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OPTIMIZATION APPROACH TO THE VELOCITY FIELD DETERMINING PROBLEM

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Abstract

A reconstruction of the velocity field is of current interest for different theoretical and practical problems. In the presented work the optimization method of the velocity field construction for successive images is developed and generalized using the concepts of trajectory beam (an ensemble of trajectories), a distribution density function (brightness), and a quality functional. A new formulation of the problem is proposed with the splitting of the initial image to subsets and taking into account a temporal variation.

Analytical expressions of the variation and gradient of the investigated functional are given, allowing one to construct various directed optimization methods. The proposed approach can be used for determining the velocity field within different tasks, particularly, for the problems of diagnostic image processing.

Key words

Velocity field, optical flow, trajectory beam, quality functional, functional variation, optimization, image processing.

1 Introduction

A problem of the velocity field reconstruction appears when solving many theoretical and applied tasks. There are physical experimental problems, investigations of complex mechanical systems, problems of electrodynamics and so on. Concerning image processing tasks the given problem is known as the problem of optical flow construction for successive images [Horn and Schunck,1981; Lucas and Kanade, 1981; Anandan, 1989; Fleet and Weiss, 2005; Papenberg, et.al., 2006]. Images processing and analysis including determining the velocity field are actual for medical diagnostics, robotics, computer vision, which includes methods of tracing objects in digital images, space researches, investigations of the arctic ice movement, as well as an analysis of transport flows [Kotina, Ploskikh and Shirokolobov, 2022; Gecha, et.al., 2020; Kopenkov and Myasnikov, 2014].

Different approaches are known from publications and widely used in practice [Barron and Fleet, 1994; Bruhn, Weickert and Schnorr, 2005; Sun, Roth and Black, 2010; Tu, et.al., 2019]. In the current work the problem of constructing the velocity field is considered as control and optimization problem [Ovsyannikov, 1980; Bazhanov, et al., 2018; Kotina and Ovsyannikov, 2021].

In our previous works [Ovsyannikov and Kotina, 2012; Kotina and Pasechnaya, 2015; Bazhanov, et al., 2018; Ovsyannikov and Kotina, 2012; Kotina, Leonova and Ploskikh, 2022] the problem of determining the velocity field was considered on the basis of continuous as well as discrete systems, particularly, for radionuclide medical images. In the current work, the problem of determining the velocity field is generalized on the basis of systems with continuous right part using fragmentation of an image to subareas and time variation. Analytical representations of a variation and a gradient of the minimized functional are given.

2 Problem statement

A problem of the velocity field reconstruction for a complex structure is investigated. Let a set $M_0 \subset R^n$, let us consider sets having nonzero Lebesgue measure M_0^i, $i = \overline{1, N}$, such that

$$\bigcup_i M_0^i \subset M_0,\ M_0^i \bigcap_{i \neq j} M_0^j = \oslash.$$

We assume that the velocity field for every M_0^i subset, is determined by a separate system of differential equations

$$\dot{x} = f(t, x, u^i), i = \overline{1, N}. \quad (1)$$

Here t is a time, $t \in [0, T]$, x is a vector of spatial coordinates, $x \in R^n$, u^i is a vector of parameters, $u^i \in U$, U is a compact in R^r, $f = f(t, x, u^i)$ is a n-dimensional vector function, continuous with its partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial \operatorname{div}_x f}{\partial x}$, ($\operatorname{div}_x f = \sum_{i=1}^{n} \frac{\partial fi}{\partial x_i}$), with respect to the all arguments.

For every system (1) initial conditions are given

$$x^i(0) = x_0^i \in M_0^i, i = \overline{1, N}, \quad (2)$$

where M_0^i is a set of initial values for the corresponding i for the system (1). We denote solutions of the system (1) with initial conditions (2) as

$$x_t^i = x^i(t) = x(t, x_0^i, u^i), x_0^i \in M_0^i, \\ i = \overline{1, N}. \quad (3)$$

We will refer to the set of this solutions with every i as a trajectory beam (or a beam), where trajectories outgo from the M_0^i set with a given parameter vector u^i.

For every i ($i = \overline{1, N}$) we will introduce the density distribution function $\rho^i = \rho^i(t, x)$, which plays a role of mass or charge density distribution in different tasks of mechanics and electrodynamics. It also may be considered as quantitative characteristics of an image (a brightness) depending on spatial coordinates and a time, or density of radiopharmaceutical when data of radionuclide studies are processed [Kotina and Ovsyannikov, 2018].

