# Methods and Models in Dynamics of Stabilization and Orientation Systems[†]

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**Abstract:** The aim of research is the development of modelling methods and analysis in dynamics of complex systems, in reference to peculiarities of aerospace systems. Various aspects of mathematical modelling for systems of stabilization-orientation with gyroscopic controlling elements are discussed. The reduction principle is developed, the problem of decomposition for such systems in the general qualitative analysis is solved on the basis of A.M. Lyapunov methods. For systems of gyroscopic stabilization-orientation it generates singular perturbed problems, various types of singularities, critical cases. Here the original models are treated as singularly perturbed ones, here nonlinear approximate systems are introduced, which are also singular systems.

***Keywords:*** nonlinear dynamics, modelling, stabilization and control systems

## 1. INTRODUCTION

The specific problems of avia-, aerospace systems, with reference to problems of the mathematical modelling, analysis and synthesis for the systems of stabilization, orientation and control, with gyroscopic controlling elements, are examined in this research. Also decomposition problems at system level and sub-system level, for complex systems of gyroscopic stabilization (SGS) are considered.

Nonlinearity, high dimensionality, multi-connectivity are causing the impediments in obtaining exact solution by analytical and analytic-computer methods in designing and control. It leads to the necessity of the reducing for original model, with the subsequent transition to the decomposed systems, to reduced submodels, with designing of separated subsystems (Lyapunov, 1892, 1956; Chetayev, 1957). Effective approach is worked out, here new theoretical and applied problems are revealed, that are considered in this work, supplementing and extending early work statements, results (Chetayev, 1936; Persidsky, 1951; Merkin, 1956; Kuzmin, 1957; Magnus, 1971; Raushenbakh, Tokar, 1974; Ishlinsky, 1976; Campbell, 1980; Voronov, 1985; Kuzmina, 1988, 1991, 1996, 1997, 1998).

In regard to the stabilization and orientation systems with the gyroscopic controlling elements, it leads to the singularly perturbed problems with the different singularities types, with critical cases, with the nonlinear singular generating systems. Here formulated problems are solved by developed method, following to the ideology of stability theory. General approach, based by A.M. Lyapunov, applied by N.G. Chetayev to mechanics problems, is extended here. The understanding these problems via singularly perturbed systems approach gives the perspective results both for theory and for applications, with revealing a constructiveness of Lyapunov stability methods as effective unified mathematical tool. As illustration there is considered the family of the stabilization and orientation systems models with gyroscopic controlling elements (including the models for small satellites, for large stabilized objects, …). The cases of full mathematical decomposition (Siljak, 1991) for original model of considered systems are examined.

## 2. INITIAL STATEMENT

The work is formed on the accepted basic assumption about global methodological connection between modelling problems and methods of A.M. Lyapunov stability theory (Chetayev, 1936; Kuzmin, 1957; Kuzmina, 2006, 2008). Such approach ascends to a stability postulate (Chetayev, 1936) and to property of stability with parametrical perturbations (Kuzmin, 1957). The developed method establishes the uniform approach based on a singularity postulate (Kuzmina, 2006), with interpretation of examined objects as singular class systems, with combining of methods of the stability theory and perturbations theory. The state of initial object is described by mathematical model with singular perturbations. Two basic principles (stability postulate and singular postulate) are accepted here as main axioms. From these points the systems of gyroscopic stabilization-orientation are treated as singular systems; working models for them are shortened models of less order. In practice these models are introduced on intuition, without the strict mathematical analysis of influence of the rejected members on dynamic properties. The problems of a

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correctness and qualitative equivalence are not discussed. The experiment is accepted as criterion of legality of these models "of an intuitive level". But the general theory is necessary (Andronov et al., 1959; Chetayev, 1957; Kuzmina, 2001). Different examples of a concrete physical-technical matter are considered. Let consider the system of one-axis gyroscopic stabilization (OGS); let full initial model (IM) (equations in the Lagrange form), have the form:

$$\text{IM} \qquad \begin{array}{r|l} & A\ddot{\beta} - H\dot{\theta} - H\dot{\alpha} = -b_1\dot{\beta} + \ldots \\ J\ddot{\alpha} + B\ddot{\theta} \; + & H\dot{\beta} = -b_2\dot{\alpha} - e\beta + \ldots \\ B(\ddot{\theta} + \ddot{\alpha}) \; + & H\dot{\beta} = -b_3\dot{\theta} - c\,\theta + \ldots \end{array} \qquad (1)$$

