MINIMUM ENERGY CONTROL OF FRACTIONAL-ORDER DIFFERENTIAL-ALGEBRAIC SYSTEM

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Abstract

This paper discusses the minimum energy control problem of fractional-order differential-algebraic system. The main aim of this paper is to find the minimum energy that drives an initial state of the fractional-order differential-algebraic system to the zero state such that an index performance is minimized. The method of solving is to convert the minimum energy control problem of fractional-order differential-algebraic system into the standard fractional-order linear quadratic optimization problem by using a transformation and further solve the standard fractional-order linear quadratic optimization using the available theory in the literature. Under some particular conditions, we find the explicit formulas of the minimum energy control of fractional-order differential-algebraic system in Mittag-Leffler terms.

Key words

Minimum energy control, Fractional-order, differential-algebraic system, Mittag-Leffler.

1 Introduction

Issue on the minimum energy control for standard control system is a classical topic in dynamic optimization field. Several literatures review about this issue are given [Lewis et all, 2012] and [Naidu, 2003]. Some new developments and their applications are presented in [Baggio et all, 2019], [Lindmark, 2018] and [Glizer, 2021].

Along with the development of the fractional-order differential equation, recently the issue on minimum energy control of the fractional-order system and its application are widely discussed by researchers, see [Sajewski, 2016], [Kaczorek, 2016(a)], [Kaczorek, 2016(b)]. The solved problem is to find the least possible energy which drive the initial state $\mathbf{p}(0) = \mathbf{p}_0$ of the following fractional-order dynamic system

$$\mathcal{D}^{(\delta)}\mathbf{p} = A\mathbf{p} + B\mathbf{u} \tag{1}$$

to the desired state $\mathbf{p}(t_f) = \mathbf{p}_f$ in a time interval $[0, t_f]$ such that the performance index

$$\mathcal{J}_{\mathbf{u},\mathbf{p}} = \int_0^{t_f} \mathbf{u}^\intercal Q\mathbf{u}\, dt \tag{2}$$

is minimized. In the equation (1) and (2), $\mathbf{p} = \mathbf{p}(t) \in \mathbb{R}^n$ denotes the state, $\mathbf{u} = \mathbf{u}(t) \in \mathbb{R}^r$ denotes the energy, $\mathcal{D}^{(\delta)}\mathbf{p}$ denotes the fractional derivative of order δ of variable $\mathbf{p}$ with respect to t, with $\delta \in (m-1, m)$ for $m \in \mathbb{N}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and $Q \in \mathbb{R}^{r\times r}$ is a symmetric positive definite. It is clear that when $\delta = 1$, the problem (1) and (2) constitute a standard minimum energy control problem that have been discussed in [Lewis et all, 2012] and [Naidu, 2003].

Recently, Kaczorek and Borawski in [Kaczorek and Borawski, 2016] extends the minimum energy control problem for fractional-order descriptor system. Moreover, Sajewski in [Sajewski, 2017] continues the minimum energy control problem for two different fractional-order descriptor system. In both of these work, the considered system is the discrete type. Several other studies on the minimum energy control problem for fractional-order discrete-time descriptor system are given in [Klamka, 2010], [Kaczorek, 2017] and the references therein .

In this paper we study the minimum energy control problem for fractional-order differential-algebraic system that constitutes the continuous-time type of descriptor system. The fractional-order differential-algebraic systems have attracted the attention of many researcher in the past years due to the fact that, in some cases, they describe the behavior of physical systems better than standard systems. They can preserve the structure of physical systems and include a non dynamic constraint and an impulsive element. Systems of this kind have many important applications, e.g., in electrical circuit [Gomez et all, 2013], in mechanical system [Martinez et all, 2020]. Therefore, it is fair to say that the fractional-order differential-algebraic systems give a more complete class of dynamical models than conventional dynamic systems.

The minimum energy control problem for the fractional-order differential-algebraic systems can be stated as follows: Given the following fractional-order differential-algebraic systems:

$$E\mathcal{D}^{(\delta)}\mathbf{p} = A\mathbf{p} + B\mathbf{u},\ \ \mathbf{p}(0) = \mathbf{p}_0,\ t \in [0, t_f], \qquad (3)$$

where $\mathcal{D}^{(\delta)}$ is the fractional derivative operator of Caputo type of order δ with $\delta \in (0,1)$, $E \in \mathbb{R}^{n\times n}$ with rank$(E) < n$, find the least energy $\mathbf{u} \in \mathbb{R}^r$ that drives the initial state $\mathbf{p}(0) = \mathbf{p}_0$ of the system (3) to the state $\mathbf{p}(t_f) = \mathbf{0}$ such that the performance index (2) is minimized.

