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GENERALISED ENTROPY OF CURVES AND EVALUATION OF CHAOTIC BEHAVIOURS

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Abstract

An Entropy of Curves based Indicator (ECI) is used to evaluate and compare chaotic deterministic dynamical systems. ECI is computed according to a methodology that similarly to Monte Carlo calculations exploits a set of random realizations of the dynamical system, where randomness is with respect to the choice of the initial conditions. Each sampled initial condition evolves in time according to the deterministic state dynamics and the generalised entropy of the curve connecting sequentially all the points is computed at each time step. According to this procedure, all linear dynamical systems are characterised by a zero constant ECI, while higher values of the ECI reveal the nonlinear behaviour of the dynamical system. In this paper the analysis of chaotic systems is performed, and ECIs of well known chaotic systems are computed and compared. ECI is also used to infer the region of the initial conditions in the case of dynamical systems where ordered and chaotic regions coexist.

Key words

Thermodynamics of plane curves, Lyapunov exponents, nonlinear dynamical systems.

1 Introduction

Mendès France developed a new theory called thermodynamics of plane curves, [Mendès France, 1983], [Dupain, Kamae and Mendès France, 1986], where the equivalent of several thermodynamics quantities like *entropy*, *temperature* and *pressure* were defined in association to plane curves. The main concept arising from the proposed theory is that the *temperature of a curve is* 0 *only if the curve is a straight line, and increases as the curve becomes more "wiggly"*, the last being a quote from Wolfram [MathWorld]. Moreover, straight lines represented by zero temperature also have zero entropy, according to Nernst's thermodynamic assumption, still in accordance with classic thermodynamics. Thermodynamics of plane curves has been investigated by other authors in the literature, [Jumarie, 1997] and has been used for geophysical applications [Denis and Crémoux, 2002] and [Denis et al., 2005]. More recently, the authors of this paper have extended the definition of the entropy of plane curves to higher dimensions and showed that the concept could be used conveniently to analyse and classify dynamical systems. The generalised entropy of curves could be used in a general $\mathbb{R}^n$ space, while preserving the same properties of the original entropy in the planar case, [Balestrino, Caiti and Crisostomi, 2009]. Analysis and classification of dynamical systems can be performed according to the generalised Entropy of Curves Indicator (ECI) that is computed iteratively on the basis of geometrical properties of random state trajectories. The idea behind the proposed approach is to embed a straight curve (therefore characterised by zero entropy) inside a dynamical system and check at each time step the value of its entropy. In case of linear system its entropy is known to be constantly zero, while in the case that the system dynamics fold, bend or stretch the line, the entropy increases reaching an asymptotic value. Therefore the sample line can be thought of as a thermometer which is immersed in the dynamical system to evaluate its complexity.

In [Balestrino, Caiti and Crisostomi, 2009] several classic dynamical systems were classified according to their ECI value, and it was shown that the ECI not only separates linear from nonlinear systems, but among nonlinear systems it correctly distinguishes mild nonlinear, highly nonlinear, chaotic and noisy dynamical systems. As already acknowledged in [Balestrino, Caiti and Crisostomi, 2009], such a classification is not always possible, since the same dynamical system might show very different behaviours depending on the initial conditions, therefore it does not always make sense to associate one only value of the ECI to the overall dynamical system.

The novelty of this paper is the use of the ECI, which is devoted to the analysis of chaotic systems, with special concern on dynamical systems where both order and chaos coexist. In this case, the final objective is to verify whether this indicator correctly identifies if the initial conditions belong to the chaotic region or not. In principle, it is desired that the ECI evolves differently depending on the initial conditions, regardless of the fact that the dynamical equations are the same.

The paper is organised as follows: next section recalls the algorithmic procedure to compute ECI and to list the main properties of the indicator. In section 3, a comparison is performed between the proposed indicator and other methods already available from the related literature. ECI values associated to well known chaotic dynamical systems are provided in section 4, where examples of dynamical systems presenting both chaotic and ordered behaviours are shown. Finally, in section 5 we summarise our results and report the conclusions.

