ENERGY EXCHANGE IN WEAKLY COUPLED FPU CHAINS

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Abstract
We consider two limiting cases in the problem of energy exchange between two weakly Fermi-Pasta-Ulama (FPU) chains. The case of weak coupling demonstrates full absence of competition between the processes if intra-chain excitation and interchain energy transfer because these ones are separated in the time scale. Another limit of "superweak" bond is more complicated since the processes mentioned above proceed in the same time scale. Both limits are studied by asymptotic method. It is shown that the main peculiarities of the processes can be understood due to close analogy with the system of two coupled nonlinear oscillators. The conditions which determine the transition from full energy exchange to confinement of excitation on one of chains are discussed. Main conclusions of analytical study are in a good agreement with computer simulation data.

Key words
Nonlinear chain, breather, energy exchange, limiting phase trajectory.

1 Introduction
The coupled oscillatory chains is a subject of growing interest both in mechanics and physics [1-6]. Besides of numerous applications the reason of such interest is the richness of dynamical properties, especially of those manifesting in the beating phenomenon.
Unfortunately, the complexity of the problem does not allow solving it exactly as it is possible in the case of coupled nonlinear oscillators [2]. Therefore we distinguish two limiting regimes: the case of dominating coupling (over nonlinearity) – the weak coupling limit, and the case when the coupling and nonlinearity have the same order – limit of superweak coupling. In the first case the local approach turns out to be possible, while in the second one there is necessity in the consideration of integral quantities, that allows reduction to the system of two coupled nonlinear oscillators. The conditions of transition from intensive interchain energy exchange to energy localization on one of chains are formulated.

2 The model
We consider two nonlinear oscillatory chains with a weak harmonic interaction. The equations of motion in the short-wave continuum limit are [6]:

$$\frac{\partial^2 u_j}{\partial \tau^2} + \frac{\partial^2 u_j}{\partial x^2} + u_j + 16\beta u_j{}^3 - \varepsilon\gamma u_{3-j} = 0 \qquad (1)$$
$$\tau = \omega t,\ \omega^2 = 4 + \varepsilon\gamma$$

where u_j is a modulation of the displacement of j-th chain (j=1,2), β - parameter of nonlinearity and γ - coupling constant. Small parameter ε characterizes a weakness of inter-chain coupling, τ is the normalized time variable, x – dimensionless space variable, ω - eigenfrequency. It is useful for further analysis to introduce the complex variables:

$$\Psi_j = \frac{1}{\sqrt{2}}\left(\frac{\partial u_j}{\partial \tau} + iu_j\right), \quad \overline{\Psi}_j = \frac{1}{\sqrt{2}}\left(\frac{\partial u_j}{\partial \tau} - iu_j\right) \qquad (2)$$

The line over the symbol means a complex conjugation. So, the starting point of our analysis will be Eq. (3) for complex amplitudes Ψ_j:

$$i\frac{\partial}{\partial \tau}\Psi_j + \Psi_j + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\Psi_j - \overline{\Psi}_j) - 4\beta(\Psi_j - \overline{\Psi}_j)^3 - \varepsilon\frac{\gamma}{2}(\Psi_{3-j} - \overline{\Psi}_{3-j}) = 0 \qquad (3)$$

3 Weak coupling limit
Let us consider the case of small amplitude oscillation when complex amplitude |Ψ|~ε. It means that coupling forces in Eqs. (3) are stronger than nonlinear ones. Now we can construct expansions of Ψ_j by parameter ε<<1:

$$\Psi_j = \varepsilon(\psi_j + \varepsilon\psi_{j,1} + \varepsilon^2\psi_{j,2} + ...) \qquad (4)$$

and define, alongside with "fast" time, the "slow" time and space variables

$$\tau_0 = \tau, \quad \tau_1 = \varepsilon\tau, \quad \tau_2 = \varepsilon^2\tau, \quad \xi = \varepsilon x \qquad (5)$$