We assume in this paper that $\det(s^\delta E - A) \neq 0$ for some scalar $s \in \mathbb{C}$ to guarantee the existence and uniqueness of the solution of (3) [Muhafzan et all, 2019]. Moreover, we also assume that the system (3) is controllable.

To the best of the author's knowledge, this issue has not been solved yet to date. Therefore the results of this work constitute a novelties at once a new contribution in the field of dynamic optimization subject to fractional-order differential-algebraic system.

The rest of the paper is organized as follows. Section 2 discusses some useful materials related to the desired results, such as the information about the Caputo fractional-order derivative, Mittag-Leffler function and fractional-order differential equation systems. The main result of this article is given in section 3. Section 4 concludes the paper.

2 Some Useful Results

In this part, we present some mathematical tools used in this study. Suppose that $\mathbf{y} : [0,\infty) \to \mathbb{R}^n$ is an integrable function and derivative order m for $m \in \mathbb{N}$ of the function $\mathbf{y}$, that is $\mathcal{D}^{(m)}\mathbf{y}$, exists . The fractional derivative of Caputo type of order δ with $\delta \in (m-1, m)$, is defined by:

$$\mathcal{D}^{(\delta)}\mathbf{y}(t) = \frac{1}{\Gamma(m-\delta)} \int_0^t \frac{\mathbf{y}^{(m)}(\tau)}{(t-\tau)^{\delta-m+1}} d\tau, \qquad (4)$$

where $\Gamma(.)$ is the Gamma function [Podlubny, 1999]. The physical interpretationof the fractional derivative of Caputo type has been already discussed by several researchers, see [Aguilar et all, 2014].

The one parameter Mittag-Leffler function with parameter $\beta > 0$ is defined as the following infinite series [Podlubny, 1999]:

$$\mathcal{E}_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)},\ z \in \mathbb{C}. \qquad (5)$$

Meanwhile, the two parameters Mittag-Leffler function with parameters $\beta, \gamma > 0$ are defined by [Aguilar et all, 2014]:

$$\mathcal{E}_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + \gamma)},\ z \in \mathbb{C}. \qquad (6)$$

The Mittag-Leffler functions (5) and (6) are convergent series [Podlubny, 1999]. One can replace variable z in (6) by Fz for an arbitrary square matrix F, such that

$$\mathcal{E}_{\beta,\gamma}(Fz) = \sum_{k=0}^{\infty} \frac{(Fz)^k}{\Gamma(k\beta + \gamma)}. \qquad (7)$$

Using the definition of the Laplace transformation, one can observe that

$$\mathcal{L}[\mathcal{D}^{(\delta)}\mathbf{x}(t)] = s^\delta X(s) - \sum_{k=0}^{m-1} s^{\delta-k-1}\mathbf{x}^{(k)}(0), \qquad (8)$$

where $X(s) = \mathcal{L}[\mathbf{x}(t)]$ with $\mathcal{L}$ denotes the Laplace operator and $\delta \in (m-1, m)$ [Podlubny, 1999].

Using the Laplace transformation (8) and the Mittag-Leffler function (7), the solution of the fractional-order system (1) for $\delta \in (0,1)$ is given by the following equation:

$$\mathbf{p}(t) = \mathcal{E}_\delta(At^\delta)\mathbf{p}_0 + t^\delta \mathcal{E}_{\delta,1+\delta}(At^\delta) \circledast B\mathbf{u}, \qquad (9)$$

where $\circledast$ denotes the convolution on $[0,t]$[Li and Chen, 2008].