2 ECI

Given a set of ordered N points in $\mathbb{R}^n$, the generalised entropy H of the piecewise linear curve connecting sequentially the points was defined in [Balestrino, Caiti and Crisostomi, 2009] as

$$H = \frac{\log\left(\frac{L}{d}\right)}{\log\left(N-1\right)}, \qquad (1)$$

where L is the length of the curve and d is the diameter of the smallest hypersphere including all the points. Equation (1) was proposed to extend the definition of the entropy of a plane curve to $\mathbb{R}^n$ space, while preserving the typical properties of the entropy of plane curves [Mendès France, 1983]. In particular, let Γ be a curve in $\mathbb{R}^n$, then the following properties regarding the entropy of Γ hold: (proofs can be found in [Balestrino, Caiti and Crisostomi, 2009]).

- The entropy of Γ is always included between 0 and 1.
- The entropy of Γ is 0 if and only if Γ is a straight line.
- The entropy of Γ is insensitive to scale changes, rotations and translations.

Informally, the generalised entropy of a curve measures the irregularity of a curve, thus straight lines have zero entropy, while more tortuous and wiggly curves have higher entropies.

The idea here is to use equation (1) to analyse deterministic dynamical systems: an entropy of curves indicator records at each time step the entropy (1) associated to a curve that evolves in the phase space according to the discrete dynamics of an underlying dynamical system. System equations can be described according to the general notation

$$x\left(k+1\right) = f\left(x\left(k\right), k\right) \qquad x \in \mathbb{R}^n, \qquad (2)$$

and the indicator ECI can be computed iteratively according to the following procedure

Algorithm 1*:*

1. **Initialisation:** $k = 0$
   (a) *Choose* N *points* $x_1\left(0\right), ..., x_N\left(0\right)$ *ordered along a straight line*
   (b) $ECI\left(0\right) = 0$
2. **Evolution:** step k
   (a) *Compute the next state for each point* $x_1\left(k+1\right), ..., x_N\left(k+1\right)$ *according to (2)*
   (b) *Consider the curve that connects sequentially all the points and take its length* $L\left(k\right)$
   (c) *Compute the smallest hypersphere that contains all the points, and take its diameter* $d\left(k\right)$
   (d) *The value of our indicator* $ECI\left(k\right)$ *is equivalent to the entropy of the curve at that time step, computed according to (1)*
   (e) *Go to next step (* $k = k+1$ *).*

Initial ECI at step 1.b is zero because the curve is a straight line, and since collinearity is preserved under affine transformations, it remains zero if the dynamical system is linear. Deterministic inputs can be included in equation (2) without significant changes, and have not been considered here for the sake of simplicity. The algorithmic procedure of computing the entropy of a curve is represented in figure 1 where a curve evolving according to well known Lorenz equations is shown every 10 steps. Clearly, its entropy is increasing. The proposed algorithmic procedure to compute the ECI has three other important properties: (proofs can be found again in [Balestrino, Caiti and Crisostomi, 2009])

- Let us consider the special case where equations (2) are linear. Then according to *Algorithm 1*, if $ECI\left(0\right) = 0$ then $ECI\left(k\right) = 0, \forall k \in \mathbb{N}$.
- Let us consider the special case where system (2) is one-dimensional. Then, if the state transition function is monotone and $ECI\left(0\right) = 0$, then $ECI\left(k\right) = 0, \forall k \in \mathbb{N}$.
- Even if the curve remains the same, but at least two points exchange their positions, then the associated ECI changes.

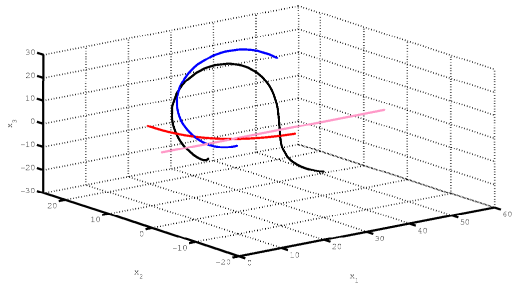

Figure 1. Evolution of 10 points in the phase space, and transformation of the curve connecting sequentially all of them. Initially points are collinear, and they later evolve according to Lorenz equations. The curve connecting them is a straight line at the beginning, and it becomes more irregular in time (the sequence is from lighter to darker colors) and the ECI increases accordingly.

