\hat{\omega}_k D_1 R_{1k}\, sink\pi z_0 \\ -2k\pi\left(\hat{\beta} - 2k^2\pi^2\hat{\alpha}\right)\Delta_1 S_{1k}\cos k\pi z_0 \end{bmatrix}\cos\hat{\omega}_k\tau_0 \\
&+ \begin{bmatrix} 2\hat{\omega}_k D_1 S_{1k}\, sink\pi z_0 \\ -2k\pi\left(\hat{\beta} - 2k^2\pi^2\hat{\alpha}\right)\Delta_1 R_{1k}\cos k\pi z_0 \end{bmatrix} sin\,\hat{\omega}_k\tau_0
\end{aligned} \qquad (16)$$

From equation (16), the solvability condition, that is, the elimination of secular terms, requires that $D_1 R_{1k} = D_1 S_{1k} = \Delta_1 R_{1k} = \Delta_1 R_{1k} = 0$. Note that the solution v_{2k} is already included in v_{1k}, so that it will not be kept in the asymptotic expansions further on.

3.2 Order ε^3 solution

The equation of order ε^3 leads to:

$$\begin{aligned}
&D_0^2 v_{3k} + \hat{\alpha}\Delta_0^4 v_{3k} - \hat{\beta}\Delta_0^2 v_{3k} = \\
&-2D_0 D_2 v_{1k} - 4\hat{\alpha}\Delta_0^3\Delta_2 v_{1k} + \hat{\gamma}\left(\frac{1}{2} - z_0\right)\Delta_0^2 v_{1k} \\
&-\hat{\gamma}\Delta_0 v_{1k} + \hat{\mu}\Delta_0^2 v_{1k}\int_0^1 \left(\Delta_0 v_{1k}\right)^2 dz \\
&= \left\{ \begin{aligned} &\begin{bmatrix} -2\hat{\omega}_k D_2 R_{1k} + k^2\pi^2\hat{\gamma}S_{1k}\left(\frac{1}{2} - z_0\right) \\ -\frac{3}{8}k^4\pi^4\hat{\mu}S_{1k}\left(S_{1k}^2 + R_{1k}^2\right) \end{bmatrix} sink\pi z_0 \\ &+ k\pi\begin{bmatrix} 2\left(\hat{\beta} + 2k^2\pi^2\hat{\alpha}\right)\Delta_2 S_{1k} \\ -\hat{\gamma}S_{1k} \end{bmatrix}\cos k\pi z_0 \end{aligned} \right\}\cos\hat{\omega}_k\tau_0 \\
&+ \left\{ \begin{aligned} &\begin{bmatrix} 2\hat{\omega}_k D_2 S_{1k} + k^2\pi^2\hat{\gamma}R_{1k}\left(\frac{1}{2} - z_0\right) \\ -\frac{3}{8}k^4\pi^4\hat{\mu}R_{1k}\left(S_{1k}^2 + R_{1k}^2\right) \end{bmatrix} sink\pi z_0 \\ &+ k\pi\begin{bmatrix} 2\left(\hat{\beta} + 2k^2\pi^2\hat{\alpha}\right)\Delta_2 R_{1k} \\ -\hat{\gamma}R_{1k} \end{bmatrix}\cos k\pi z_0 \end{aligned} \right\} sin\,\hat{\omega}_k\tau_0 \\
&+ \frac{1}{8}k^4\pi^4\hat{\mu}S_{1k}\left(3R_{1k}^2 - S_{1k}^2\right) sink\pi z_0\cos 3\hat{\omega}_k\tau_0 \\
&- \frac{1}{8}k^4\pi^4\hat{\mu}R_{1k}\left(3S_{1k}^2 - R_{1k}^2\right) sink\pi z_0\, sin 3\hat{\omega}_k\tau_0
\end{aligned} \qquad (17)$$

Equation (17) requires the solvability conditions:

$$\begin{aligned}
&-2\hat{\omega}_k D_2 R_{1k} + k^2\pi^2\hat{\gamma}S_{1k}\left(\frac{1}{2} - z_0\right) \\
&-\frac{3}{8}k^4\pi^4\hat{\mu}S_{1k}\left(S_{1k}^2 + R_{1k}^2\right) = 0 \\
&2\hat{\omega}_k D_2 S_{1k} + k^2\pi^2\hat{\gamma}R_{1k}\left(\frac{1}{2} - z_0\right) \\
&-\frac{3}{8}k^4\pi^4\hat{\mu}R_{1k}\left(S_{1k}^2 + R_{1k}^2\right) = 0 \\
&2\left(\hat{\beta} + 2k^2\pi^2\hat{\alpha}\right)\Delta_2 S_{1k} - \hat{\gamma}S_{1k} = 0 \\
&2\left(\hat{\beta} + 2k^2\pi^2\hat{\alpha}\right)\Delta_2 R_{1k} - \hat{\gamma}R_{1k} = 0
\end{aligned} \qquad (18)$$