3 Main Result

Reconsider the energy control problem for fractional-order differential-algebraic systems (3) and (2), and denote it as Ω for shorthen. A pair $(\mathbf{u}, \mathbf{p})$ is called admissible for Ω if it satisfies the system (3) and $\mathcal{J}_{\mathbf{u},\mathbf{p}} < \infty$ for an initial state $\mathbf{p}_0 \in \mathbb{R}^n$. Let us define the admissible pairs set for problem Ω by

$$\mathcal{S} \triangleq \{(\mathbf{u},\mathbf{p}) |\ \mathbf{u} \text{ and } \mathbf{p} \text{ are the continuous functions that satisfy (3) and } \mathcal{J}_{\mathbf{u},\mathbf{p}} < \infty\}.$$

A pair $(\mathbf{u}^*, \mathbf{p}^*) \in \mathcal{S}$ is called an optimal pair for Ω if $\mathcal{J}_{\mathbf{u}^*,\mathbf{p}^*} \leq \mathcal{J}_{\mathbf{u},\mathbf{p}}$ for all $(\mathbf{u}, \mathbf{p}) \in \mathcal{S}$. It is clear that $\mathcal{S} \neq \varnothing$. We will find the explicit formulation of the optimal pairs $(\mathbf{u}^*, \mathbf{p}^*) \in \mathcal{S}$, where $\mathbf{u}^*$ is the minimum energy that drives the initial state $\mathbf{p}(0) = \mathbf{p}_0$ of fractional-order differential-algebraic system (3) to the state $\mathbf{p}(t_f) = \mathbf{0}$ in a time interval $[0, t_f]$ such that the performance index (2) is minimized.

First of all, let us transform the minimum energy control problem Ω into the fractional linear quadratic problem. For this purpose, we adopt Definition 1 in [Fang et all, 2014] and the Singular Value Decomposition principle in [Muhafzan, 2010] to find a restricted system equivalent (r.s.e.) to the system (3).

Definition 1 *A fractional-order differential-algebraic system*

$$\breve{E}\mathcal{D}^{(\delta)}\breve{\mathbf{p}} = \breve{A}\breve{\mathbf{p}} + \breve{B}\mathbf{u},\ \breve{\mathbf{p}}(0) = \breve{\mathbf{p}}_\mathbf{0},$$

is said to be a restricted system equivalent (r.s.e.) to the system (3) *if there exists two nonsingular matrices* $M, N \in \mathbb{R}^{n\times n}$ *such that* $MEN = \breve{E}$, $MAN = \breve{A}$, $MB = \breve{B}$ *and* $\mathbf{p} = N\breve{\mathbf{p}}$.

Obviously, the restricted system equivalence is an equivalent relationship and it is consistent with Definition 1 in [Fang et all, 2014] for the standard differential-algebraic systems.

Let $\text{rank}(E) = p < n$. Based on the differential-algebraic Value Decomposition principle, there exists nonsingular matrices $M, N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_p & O \\ O & O \end{bmatrix}, \tag{10}$$

where I_p is an identity matrix of size $p\times p$ and O is a zero matrix of suitable size. Using these M and N matrices, we have

$$MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and

$$N^{-1}\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}, \tag{11}$$

with $A_{11} \in \mathbb{R}^{p\times p}$, $B_1 \in \mathbb{R}^{p\times r}$, $\mathbf{p}_1 \in \mathbb{R}^p$ and $\mathbf{p}_{10} = \begin{bmatrix} I_p & O \end{bmatrix} N^{-1}\mathbf{p}_0$. Thus we have the following fractional-order differential-algebraic system

$$\begin{bmatrix} I_p & O \\ O & O \end{bmatrix}\begin{bmatrix} \mathcal{D}^{(\delta)}\mathbf{p}_1 \\ \mathcal{D}^{(\delta)}\mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\mathbf{u}, \tag{12}$$

with $\mathbf{p}_1(0) = \mathbf{p}_{10}$, which is r.s.e. to the fractional-order differential-algebraic system (3). The equation (12) can be written as

$$\mathcal{D}^{(\delta)}\mathbf{p}_1 = A_{11}\mathbf{p}_1 + A_{12}\mathbf{p}_2 + B_1\mathbf{u}, \tag{13}$$
$$\mathbf{0} = A_{21}\mathbf{p}_1 + A_{22}\mathbf{p}_2 + B_2\mathbf{u}. \tag{14}$$