In particular the last property is due to the fact that points are interpolated sequentially, and it permits to identify correctly the nonlinear behaviour of a system that performs folding operations.

3 Comparison with other chaotic indicators

Several well-established methods are known to provide quantitative evaluations of chaos, including for instance the Lyapunov Exponents (LEs), the autocorrelation function and the power spectrum. Other less conventional indicators have been introduced in the recent literature, as for instance papers [Froeschlé, Lega and Gonczi, 1997], [Voglis, Contopoulos and Efthymiopoulos, 1999], [Skokos, 2001], [Bonasera et al., 2003] and [Lukes-Gerakopoulos, Voglis and Efthymiopoulos, 2008]. There are two main differences between the ECI introduced here and the LEs:

1. LEs give a quantitative characterization of the exponential divergence of initially nearby trajectories, thus taking into account the stretching effect of the underlying dynamical system. For this reason both linear unstable dynamical systems and chaotic systems might share the same LEs. In contrast, ECI strongly differentiates linear from chaotic systems, and takes into account simultaneously stretching and folding aspects of the underlying dynamical systems.
2. It is not straightforward to extend the theory of LEs to the case of discrete-state systems, while ECI can be still computed according to *Algorithm 1* without changes.

The proposed method is also similar to the d_∞ parameter introduced in [Bonasera et al., 2003], where pairs of random trajectories are generated in a Monte-Carlo way. In the assumption that distances among trajectories reach an average common asymptotic value that has lost memory of the initial conditions [Gade and Amriktar, 1990], it is possible to relate d_∞ to the asymptotic value of the ECI, here called ECI_∞ by analogy:

$$ECI_\infty = \frac{\log\left(\frac{(N-1)\,d_\infty}{diam}\right)}{\log(N-1)}, \tag{3}$$

where the diameter $diam$ of the bounding hypersphere has been indicated more extensively so to avoid misunderstandings with d_∞. The main difference with ECI is that d_∞ only takes into account asymptotic average distances between trajectories, while ECI also exploits information regarding the smallest hypersphere including all points, thus mixing local with global information. Moreover, the two methods provide completely different results when applied to non chaotic systems.

4 Examples

The first example is provided to compare three chaotic dynamical systems: Lorenz equations (4), Hénon map (5) and the Logistic equation (6). Results are shown in figure 2. In the Lorenz equations

$$\begin{cases} \dot{x_1} = -10\,(x_1 - x_2) \\ \dot{x_2} = 28x_1 - x_2 - x_1x_3 \\ \dot{x_3} = x_1x_2 - 8/3x_3 \end{cases}, \tag{4}$$

a discretisation step of 0.01 s is used. In the Hénon map

$$\begin{cases} x_1\,(k+1) = x_2\,(k) + 1 - ax_1^2\,(k) \\ x_2\,(k+1) = bx_1\,(k) \end{cases} \tag{5}$$

the typical values $a = 1.4$ and $b = 0.3$ are used. In the logistic equation

$$x\,(k+1) = x_1\,(k) \cdot r\,(1 - x_1\,(k))\,, \tag{6}$$

$r = 4$ is chosen. Figure 2 shows that although very close asymptotic values of the ECI are reached, thus suggesting that all systems present comparable contents of nonlinearity, yet different values are obtained, as can be seen clearer in the bottom of figure 2. In particular the logistic equation is characterised by a higher ECI, while Lorenz system by the smallest. However, one can argue that the previous classification is not unique, as the proposed dynamical systems can be very different if their parameters are modified. For instance, Lorenz equations can give rise to periodic trajectories, or, as it is well known, the logistic equation is completely different if other values of r are chosen within the interval $[0 \quad 4]$.