After substitution of expressions (4-5) into Eq.(3), we get equations of main approximation for various orders of ε:

$$\varepsilon^1 \quad i\partial_{\tau_0}\psi_j + \psi_j = 0, \quad \psi_j = \chi_j e^{i\tau_0} \qquad (6)$$

$$\varepsilon^2 \quad i\partial_{\tau_0}\psi_{j,1} + i\partial_{\tau_1}\psi_j + \psi_j - \frac{\gamma}{2}(\psi_{3-j} - \overline{\psi}_{3-j}) = 0$$
$$\psi_{j,1} = \chi_{j,1}e^{i\tau_0} \qquad (7)$$
$$i\partial_{\tau_0}\chi_{j,1} + i\partial_{\tau_1}\chi_j - \frac{\gamma}{2}(\chi_{3-j} - \overline{\chi}_{3-j}e^{-2i\tau_0}) = 0$$

The lasts of Eqs. (7) lead to following relations between main amplitudes χ_j and their first corrections $\chi_{j,1}$:

$$\chi_{j,1} = \frac{\gamma}{4}\bar{\chi}_{3-j}e^{-2i\tau_0}\,. \tag{8}$$

The equations defining the dynamics in the time scale τ_1 can be written as follows:

$$i\partial_{\tau_1}\chi_j - \frac{\gamma}{2}\chi_{3-j} = 0\,. \tag{9}$$

Let us note that Eqs. (9) are essentially local ones. We can write the solution of Eqs. (9) in the form:

$$\chi_1 = \frac{1}{\sqrt{2}}[X_1\cos(\frac{\gamma}{2}\tau_1) - iX_2\sin(\frac{\gamma}{2}\tau_1)], \qquad \chi_2 = \frac{1}{\sqrt{2}}[X_2\cos(\frac{\gamma}{2}\tau_1) - iX_1\sin(\frac{\gamma}{2}\tau_1)] \tag{10}$$

where the functions X_1, X_2 depend on the "slow" time and space variables τ_2, ξ.

In the next order by small parameter ε we get resulting equations for amplitudes χ_j, taking into account the relation (8):

$$i\partial_{\tau_2}\chi_j + \frac{1}{2}\partial_\xi^{\,2}\chi_j - \frac{\gamma^2}{8}\chi_j + 12\beta|\chi_j|^2\chi_j = 0 \tag{11}$$

First of all, we can see, that Eqs.(11) are decoupling ones. But the unknown functions χ_j depend on "fast" time τ_1. To overcome this problem we have to integrate them with respect to the "fast" time τ_1 over the period $T_1=4\pi/\gamma$. Then we get resulting equations for main approximation:

$$i\partial_{\tau_2}X_j + \frac{1}{2}\partial_\xi^{\,2}X_j - \frac{\gamma^2}{8}X_j + \frac{3\beta}{8}(3|X_j|^2X_j + 2|X_{3-j}|^2X_j - X_{3-j}^{\,2}\bar{X}_j) = 0 \tag{12}$$

where the functions X_j are defined by Eqs.(10). It is splendid point, that Eqs. (12) allow the solutions, concentrated on one of the chains only. Taking into account Eqs. (10), we get a full transition of initial excitation (*e.g.*, X_1) from "parent" chain to another one and backwards. It is worth to note, that such a result is valid for localized soliton-like excitations (breathers) as well as for anharmonic plane waves. This conclusion is in a good agreement with computer simulation data, some examples being shown in Figs. 1,2.