One can observe that the transformations (10) and (11) imply that the fractional-order differential-algebraic system (12) is impulse controllable, see [Muhafzan et all, 2020], [Nazra et all, 2020], [Yulianti et all, 2019], and this implies $\text{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n - p$. Therefore, the solution of equation (14) is

$$\begin{bmatrix} \mathbf{p}_2 \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} -\widehat{A}^\dagger A_{21} & \Psi \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{v} \end{bmatrix}, \tag{15}$$

for some full rank matrix $\Psi \in \mathbb{R}^{(n-p+r)\times r}$ with $\Psi \in \ker\begin{bmatrix} A_{22} & B_2 \end{bmatrix}$, and for some $\mathbf{v} \in \mathbb{R}^r$ with

$$\widehat{A}^\dagger = \begin{bmatrix} A_{22} & B_2 \end{bmatrix}^\top \left(A_{22}A_{22}^\top + B_2B_2^\top\right)^{-1},$$

is the generalized inverse of the matrix $\begin{bmatrix} A_{22} & B_2 \end{bmatrix}$. By using the expression (15), we can create the following transformation:

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{u} \end{bmatrix} = \left[\begin{array}{c|c} I_p & O \\ \hline -\widehat{A}^\dagger A_{21} & \Psi \end{array}\right]\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{v} \end{bmatrix}. \tag{16}$$

By defining matrix $\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}$ and $-\widehat{A}^\dagger A_{21} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ where $A_1 \in \mathbb{R}^{(n-p)\times p}, A_2 \in \mathbb{R}^{r\times p}, \Psi_1 \in \mathbb{R}^{(n-p)\times r}, \Psi_2 \in \mathbb{R}^{r\times r}$, then the transformation (16) can be written as

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{u} \end{bmatrix} = \left[\begin{array}{c|c} I_p & O \\ \hline A_1 & \Psi_1 \\ A_2 & \Psi_2 \end{array}\right]\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{v} \end{bmatrix}. \tag{17}$$

From (17), one get

$$\mathbf{u} = \begin{bmatrix} A_2 & \Psi_2 \end{bmatrix}\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{v} \end{bmatrix}. \tag{18}$$

By substituting (18) into (2), the performance index (2) can be written as

$$\mathcal{J}_{\mathbf{v},\mathbf{p}_1} = \int_0^{t_f} \left(\mathbf{p}_1^\top A_2^\top Q A_2 \mathbf{p}_1 + \mathbf{v}^\top \Psi_2^\top Q \Psi_2 \mathbf{v}\right) dt, \tag{19}$$

with the matrix Q is chosen such that $A_2^\top Q \Psi_2$ is a zero matrix. Moreover, by substituting (15) into (13) we have the fractional-order system as follows:

$$\mathcal{D}^{(\delta)}\mathbf{p}_1 = \bar{A}\mathbf{p}_1 + \bar{B}\mathbf{v},\ \mathbf{p}_1(0) = \mathbf{p}_{10}, \tag{20}$$

with $0 < \delta < 1$, where $\bar{A} = A_{11} - \left[\, A_{12}\ B_1 \,\right] \widehat{A}^{\dagger} A_{21}$ and $\bar{B} = \left[\, A_{12}\ B_1 \,\right] \Psi$.

One can see now that the performance index (19) and the fractional-order system (20) constitute a linear quadratic optimization problem for the fractional-order standard system with the state $\mathbf{p}_1$ and control $\mathbf{v}$. Likewise, the solving of the minimum energy control problem for fractional-order differential-algebraic system (3) with the performance index (2) is reduced to the solving of the linear quadratic optimization problem for the fractional-order standard system (20) with the performance index (19). Therefore, to find the solving of the minimum energy control problem for the fractional-order differential-algebraic system (3) with the performance index (2), one can solve the linear quadratic optimization problem for the fractional-order standard system (20) with the performance index (19). The problem to solve here is to find the pair $(\mathbf{v}^*, \mathbf{p}_1^*)$, which satisfy the fractional system (20) such that the performance index (19) is minimized with boundary condition is $\mathbf{p}_1(0) = \mathbf{p}_{10}$ and $\mathbf{p}_1(t_f) = \mathbf{0}$.