Here IM has 3 freedom degrees ($k$ = 3); it is simulated as mechanical system with inertialess following drive; $H$ is eigen angular momentum of gyroscope; $q = (\beta, \alpha, \theta)$; $\beta$ is precession angle, $\alpha$ is stabilization angle, $\theta$ is deformation angle of an axis of gyroscope mount (it is accepted, that these elements are have not absolute rigidity); $A$, $J$, $B$ are corresponding moments of inertia; $b_1$, $b_2$, $b_3$ are appropriate coefficients of friction; $c$, $e$ are coefficients of the generalized forces of potential and non-potential character.

In engineering practice the various shortened models (SM) are used as working models, in the assumption about fast gyroscope ($H$ is big parameter): (2), (3); these models are introduced an intuitive level.

$$\text{SM} \qquad \begin{array}{l} -H\dot{\theta} - H\dot{\alpha} = -b_1\dot{\beta} + \ldots \\ H\dot{\beta} = -b_2\dot{\alpha} - e\beta + \ldots \\ H\dot{\beta} = -b_3\dot{\theta} - c\theta + \ldots \end{array} \qquad (2)$$

$$\text{SM} \qquad \begin{array}{l} -H\dot{\theta} - H\dot{\alpha} = -b_1\dot{\beta} + \ldots \\ \tilde{J}\ddot{\alpha} + H\dot{\beta} = -b_2\dot{\alpha} - e\beta + \ldots \\ H\dot{\beta} = -b_3\dot{\theta} - c\theta + \ldots \end{array} \qquad (3)$$

Model (2) is shortened model with 1,5 degrees of freedom ($k_S$= 1,5), (2) is a precession model, known in the applied theory of fast gyroscopes. But (2) is the only formalized mathematical abstraction, $k_S < k$, $k_S$ is non-integer number. It is necessary to obtain a strict substantiation for acceptability, with estimations of required values of $H$ for legality of this shortened model.

Model (3) is other shortened model, with $k_S$ = 2. This model has not the members corresponding to mass and the moments of inertia for gyroscope and mount; only the members appropriate to the inertia moments of stabilized object (platform) are kept. Shortened model (3) is also formal mathematical formalization ($k_S < k$). In engineering practice this model also is named as precession model; the researchers assume, that the acceptability of (3) can be based on same assumption about fast rotating gyroscope (big parameter $H$). The detailed analysis shows, that it is non-correct assumption. The acceptability of (3) should be based on the other physical property, it is corresponding to big mass of stabilized object.

The main questions in reference to the theory of systems of gyroscopic stabilization (SGS):
- building of this shortened model by a strict analytical way;
- condition of a correctness of the shortened model;
- physical interpretation of these shortened models.

The similar problems take place and in generally, for systems of gyroscopic stabilization-orientation, modelled as electromechanical system, when the transient processes in following systems are taken into consideration; initial mathematical model is the Lagrange-Maxwell (Gaponov) equation (4). The standard designations (Ishlinsky, 1976; Raushenbakh and Tokar, 1974; Kuzmina, 1997) here are used

$$\text{IM} \qquad \frac{d}{dt} a\dot{q}_M + (b+g)\dot{q}_M = Q'_M + Q_{ME} + Q''_M \qquad (4)$$

$$\frac{d}{dt} L\dot{q}_E + R\dot{q}_E = Q'_E + Q_{EM} + Q''_E\,; \qquad \frac{dq_M}{dt} = \dot{q}_M$$

$$Q_{ME} = A_M\dot{q}_E;\; Q_{EM} = B_E\dot{q}_M;\; Q'_E = -\omega q_1 - \Omega\dot{q}_E$$