The last two of equations (18) yield:

$$\begin{aligned}
R_{1k} &= R_{0k}e^{\hat{\eta}_k z_k} \\
S_{1k} &= S_{0k}e^{\hat{\eta}_k z_k} \\
\hat{\eta}_k &= \frac{\hat{\gamma}}{2\left(\hat{\beta} + 2k^2\pi^2\hat{\alpha}\right)}
\end{aligned} \qquad (19)$$

From (19) and the first two of equations (18), it can be seen that:

$$R_{0k}\left(D_2 R_{0k}\right) + S_{0k}\left(D_2 S_{0k}\right) = 0 \qquad (20)$$

which, after integration gives:

$$R_{0k}^2 + S_{0k}^2 = A_k^2 = const \tag{21}$$

Let thus θ_k be such that:

$$\begin{aligned} sin\theta_k &= \frac{R_{0k}}{A_k} \\ \cos\theta_k &= \frac{S_{0k}}{A_k} \end{aligned} \tag{22}$$

which obviously satisfies (21). Taking (22) and (19) in (18), it can be seen that the following differential equation in $\theta_k(\tau_2)$ remains to be integrated:

$$D_2(sin\theta_k) = -\frac{k^2\pi^2}{2\hat{\omega}_k}\cos\theta_k\left[\hat{\gamma}\left(\frac{1}{2} - z_0\right) + \frac{3k^2\pi^2}{8}\hat{\mu}A_k^2 e^{2\eta_k z_2}\right] \tag{23}$$

or

$$D_2\theta_k = -\frac{k^2\pi^2}{2\hat{\omega}_k}\left[\hat{\gamma}\left(\frac{1}{2} - z_0\right) + \frac{3k^2\pi^2}{8}\hat{\mu}A_k^2 e^{2\eta_k z_2}\right] \tag{24}$$

and finally

$$\theta_k = \theta_{0k} - \frac{k^2\pi^2}{2\hat{\omega}_k}\left[\begin{matrix} \hat{\gamma}\left(\frac{1}{2} - z_0\right) \\ + \frac{3k^2\pi^2}{8}\hat{\mu}A_k^2 e^{2\eta_k z_2} \end{matrix}\right]\tau_2. \tag{25}$$

A particular solution of (17) can now be written in the form:

$$\begin{aligned} v_{3k} &= \left(R_{3k} sin3\hat{\omega}_k\tau_0 + S_{3k}\cos 3\hat{\omega}_k\tau_0\right) sink\pi z_0 \\ R_{3k} &= \frac{k^4\pi^4}{64\hat{\omega}_k^2}\hat{\mu}A_k^3 e^{3\eta_k z_2} sin3\theta_k \\ S_{3k} &= \frac{k^4\pi^4}{64\hat{\omega}_k^2}\hat{\mu}A_k^3 e^{3\eta_k z_2}\cos 3\theta_k \end{aligned} \tag{26}$$

The nonlinear normal mode *k*, up to terms of order ε^3 can now be written from (11), (14), (15), (19), (25) and (26):

$$\begin{aligned} v_k &= (\varepsilon A_k) e^{\varepsilon^2\hat{\eta}_k z}\left[\begin{matrix} \cos\left(\hat{\Omega}_k\tau - \theta_{0k}\right) + \\ \frac{k^4\pi^4}{64\hat{\omega}_k^2}\hat{\mu}(\varepsilon A_k)^2 e^{2\varepsilon^2\hat{\eta}_k z}\cos 3\left(\hat{\Omega}_k\tau - \theta_{0k}\right) \end{matrix}\right] sink\pi z \\ \hat{\Omega}_k &= \hat{\omega}_k\left\{1 + \frac{k^2\pi^2}{2\hat{\omega}_k^2}\left[\hat{\gamma}\left(\frac{1}{2} - z\right) + \frac{3k^2\pi^2}{8}\hat{\mu}(\varepsilon A_k)^2 e^{2\varepsilon^2\hat{\eta}_k z}\right]\right\} \end{aligned} \tag{27}$$

It is understood that these nonlinear normal modes are similar to the linear ones, as far as the transversal motion is concerned. In other words, the invariant manifold that characterises the nonlinear normal mode coincides with the corresponding linear-mode eigenplane, though a nonlinear oscillator rules the system dynamics — see Shaw and Pierre [1991, 1993] and Shaw, Pierre and Pesheck [1999].