In order to find the pair $(\mathbf{v}^*, \mathbf{p}_1^*)$, we use the theory of the standard fractional-order linear quadratic optimization problem as introduced in [Fuentes et all, 2018], [Matychyn and Onyshchenko, 2018] by making some modifications. First of all, based on the performance index (19) and the fractional-order system (20), let us consider the following augmented performance index:

$$\mathcal{J}^a_{\mathbf{v},\mathbf{p}_1} = \int_0^{t_f} \Big(\mathbf{p}_1^\top A_2^\top Q A_2 \mathbf{p}_1 + \mathbf{v}^\top \Psi_2^\top Q \Psi_2 \mathbf{v} + \mathbf{q}^\top (\bar{A}\mathbf{p}_1 + \bar{B}\mathbf{v} - \mathbf{p}_1^{(\delta)}) \Big) dt, \tag{21}$$

where $\mathbf{q} = \mathbf{q}(t) \in \mathbb{R}^p$ is a costate variable. By defining the Hamiltonian:

$$\mathcal{H} = \mathbf{p}_1^\top A_2^\top Q A_2 \mathbf{p}_1 + \mathbf{v}^\top \Psi_2^\top Q \Psi_2 \mathbf{v} + \mathbf{q}^\top (\bar{A}\mathbf{p}_1 + \bar{B}\mathbf{v}), \tag{22}$$

the augmented performance index (21) can be written as:

$$\mathcal{J}^a_{\mathbf{v},\mathbf{p}_1} = \int_0^{t_f} f(\mathbf{p}_1, \mathbf{v}, \mathbf{q}, \mathcal{D}^{(\delta)}\mathbf{p}_1, \mathcal{D}^{(\delta)}\mathbf{v}, \mathcal{D}^{(\delta)}\mathbf{q},\ t)dt, \tag{23}$$

where

$$f(\mathbf{p}_1, \mathbf{v}, \mathbf{q}, \mathcal{D}^{(\delta)}\mathbf{p}_1, \mathcal{D}^{(\delta)}\mathbf{v}, \mathcal{D}^{(\delta)}\mathbf{q}, t) = \mathcal{H} - \mathbf{q}^\top \mathcal{D}^{(\delta)}\mathbf{p}_1. \tag{24}$$

By using the procedure in [Fuentes et all, 2018], [Matychyn and Onyshchenko, 2018], we find the following Euler-Lagrange equation:

$$\frac{\partial f}{\partial \mathbf{v}} - \mathcal{D}^{(\delta)}\Big[\frac{\partial f}{\partial \mathcal{D}^{(\delta)}\mathbf{v}}\Big] = \mathbf{0}, \tag{25}$$

$$\frac{\partial f}{\partial \mathbf{q}} - \mathcal{D}^{(\delta)}\Big[\frac{\partial f}{\partial \mathcal{D}^{(\delta)}\mathbf{q}}\Big] = \mathbf{0}, \tag{26}$$

$$\frac{\partial f}{\partial \mathbf{p}_1} - \mathcal{D}^{(\delta)}\Big[\frac{\partial f}{\partial \mathcal{D}^{(\delta)}\mathbf{p}_1}\Big] = \mathbf{0}. \tag{27}$$

After some calculations, the equations (25), (26) and (27) result:

$$\mathbf{v} = -(\Psi_2^\top Q \Psi_2)^{-1} \bar{B}^\top \mathbf{q}, \tag{28}$$

$$\mathcal{D}^{(\delta)}\mathbf{p}_1 = \bar{A}\mathbf{p}_1 + \bar{B}\mathbf{v}, \tag{29}$$

$$\mathcal{D}^{(\delta)}\mathbf{q} = -A_2^\top Q A_2 \mathbf{p}_1 - \bar{A}^\top \mathbf{q}. \tag{30}$$

By substituting equation (28) into (29), we have the following boundary value problem:

$$\begin{bmatrix} \mathcal{D}^{(\delta)}\mathbf{p}_1 \\ \mathcal{D}^{(\delta)}\mathbf{q} \end{bmatrix} = \begin{bmatrix} \bar{A} & -\bar{B}(\Psi_2^\top Q \Psi_2)^{-1}\bar{B}^\top \\ -A_2^\top Q A_2 & -\bar{A}^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{q} \end{bmatrix}, \tag{31}$$

with boundary conditions

$$\mathbf{p}_1(0) = \mathbf{p}_{10} \ \text{ and } \ \mathbf{p}_1(t_f) = \mathbf{0}. \tag{32}$$