The last case is well illustrated in the next example where the one dimensional state of the classic logistic equation is extended to include the fixed parameter, so

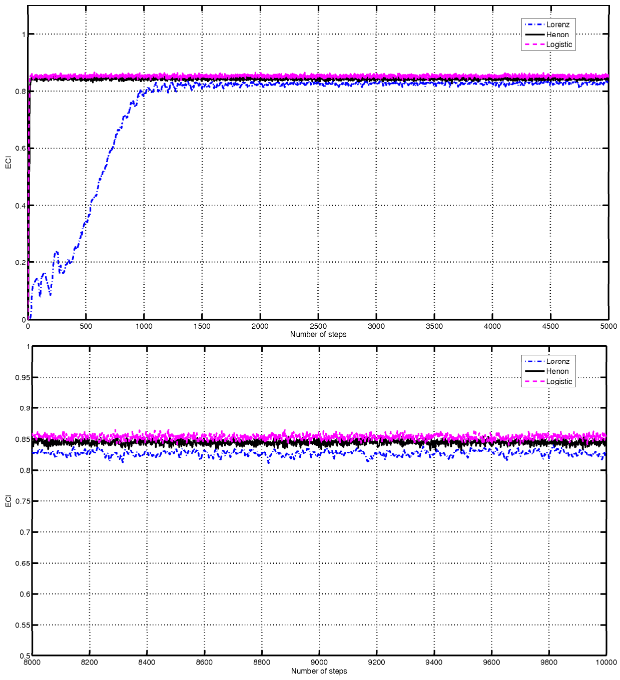

Figure 2. Comparison among the three different chaotic systems: Lorenz equations, Hénon map and the Logistic equation. The bottom figure zooms on the "steady state" values of the ECI, to enable a visual classification of the chaotic systems based on the correspondent ECI.

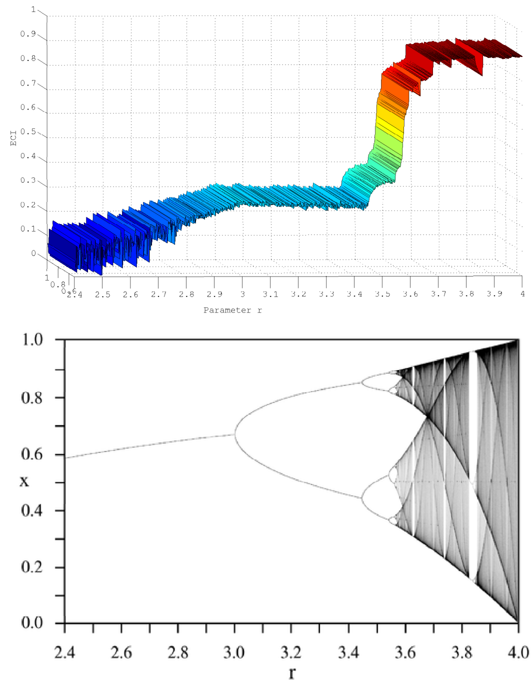

Figure 3. ECI computed for the logistic map as a function of the parameter is in accordance to the well known bifurcation diagram (taken from Wikipedia http://en.wikipedia.org/wiki/Logistic_map).

that the final behaviour of the system depends heavily on the initial conditions:

$$\begin{cases} x_1(k+1) = x_2(k) \cdot x_1(k)(1 - x_1(k)) \\ x_2(k+1) = x_2(k) \end{cases} . \quad (7)$$

Figure 3 shows the asymptotic value of the ECI as a function of the initial condition of the second state (i.e. the parameter) and proves that ECI quantifies the nonlinear content in accordance to the well known behaviour of the nonlinear map. The bifurcation diagram for interesting values of the parameter r (i.e. initial condition of the extended state x_2) is shown in the second part of figure 3 (taken from Wikipedia http://en.wikipedia.org/wiki/Logistic_map).

Another similar example is dedicated to the analysis of a conservative discrete dynamical system, the standard map [Contopolus and Voglis, 1999], described by equations