Liouville (transport) equation, which determines a variation of the density function in space over time when vector function $f(t, x, u^i)$ is given, has the form [Ovsyannikov, 1980; Brockett, 2007]:

$$\frac{\partial \rho^i(t, x)}{\partial t} + \frac{\partial \rho^i(t, x)}{\partial t} f(t, x, u^i) + \\ + \rho^i(t, x) \operatorname{div}_x f(t, x, u^i) = 0. \quad (4)$$

For this equation, the initial conditions are set:

$$\rho^i(0, x) = \rho_0^i(x), x \in M_0^i, \quad (5)$$

where $\rho_0^i(x)$ are the given functions, $i = \overline{1, N}$. For every i we assume that $\rho^i = \rho^i(t, x)$ function is given and it is necessary to reconstruct $f(t, x, u^i)$ functions, in which parameters $u_1^i, u_2^i, \ldots, u_r^i$ are considered to be unknown. So, we have to determine N parameter vectors u^i, $i = \overline{1, N}$.

Let us consider a problem of u^i parameter vectors reconstruction as an optimization task. For this purpose we will use optimization methods for charged particle beams dynamics presented in [Ovsyannikov, 1980; Ovsyannikov, 1990].

We denote a trajectory beam cross-section of system (1) at timepoint t for the fixed u^i vector as M_{t,u^i}^i, i.e., the set

$$M_{t,u^i}^i = \left\{ x^i(t) = x(t, x_0^i, u^i), x_0^i \in M_0^i \right\}, \\ i = \overline{1, N}. \quad (6)$$

Let $\rho_0(x)$ be known density distribution at timepoint $t = 0$ in M_0, while $\rho_0^i(x)$ is a narrowing of the given function in the M_0^i set. Then we assume that we know the density $\hat{\rho}(x)$, $x \in R^n$, characterizing the density changed in time Δt. We denote the timepoint as $T = \Delta t$.

The task is to find such u^i parameter vectors that at the timepoint T the density calculated using equation (4) under the condition of (5) is equal to the $\hat{\rho}(x)$ density in M_{T,u^i}^i, i.e.

$$\rho^i(T, x) = \hat{\rho}(x), x \in M_{T,u^i}^i, i = \overline{1, N}. \quad (7)$$

Let us formulate the optimization problem. For this purpose, we introduce a functional

$$J(u) = \sum_{i=1}^{N} \int_{M_{T,u}^i} g(x_T, \rho^i(T, x_T)) \, \mathrm{d}x_T, \quad (8)$$

here u is a vector composed from vectors u^i, $u = (u^1, u^2 \ldots, u^N)$, M_{T,u^i}^i is a trajectory beam cross-section at the timepoint $t = T$, $g = g(x_T, \rho^i(T, x_T))$ is a non-negative, continuously differentiable function with respect to the all arguments, characterizing the condition (7) in one form or another.

The problem of functional (8) minimization is stated. We assume that the timepoint T is not fixed here and will be varied as well.

Solving functional (8) minimization task and determining parameter vector u we solve the task of $f(t, x, u^i)$, $i = \overline{1, N}$, functions restoring, i.e., we determine the velocity field for the all subdomains.

3 Functional gradient

An increment of the functional has a following form

$$\Delta J = \delta J + o(\|u\| + |\Delta T|),$$

where $\|u\|$ is a vector norm.

Following the work [Ovsyannikov, 1990] let us present the functional (8) variation in the form

$$\delta J = -\sum_{i=1}^{N} \int_0^T \int_{M_{T,u^i}^i} (\psi^{i^*}(t, x_t) \Delta_{u^i} f(t, x_t, u^i) + \\ + \lambda^i(t, x_t) \Delta_{u^i} \operatorname{div}_x f(t, x_t, u^i)) \, \mathrm{d}x_t \mathrm{d}t - \\ - \Delta T \sum_{i=1}^{N} \int_{M_{T,u^i}^i} H^i(T, x_T, \lambda^i(T, x_T), \psi(T, x_T), u^i) \, \mathrm{d}x_t. \quad (9)$$

Here

$$H^i = \psi^{i^*}(T, x^i(T)) f(T, x^i(T), u^i) + \\ +\lambda^i(t, x^i(T)) \operatorname{div}_x f(t, x^i(T), u^i).$$