Here $a$ is symmetric matrix of quadric form of kinetic energy of system mechanical part, $L$ is symmetric matrix of quadric form of the system electromagnetic energy; $b$ is symmetric matrix of quadric form in expansion of dissipative function of viscous friction forces (external and internal); $g$ is skew-symmetric matrix of gyroscopic forces coefficients; $R$ is symmetric matrix of quadric form in expansion of dissipative function of electrical currents (that is characterizing the loss on Joule heat); $Q_{ME}$ is vector of mechanical generalized forces of electromagnetic nature; $Q_{EM}$ is vector of electrical generalized forces of mechanical nature; $Q_M$ and $Q_E$ are generalized mechanical and electrical forces; $q_M = \|q_1, q_2\|^T$; $q_M$ is $n$-vector of the mechanical generalized coordinates, $q_1$ is $l$-vector of mechanical controlling coordinates (angles of precessions); $q_E$ is $u$-vector of the electrical generalized coordinates.

In engineering practice, for IM of ($2n$+$u$) order, the authors use for (4) the various shortened idealized models, SM of less order, as working models: SM of the ($n$+$u$) order, SM of the ($n$) order, SM of the ($2n$-l + $u$) order, … These SM of the less order are used in the solving of problems of the analysis and synthesis for initial model, though without strict substantiation for such transition. These SM are obtained at an intuitive level, but ones are only formalized mathematical abstractions, which are not generally qualitatively equivalent to initial model (Moiseev, 1981; Ljung, 1987).

The main purposes of research: the constructing of the optimal shortened mechanical-mathematical models; strict manners of idealization; legality of the short models in dynamics; conditions of qualitative equivalence.

## 3. GENERAL PRINCIPLES

Consider IM in the form (4). Main tasks: (a) modelling problem (constructing of SM by a strict mathematical way); (b) acceptability problem (the conditions of models equivalence); (c) estimations problem (finding of allowable values areas of parameters); (d) problem of constructing of the minimal model (in N.N. Moiseev sense). The powerful tool here is methods of A.M. Lyapunov theory with extending of property of parametrical stability on an irregular case (N.G. Chetayev, P.A. Kuzmin). The main hypothesis: all examined objects are ones of singular perturbed class

(Kuzmina, 2006), and always there exists such transformation of variables $(q,\ \dot{q})\to y$, with which the initial mathematical model can be presented in the standard form of system with irregular perturbations in form (5), initial system (IS).

$$\text{IS} \qquad M(\mu)\, dy/dt = Y(t, \mu, y) \tag{5}$$

Here $\mu$ is small positive dimensionless parameter; $M(\mu)=\|M_{ij}(\mu)\|$; $M_{ii}(\mu) = \mu^{\alpha\, i}\, I$, $0 \le \alpha_I \le r$, $I$ is identity matrices; $y$ is $N$-vector of new variables; $Y(t,\ \mu,\ y)$ is nonlinear $N$-vector-function with the appropriate properties.

The system (5), named here initial system (IS), has the $N$ order ($N = 2n + u$). Let shortened system (SS) for (5) is simplified system (6).

$$\text{SS} \qquad M_S(\mu)\, dy/dt = Y_S(t, \mu, y) \tag{6}$$

The system (6) has $N_S$ order ($N_S < N$). (6) is obtained from (5) as system of $s$-approximation on $\mu$ parameter. We shall name it shortened system of a $s$-level (s-system, $SS_S$); (5), (6) are singularly perturbed systems (SPS). Coming back in (6) to old variables $(q, \dot{q})$, we shall get shortened model of $s$-level ($s$-model, $SM_S$). $SM_S$ has $k_S$ of degrees of freedom ($k_S = N_S/2$). From the view point of the mechanics SM is some idealized model (idealization on the chosen physical properties, corresponding to small parameter $\mu$). By such approach all family of possible models for considered SGS may be constructed.

Developed method gives regular algorithm for building of the shortened mathematical models (working ones) of object (IO) by strict mathematical way:

$$\text{IO} \to \underset{(q,\, \dot{q})}{\text{IM}} \to \underset{(y)}{\text{IS}} \to \underset{(y)}{\text{SS}} \to \underset{(q,\, \dot{q})}{\text{SM}} \tag{7}$$

Here the decomposition of initial model on sub-models is automatically carried out, and the initial state variables are divided on different-frequency groups; the initial parameters are divided on essential and non-essential, the main degrees of freedom are automatically revealed by this method.