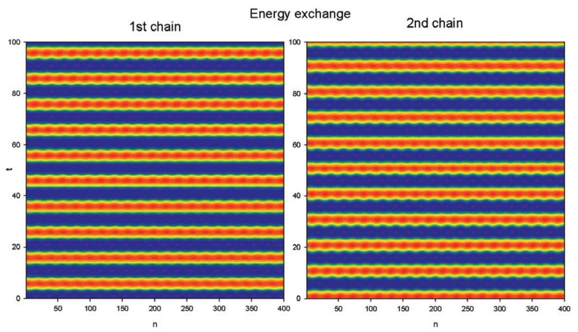

Figure 1. "Energy map" of small amplitude anharmonic waves in the system of two coupled chains. Initial conditions correspond to wave, located on the second chain. Light (red) stripes correspond to large energy and dark (blue) stripes – to small one.

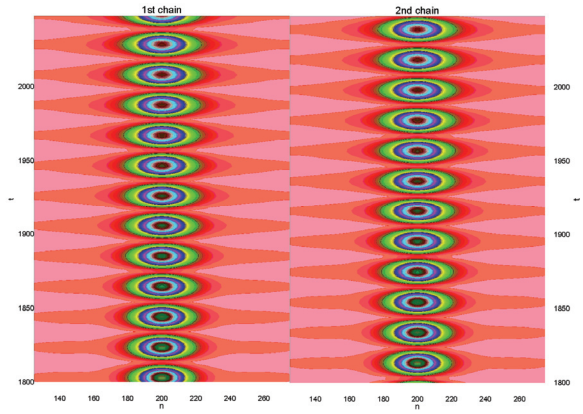

Figure 2. Energy map of standing breather. The fragment of computer simulation. The bright circles correspond to breather location.

4 Superweak coupling

Now we consider Eqs. (3) under conditions, when the coupling and nonlinearity forces have the same order by parameter ε. In such a case we have to assume that amplitude $\Psi \sim \varepsilon^{1/2}$. The appropriate expansions of Ψ_j can be written as

$$\Psi_j = \varepsilon^{1/2}(\psi_j + \varepsilon\psi_{j,1} + \varepsilon^2\psi_{j,2} + ...) \tag{13}$$

and slow time and space variables have the following view:

$$\tau_0 = \tau, \quad \tau_1 = \varepsilon\tau, \quad \tau_2 = \varepsilon^2\tau, \quad \xi = \sqrt{\varepsilon}x \tag{14}$$

Substituting expansions (13) to Eqs. (3), after some standard manipulations we get resulting equations for main approximation:

$$i\partial_{\tau_1}\chi_j + \frac{1}{2}\partial_\xi^{\,2}\chi_j - \frac{\gamma}{2}\chi_{3-j} + 12\beta|\chi_j|^2\chi_j = 0\,. \tag{15}$$

Contrary to previous case of weak coupling (see Eqs.(11)), the equations obtained are bound up. It leads to competition between energy exchange and formation of excitation (localized or not). Eqs.(15) have two symmetric solutions:

$$\chi_1 = \chi_2,\ \chi_1 = -\chi_2\,. \tag{16}$$

First of them corresponds to in-phase mode, and second one – anti-phase mode. It is clear that there is no solution localized on the one chain only.

Let us consider the solutions of Eqs.(15) in the form:

$$\chi_j(\xi,\tau_1) = A_j(\xi - v\tau_1)\exp(i(\omega\tau_1 - q\xi)) \tag{17}$$

The assumption $A_j=a_j=$const leads to anharmonic "dispersion relations":

$$-(2\omega + q^2)a_j - \gamma a_{3-j} + 24\beta a_j^3 = 0\,, \tag{18}$$

that define the dependence between the frequency of plane waves ω and their amplitudes a_j. The amplitudes corresponding to stationary points of the system (when ω and q are fixed) are the solutions of Eqs. (18).

To analyze a temporary evolution of plane wave we assume that the amplitudes A_j are the functions of time τ_1 only: $A_j=A_j(\tau_1)$. In such a case Eqs. (15) take the form:

$$i\frac{dA_j}{d\tau_1}-(\omega+\frac{q^2}{2})A_j-\frac{\gamma}{2}A_{3-j}+ \quad (19)$$
$$12\beta|A_j|^2A_j=0 .$$

These equations are fully analogous to those of two nonlinear oscillators considered in [2] in detail.