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k+1) & (\mod 1) \\ x_2(k+1) = x_2(k) + \frac{K}{2\pi}\sin(2\pi x_1(k)) & (\mod 1) \end{cases} , \quad (8)$$

where here $K = 2.5$ is used. In this case, we choose 20 random initial segments inside the unit square, and 1000 random points within each segment. Figure 4 shows the initial segments, and the trajectories followed by the end-points of the segments. To avoid confusion, not the whole trajectories are shown, but only points corresponding to the value of the state at each time step (in practice they are not connected with respect to time). Trajectories followed by the end-points of the segments lying initially inside the islands of stability can be followed easily by inspection, even without showing the connections, while points evolving in the chaotic region fill the region apparently homogeneously without following distinguishable orbits. In figure 5, the correspondent values of the ECI are shown, and as can be seen, it is possible to distinguish four sets of possible ECI evolutions, which have been clustered according to lighter to darker colours. It is remarkable that the value of the ECI is affected by how much the initial segment was immersed inside a chaotic region. The two segments completely inside the ordered regions, shown with pink solid line in figure 4, give rise to the two clearly lower ECI graphs in 5. The five dashed red segments (where approximately more than half of the segment was in the ordered region) and the two dashed blue segments (where approximately less than half of the segment was in the ordered region) provide the two intermediate sets of ECI values. Finally, all the other segments, represented with solid black lines, are completely inside chaotic regions and provide the highest values of the ECI. Also notice that the value of the ECI is not related to the system equations or parameters (which are the same in all the cases), or to the length of the initial random segments.

The main motivation of the indicator ECI is to quantify the amount of nonlinearity of a dynamical system;

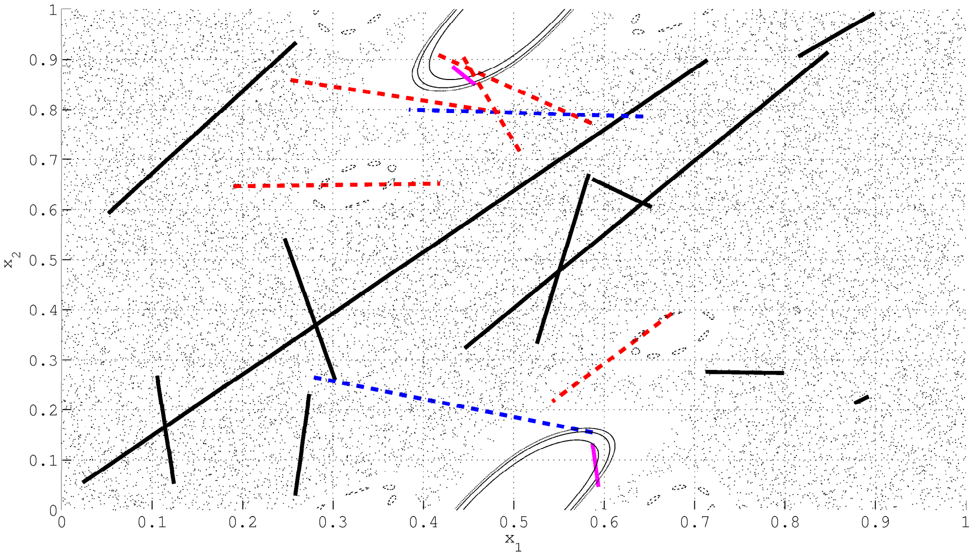

Figure 4. Initial random segments and the trajectories followed by their endpoints according to the standard map are shown. Initial points starting from non-chaotic regions clearly follow ordered trajectories.

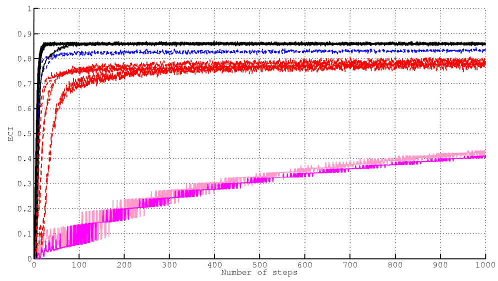

Figure 5. ECI values for each initial condition are shown. They clearly depend on the initial choice of the segment.