General results here are given for full solving on a problem (a). The following stages are (b), (c) … Not discussing all details and obtained theorems, we shall result here only some results within the framework of the used approach. On a task (b) (task of an acceptability): it is known, dynamic properties (stability, optimality, speed, …) do not possess the decomposition.

The special conditions (Kuzmina, 1996, 2008) should be realized for these sub-problems. For the solving of these questions the developed method is used. The results for a stability problem (“$s$-stability"), proximity problem ("$s$-proximity"), problem of $s$-speed, $s$-optimality may be obtained.

The application to dynamics of systems of gyrostabilization and orientation were considered. In reference to examined objects the developed approach has appeared rather fruitful. Let initial model is accepted in the form (4).

*3.1 Modeling problem for the SGS*

The SPS with fast gyroscopes. In this case in (4) it is introduced big parameter $H$ ($g = g^*H$, $H = 1/\mu$, $\mu$ is small parameter). Here the required transformation of variables is constructed, IM is presented in the form of SPS (5), the state variables are divided into three groups: $\dot{q}_M$ - high-frequency variables; $\dot{q}_E$ is middle-frequency variables; $q_M$ is low-frequency variables. Two types as of shortened models ($SM_1$, $SM_0)_\mu$ are constructed: (8), (9).

$$(b+g)\dot{q}_M = Q'_M + Q_{ME} + \overline{Q}''_M\,, \qquad \frac{dq_M}{dt} = \dot{q}_M \tag{8}$$

$$\begin{aligned} &\frac{d}{dt} L\dot{q}_E + R\dot{q}_E = Q'_E + Q_{EM} + Q''_E \\ &g\dot{q}_M = Q'_M + Q_{ME} + \tilde{Q}''_M \\ &R\dot{q}_E = Q'_E + \tilde{Q}''_E\,; \qquad \frac{dq_M}{dt} = \dot{q}_M \end{aligned} \tag{9}$$

Here (8) is linearized on $\mu$ model $SM_1$; it is model of the ($n + u$) order; (9) is $SM_0$ of the $n$-order. (9) is limit on $\mu$ model (new model). $SM_1$ is precessional model for controlled SGS.

The SGS with quick-operating follow-up systems. In this case (4) is system with small inertia electrical circuits. Here it is introduced the small parameter $\mu_1$, corresponding to small time constant of electrical circuits. For this case another required transformation of variables also is constructed; IM is presented as SPS; the state variables are divided into three groups: $\dot{q}_M$ is middle-frequency variables; $\dot{q}_E$ is high-frequency variables; $q_M$ is low-frequency variables. The shortened models $(SM_1)_{\mu 1}$ of the $2n$ order are constructed (we shall designate it by (10), not writing out), $(SM_0)_{\mu 1}$ of the $n$ order (it is limit on $\mu_1$ model) - (11).

$$\begin{aligned} &(b+g)\dot{q}_M = Q'_M + Q_{ME} + \overline{Q''_M} \\ &R\dot{q}_E = Q'_E + \tilde{Q}''_E\,, \qquad \frac{dq}{dt} = \dot{q}_M \end{aligned} \tag{11}$$

All shortened models are obtained by a regular mathematical way, on uniform algorithm; $(SM_1)_\mu$ and $(SM_1)_{\mu 1}$ are known models; but $(SM_0)_\mu$ and $(SM_0)_{\mu 1}$ are new (limit) models.