Eqs. (19) have two first integrals:

$$H=-\frac{\gamma}{2}(A_1\bar{A}_2+\bar{A}_1A_2)- \quad (20)$$
$$(\omega+\frac{q^2}{2})(|A_1|^2+|A_2|^2)+$$
$$6\beta(|A_1|^4+|A_2|^4)$$

and

$$N=|A_1|^2+|A_2|^2 . \quad (21)$$

The constancy of "occupation number" N (the integral (21)) allows to write amplitudes A_j in the form:

$$A_1=\sqrt{N}\cos\theta e^{i\delta_1} \quad (22)$$
$$A_2=\sqrt{N}\sin\theta e^{i\delta_2} .$$

"Angle" variables (θ, δ) are useful to analyze the phase plane of the system [7]. The parameter, controlling the structure of phase plane of the system is $\kappa=6\beta N/\gamma$. Up to κ=0.5, Eqs. (19) have only two symmetric stationary points (16), that correspond to in-phase ($A_1=A_2$ or $\theta=\pi/4$; $\Delta=\delta_1-\delta_2=0$) mode and anti-phase mode ($A_1=-A_2$ or $\theta=\pi/4$; $\Delta=\pi$). Attractive area of each of stationary points are circled by limiting phase trajectories (LPTs). Anti-phase mode becomes instable one at κ=0.5, that leads to creation of two new stationary points, located at Δ=π. These new asymmetric stationary points are enclosed by separatrix passing through unstable stationary point. The distance between asymmetric points is increased while parameter κ grows. LPT surrounding the anti-phase stationary point coincides with separatrix when the parameter κ reaches unity. At the moment, LPT is broken that results in creation of transit-time trajectories. Fig. 3 shows typical structure of phase plane in the terms of (θ, Δ)-variables.

Let us discuss these pictures from point of view of energy exchange. It is clear that no energy exchange exists in the stationary points of phase plane. As it follows from representation (22), the variable θ defines amplitude of oscillations of both chains. So any trajectory passing near θ=0 and θ=π/2 describes a process of intensive energy exchange. Such trajectories can exist near LPTs up to κ=1. At κ=1 the LPT surrounding the anti-phase stationary point is broken and full energy exchange is forbidden. We name this process as confinement of excitation on the one chain. But there is an additional case of confinement of excitation that is realized before discontinuity of LPT occurs. If the initial conditions of chains excitation are close one of asymmetric stationary points, the corresponding trajectory will be inside the domain bounded by the separatrix. In such a case only small part of energy can be transfer from one chain to another one. At a large values of κ only this possibility of energy exchange remains. It should be to note that in the in-phase attractive domain the essential energy exchange is feasible at any values of κ.

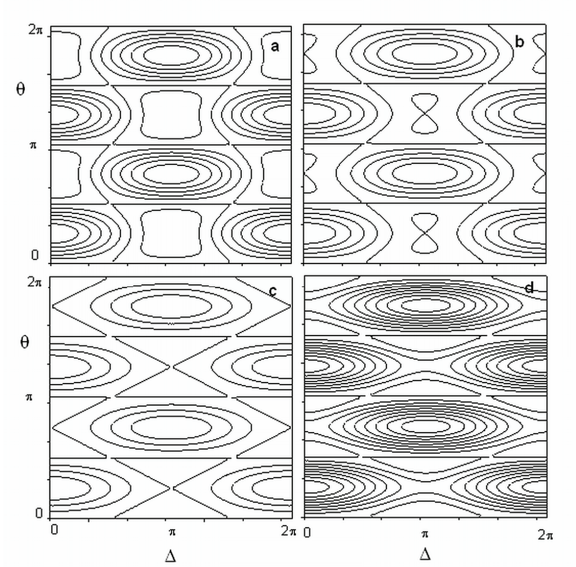

Figure 3. Typical structures of phase plane of Eqs.(19) at various values of κ: (a) κ<0.5; (b) 0.5<κ<1; (c) κ=1; (d) κ>1.