There are other important distinctive features between the linear and nonlinear modes here. It is seen that the modal frequency strongly depends on z. The normal-force decreasing with z — term $\hat{\gamma}\left(\frac{1}{2} - z\right)$ — lowers the frequency, but the nonlinear vibration-amplitude effect — term in $\frac{3}{8}k^2\pi^2\hat{\mu}(\varepsilon A_k)^2 e^{2\varepsilon^2\hat{\eta}_k z}$ — acts in the opposite way. The prevailing effect for each cross-section position depends on the system parameters. In any case, it is clear that frequency variations should be expected along the beam, for the same mode *k*. This may explain the onset of travelling waves in the nonlinear response, as opposed to the standing waves of the classical linear modes.

4 Case study

The outcomes of section 3 are applied to the analysis of a vertical riser subjected to pre-stressing and immersed weight. In the case of an immersed riser, m should account for water or oil inside it, plus the surrounding water added mass.

The system parameters are: $E = 2.1\times10^{11} N/m^2$; $A = 1.10\times10^{-2} m^2$; $I = 4.72\times10^{-5} m^4$; $\ell = 1{,}800 m$; $m = 141.24 kg/m$; $p = 727 N/m$; $N_0 = 2\times10^6 N$.

Figures 2 and 3 display the displacement and velocity time histories for the first nonlinear normal mode at three different cross sections, and initial conditions corresponding to $\theta_{01} = 0$ and $(\varepsilon A_1)\exp(0.5\eta_1) = 0.025$.

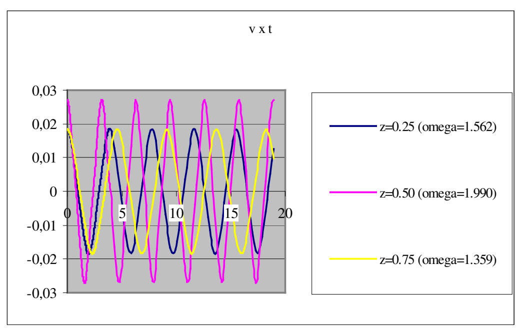

Figure 2. Displacement time-history for the first nonlinear normal mode at different cross sections.

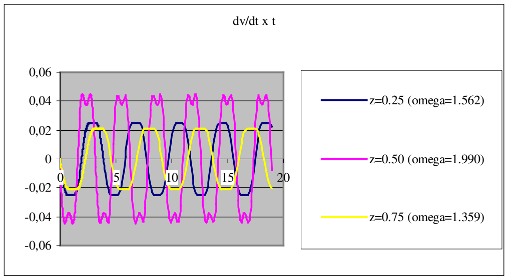

Figure 3. Velocity time-history for the first nonlinear normal mode at different cross sections.

Figure 4 displays the phase portraits corresponding to the same initial conditions and cross sections of Figures 2 and 3.

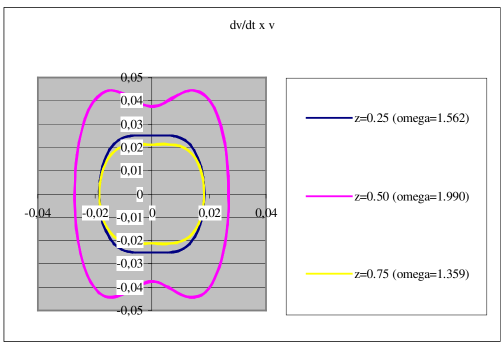

Figure 4. Phase portraits for three different cross sections.

As already pointed out, due to the frequency variation with z, the cyclic motion of distinct cross sections will last different periods and will induce travelling waves along the longitudinal direction, the effect of it can be seen in the motion projection upon the plane $v(\ell/2)\times v(\ell/6)$ of Figure 5, which shows how the first nonlinear normal mode correlates the first and the third linear modes. Here the initial conditions correspond to $(\varepsilon A_1)\exp(0.5\eta_1)=0.00\overline{5}$ and $\theta_{01}=0$.

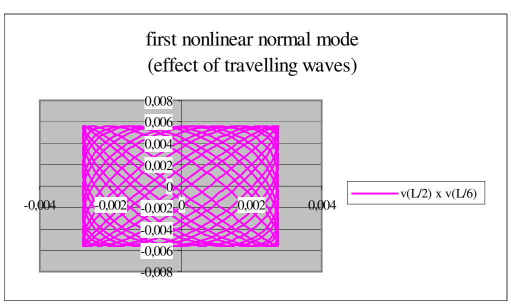

Figure 5. Phase portrait: nonlinear relationship between displacements at z=0.5 and z=0.833

5 Conclusions

The results presented in this paper at the same time that improve the findings of the analytical solution presented in Mazzilli *et al* [2008], where the average normal force was used to characterise the natural frequencies and nonlinear response, detect features that are in qualitative agreement with previous observations using finite-element models [Mazzilli *et al* , 2008], such as the travelling-wave effect along the beam, which is most probably related to the modal velocity influence onto the equation of the nonlinear modal oscillators. Extension of the approach used here to consider nonlinear multimodes should be done next.

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