The SGS with big stabilized platforms. Initial model is (4). In applied researches for this case the shortened model is introduced as the simplified mathematical model of the ($2n$-l+$u$) order (it is SM, such as model (3), named "precession model", with the references on its substantiation on "precession theory of gyroscopic systems with fast gyroscopes”). But such references are not correct for this case of big stabilized object (D.R. Merkin, A.Yu. Ishlinsky). In this case masses and inertia moments of gyroscopes and mounts are small (in comparison with the mass characteristics of stabilized platforms). Correspondingly other small parameter, $\mu_2$, should be introduced in this statement: $a_M=\|a_1, a_2\|^T$; $a_1=a_1(q_M, \mu_2)=a_1^*\mu_2$; $a_2= a_2(q_M, \mu_2)$, $a_2(q_M, 0) = \bar{a}_2 \neq 0$. On the same algorithm (7) shortened model $SM_0$ (on $\mu_2$) of ($2n$-l + $u$) order is constructed. It is new shortened model SGS with big stabilized objects; it is non-precession model. It is new result with new working SM, with new conditions of acceptability, that is different from known ones (Raushenbakh and Tokar, 1974).

*3.2 Acceptability problem for SGS*

According to the developed approach the concrete types of SGS were considered from this view point. The acceptability problem is divided on separate sub-problems; the problem of model acceptability in general statement is not correct; the concept of a model acceptability is relative one, concept "of acceptability in general" is meaningless. It should be examined within the framework of the concrete purposes of designed model (as it was noted early by L. Ljung, L.K. Kuzmina). Here within the framework of the accepted statement the theorems of decomposition for stability property (asymptotic and non-asymptotic), about proximity between the solutions of IM and SM (with estimations of N.G. Chetayev type), about decomposition of speed property, about property of the maximal degree of stability, about optimal parameters are obtained. The results were received both for the general perturbations theory and for the SGS theory. Not showing all details and proofs, the theorem for reduction-decomposition problem (4)$\rightarrow$(11) is formulated. Here (11) is corresponding to the idealized model with inertialess elements of mechanical and electrical subsystems (small parameter $\mu_1$ corresponds to small time constant of electrical circuits).

**Theorem.** If equations

$$d_1=|L\alpha+R^0+\Omega^0|=0; \qquad d_2=|a^0\beta+b^0+g^0|=0;$$

$$d_3=\left|\begin{array}{c|c|c} \multicolumn{2}{c|}{b^0_1+g^0_1} & 0 \\ \hline \multicolumn{2}{c|}{(b^0_2+g^0_2)\lambda} & -A^0 \\ \hline \omega^0 & 0 & R^0+\Omega^0 \end{array}\right|=0$$

will satisfy Hurwitz conditions, then:

- under small $\mu_1$ values the stability property (asymptotic or non-asymptotic) of system (11) trivial solution is entailing the corresponding stability property for system (4) trivial solution;
- for given in advance numbers $\varepsilon>0$, $\delta>0$, $\gamma>0$ (here $\varepsilon$, $\gamma$ are taken as much small) there is such $\mu_1^*$ value, that in perturbed motion for all $\mu_1<\mu_1^*$, $t\geq t_0+\gamma$ it will be valid

$$\left\|\dot q_M-\dot q_M^*\right\|<\varepsilon, \left\|q_M-q_M^*\right\|<\varepsilon, \left\|\dot q_E-\dot q_E^*\right\|<\varepsilon, \qquad \text{if for } t_0$$

$$q_{M0}=q_{M0}^*, \left\|\dot q_{M0}-\dot q_{M0}^*\right\|<\delta, \left\|\dot q_{E0}-\dot q_{E0}^*\right\|<\delta.$$

Here the system (11) solution is marked by index"*"; the system (7) solution is without index "*".

The theorem gives conditions of an acceptability for new SM (11), more simple, than known (10). Other cases and singular problems were considered; the appropriate theorems may be obtained.

## 4. CONCLUSIONS

The developed methodology, based on A.M. Lyapunov theory, perturbations theory, postulates of stability and singularity (LPSS-approach), allows to obtain the constructive algorithm of engineering level, which gives the regular schemes, which allow to study complex systems (SGS) by analytical (computer-analytical) methods. In reference to problems multi-channel, multi-axis of stabilization systems with gyroscopic controlling elements this approach allows: to develop systematic procedures of decomposition for initial models, with division of original state variables on different-scales (on time) components; to determine conditions of acceptability for decomposed models; to prove the legality of the approximate theory; to get conditions of decomposition of initial nonlinear models with the possibility of division on the channels of control and stabilization, in problems of synthesis and analysis, including specific cases of stabilized spacecraft.

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