What we can say about energy exchange by localized excitations in this system? It is clear, that Eqs. (15) allow two symmetric soliton-like solutions in the form:

$$\chi_1(\xi,\tau_1)=\frac{1}{4}\sqrt{\frac{2\omega+q^2\pm\gamma}{3\beta}} \quad (23)$$
$$sch(\frac{1}{4}\sqrt{\frac{2\omega+q^2\pm\gamma}{6\beta}}(\xi+q\tau_1))\exp(i(\omega\tau_1-q\xi))$$

where "plus" under square root corresponds to in-phase state $\chi_1(\xi,\tau_1)=\chi_2(\xi,\tau_1)$ and "minus" – anti-phase one $\chi_1(\xi,\tau_1)=-\chi_2(\xi,\tau_1)$. Because expression (23) describes a stationary point, no energy exchange exists. To analyze the beating phenomenon in the presence of localized excitations like (23) we prepsent the solution of Eqs.(15) by following manner:

$$\chi_1=A(\xi)X_1(\tau_1),\ \chi_2=A(\xi)X_2(\tau_1) . \quad (24)$$

Integrating the Hamiltonian corresponding to Eqs.(15) with respect to space variables, we get the following:

$$H=-\frac{\gamma}{2}N(X_1\bar{X}_2+\bar{X}_1X_2)+\frac{1}{2}\mu N(|X_1|^2+ , \quad (25)$$
$$|X_2|^2)+6\beta\nu N^2(|X_1|^4+|X_2|^4)$$

where the new parameters are:

$$N=\int_{-\infty}^{\infty}A^2d\xi,\quad \mu=\int_{-\infty}^{\infty}(\partial_\xi A)^2d\xi/N \quad (26)$$
$$\nu=\int_{-\infty}^{\infty}A^4d\xi/N^2$$

In the framework of such approach, the only requirement with respect to space profile A(ξ) is its square integrability. New dependent variables X_1 and

X_2 are normalized to unity and the occupation number N has a simple physical meaning. We can see the total analogy between the expressions (20) and (26). All the conclusions made for plane wave solutions are correct for localized soliton-like solutions with change of the control parameter κ with κ`=νκ.

The computer simulation data are in a good agreement with analytical consideration. Fig.4 shows an example of the "breather" confinement.

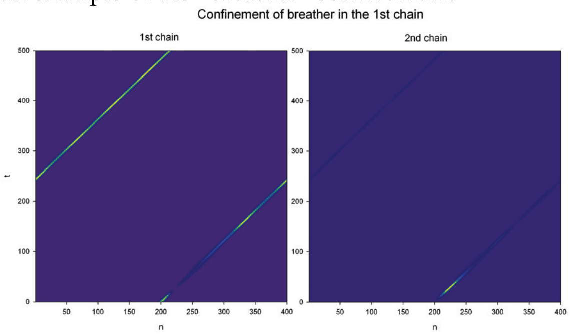

Figure 4. Confinement of localized excitation (breather) on the one of coupled chains. Light areas show the breather location. n – number of the particle, t – time measured in the eigen periods of linear oscillations. The initial conditions correspond to κ`~1.2.

Conclusion

Analytical and numerical study of wandering excitation in nonlinear chains coupled by weak harmonic interaction shows an existence of two asymptotic limits of energy transfer between different chains. First of them is characterized by quick energy transfer in comparison to processes of excitation formation. In such a case the waves in the different chains exhibit the phase shift which is equal π/2. It means that respective trajectory is closed to the LPT. Contrary, excitations with large amplitudes can show both full energy exchange near LPT and partial one near stationary points up to full confinement of excitation in one of the chains.

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