the previous example showed that this aspect can be exploited to distinguish the ordered from the chaotic regions within the same dynamical system. Next example also aims at validating the previous result, in the case of the so-called resonantly kicking oscillator [Daly and Heffernan, 1995]. Dynamical equations are

$$\begin{cases} x_1(k+1) = x_1(k)\cos(\beta) + x_2(k)\sin(\beta) + \\ \qquad +\mu\sin(2Kx_1(k))\sin(\beta) \\ x_2(k+1) = x_2(k)\cos(\beta) - x_1(k)\sin(\beta) + \\ \qquad +\mu\sin(2Kx_1(k))\cos(\beta) \end{cases}, \tag{9}$$

where parameters are chosen as $\mu = 6.5$, $K = 0.1$ and $\beta = \left(1+\sqrt{5}\right)\pi/2$. The phase space is characterised by an area around the origin characterised by ordered trajectories and surrounding regions where chaotic motion appears as shown in figure 6. Random initial segments are chosen inside a square of side 10 around the origin, and two of them are outside the ordered region, shown with dark color. The evolution of their ECI, represented with the same dark color, is clearly different from that of the other 8 segments, as it is clearly described in figure 7.

5 Conclusions

The main contribution of this paper is the use of an entropy of curves based indicator to describe and quantitatively compare well-known deterministic chaotic systems. Whenever it makes sense, the ECI asymptotic value can be associated to a dynamical system, while special care is devoted to the case when the behaviour of the system clearly depends on the initial conditions or on the values of some parameters. It is shown through the logistic equation that the second case can be seen as a special case of the first one, by extending the state and considering static dynamics for the parameters. Extensive simulations proved that ECI is correctly affected by the initial conditions, and that just by observing its evolution it is possible to infer whether the initial conditions belong to an ordered region or a chaotic one.

It is also opinion of the authors that here the asymptotic value of the ECI was used to make distinction among different dynamical systems or different initial conditions, but more information can be extracted from the ECI by observing for instance its evolution in the "transient" stage (e.g. its slope, or time required to reach the asymptotic value). Future work will focus on using this information together with the asymptotic value to infer more properties of the underlying dynamics.

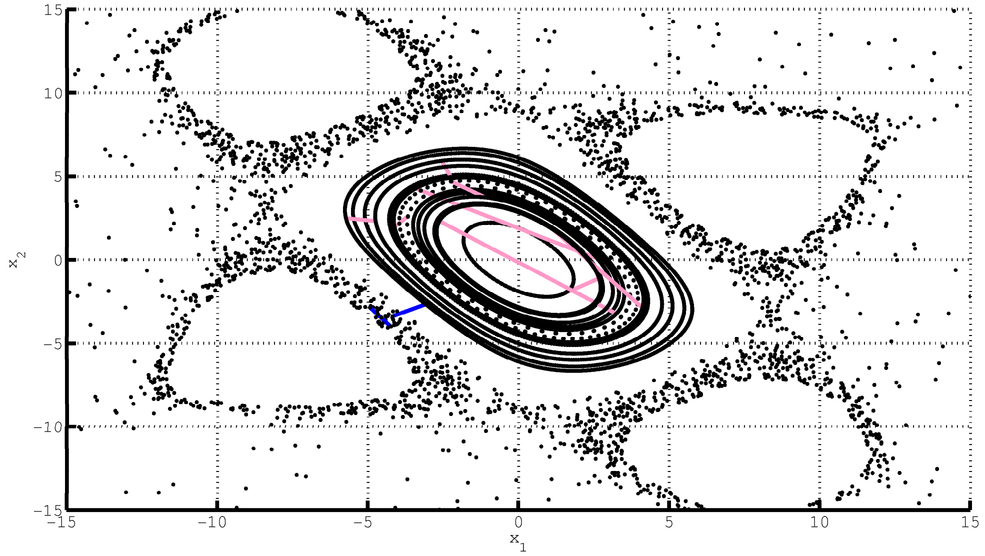

Figure 6. Initial random segments and the trajectories followed by their endpoints according to the equations of the kicking oscillator are shown.

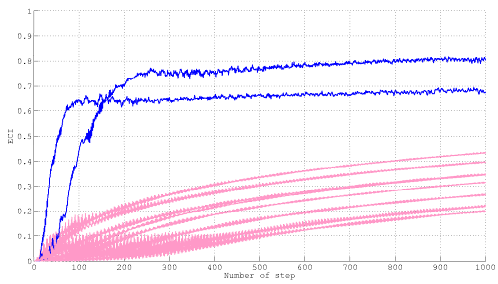

Figure 7. ECI values for each initial condition are shown. They are clearly affected by the initial choice of the segment.

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