By using the equation (9), the solution of (31) is given by

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{q} \end{bmatrix} = \mathcal{E}_\delta(\mathcal{A}t^\delta) \begin{bmatrix} \mathbf{p}_1(0) \\ \mathbf{q}(0) \end{bmatrix}, \tag{33}$$

where

$$\mathcal{A} = \begin{bmatrix} \bar{A} & -\bar{B}(\Psi_2^\top Q \Psi_2)^{-1}\bar{B}^\top \\ -A_2^\top Q A_2 & -\bar{A}^\top \end{bmatrix}. \tag{34}$$

By using the condition (32), one has

$$\mathbf{q}(0) = -\mathcal{E}_\delta((-\bar{B}(\Psi_2^\top Q \Psi_2)^{-1}\bar{B}^\top)^{-1} t_f^\delta)\mathcal{E}_\delta(\bar{A}t_f^\delta)\mathbf{p}_{10}, \tag{35}$$

thus, from (33) and (34), we have

$$\mathbf{q} = \Big(\mathcal{E}_\delta(-A_2^\top Q A_2 t_f^\delta) - Z_1\Big)\mathbf{p}_{10}, \tag{36}$$

where

$$Z_1 = \mathcal{E}_\delta(\bar{A}^\top t_f^\delta)\mathcal{E}_\delta(-(\bar{B}^{-1})^\top \Psi_2^\top Q \Psi_2 \bar{B}^{-1} t_f^\delta)\mathcal{E}_\delta(\bar{A}t_f^\delta). \tag{37}$$

Moreover,

$$\mathbf{p}_1^* = \Big(I_p - \mathcal{E}_\delta(-\bar{B}(\Psi_2^\top Q \Psi_2)^{-1}\bar{B}^\top t_f^\delta) Z_2\Big)\mathcal{E}_\delta(\bar{A}t_f^\delta)\mathbf{p}_{10}. \tag{38}$$

where

$$Z_2 = \mathcal{E}_\delta(-(\bar{B}^{-1})^\top \Psi_2^\top Q_2 \Psi_2 \bar{B}^{-1} t_f^\delta). \tag{39}$$

By using (36), we have

$$\mathbf{v}^* = -(\Psi_2^\top Q \Psi_2)^{-1} \bar{B}^\top \Big(\mathcal{E}_\delta(-A_2^\top Q A_2 t_f^\delta) - Z_1\Big)\mathbf{p}_{10}. \tag{40}$$

Note that, in accordance to the theory in [Fuentes et all, 2018], [Matychyn and Onyshchenko, 2018], the pair $(\mathbf{v}^*, \mathbf{p}_1^*)$ minimize $\mathcal{J}_{\mathbf{v},\mathbf{p}_1}$, where $\mathbf{v}^*$ and $\mathbf{p}_1^*$ are given by (40) and (38), respectively.

Furthermore, by using the transformations (11), (16) and (17), the optimal pair $(\mathbf{u}^*, \mathbf{p}^*)$ which constitutes the solution of the minimum energy control problem for fractional-order differential-algebraic systems (3) and (2) is given by

$$\mathbf{u}^* = A_2\mathbf{p}_1^* + \Psi_2\mathbf{v}^*, \tag{41}$$

and

$$\mathbf{p}^* = N\begin{bmatrix} I_p \\ A_1 \end{bmatrix}\mathbf{p}_1^* - N\begin{bmatrix} O \\ \Psi_1 \end{bmatrix}\mathbf{v}^*, \tag{42}$$

where $\mathbf{p}_1^*$ and $\mathbf{v}^*$ are given by (38) and (40), respectively.

One can see that due to $\mathbf{v}^*$ and $\mathbf{p}_1^*$ are stated in terms of Mittag-Leffler function, $\mathbf{u}^*$ and $\mathbf{p}^*$ can be stated in terms of Mittag-Leffler function as well. The equations (41) constitute the explicit formula for the minimum energy that drives the initial state $\mathbf{p}(0) = \mathbf{p}_0$ of the system (3) to the state $\mathbf{p}(t_f) = \mathbf{0}$ such that the performance index (2) is minimized.

4 Concluding Remark

We have found the explicit formula of the minimum energy that drives the initial state $\mathbf{p}(0) = \mathbf{p}_0$ of the system (3) to the state $\mathbf{p}(t_f) = \mathbf{0}$ such that the performance index (2) is minimized. The explicit formula of the minimum energy is stated in terms of Mittag-Leffler function.

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