Auxiliary functions $\psi^i(t,x)$ and $\lambda^i(t,x)$ for $i = \overline{1,N}$ satisfy equations along trajectories of the system (1)

$$\frac{d\psi^i}{dt} = -(\frac{\partial f(t, x^i(t), u^i)}{\partial x} + \\ +E \operatorname{div}_x f(t, x^i(t), u^i))^* \psi^i - \\ -\lambda^i(\frac{\partial \operatorname{div}_x f(t, x^i(t), u^i)}{\partial x})^*, \quad (10)$$

$$\frac{d\lambda^i}{dt} = -\lambda^i \operatorname{div}_x f(t, x^i(t), u^i) \quad (11)$$

under the final conditions

$$\psi^{i^*}(T, x^i(T)) = -\frac{\partial g(x^i(T), \rho^i(T, x^i(T))}{\partial x}, \quad (12)$$

$$\lambda^i(T, x^i(T)) = -g(x^i(T), \rho^i(T, x^i(T))) + \\ +\frac{\partial g(x^i(T), \rho^i(T, x^i(T))}{\partial \rho} \rho^i(T, x^i(T)). \quad (13)$$

Meanwhile

$$\Delta_{u^i} f(t, x_t, u^i) = f(t, x_t, u^i + \Delta u^i) - f(t, x_t, u^i), \\ \Delta_{u^i} \operatorname{div}_x f(t, x_t, u^i) = \operatorname{div}_x f(t, x_t, u^i + \Delta u^i) - \\ - \operatorname{div}_x f(t, x_t, u^i).$$

The derivation of the variation (9) is based on the use of a trajectory beams cross-sectional transformation [Ovsyannikov, 1980; Ovsyannikov, 1990] applying variation equations for equations (1), (4).

Based on the above considerations and the presentation of the variation (9) we can formulate the following theorem.

Theorem.

Let function f be differentiable with respect to u^i, U is a convex set. Then the expressions for the gradient components of the functional (8) has the form

$$\frac{\partial J}{\partial u^i} = -\int_0^T \int_{M^i_{T,u^i}} (\psi^{i^*}(t, x_t) \frac{\partial f(t, x_t, u^i)}{\partial u} + \\ +\lambda^i(t, x_t) \frac{\partial \operatorname{div}_x f(t, x_t, u^i)}{\partial u}) \, \mathrm{d}x_t \mathrm{d}t, \quad (14)$$

$$\frac{\partial J}{\partial T} = \\ = -\sum_{i=1}^{N} \int_{M^i_{T,u^i}} H(T, x_T, \lambda^i(T, x_T), \psi^i(T, x_T)) \, \mathrm{d}x_T, \\ i = \overline{1,N}. \quad (15)$$

The obtained equations (14)-(15) can be used to implement the optimization algorithm of the velocity field reconstruction. On the basis of the functional gradient equation one can build different methods for a directed search of the parameter vector u.

4 Linear model

Obviously, when processing images, $f(t, x, u^i)$ functions form is unknown. Let us consider function f as a linear vector function, i.e.

$$\dot{x} = A^i x + C^i, i = \overline{1,N}, \quad (16)$$

where A^i are square matrices: $A^i = \{a^i_{lm}\}^n_{l,m=1}$, C^i are vectors: $C^i = \{c^i_k\}^n_{k=1}$. The parameter vector u^i consists of matrix A^i and vector C^i components. Finding this components determine the system (16) and thereby gives us the required velocity field.

Let us further assume $n = 2$ that corresponds to planar images. Then the system (16) is a system of linear differential equations of the second order. As function g in functional (8) we consider the function

$$g = (\rho^i(T, x^i(T)) - \hat{\rho}(x^i(T)))^2, \quad (17)$$

where $\hat{\rho}(x)$ is a known density in R^2.

It should be noted that generally a problem formulation suggests a possibility of density (brightness) distribution change along the trajectories of the system (1) [Ovsyannikov and Kotina, 2012]. On the basis of equations (1), (4), this changes can be presented as follows

$$\rho^i(t, x^i(t, x^i_0, u^i)) = \rho^i_0(x^i_0) e^{-\int_0^t \operatorname{div}_x f(\tau, x^i, u^i) \, \mathrm{d}\tau}.$$

We write a functional (8) gradient with respect to required parameters for a linear case and the function g given by formula (17)

$$\frac{\partial J}{\partial a^i_{kk}} = -\int_0^T \int_{M^i_{T,u^i}} [\psi^i_k x_k + \lambda^i] \, \mathrm{d}x_t \mathrm{d}t, \\ k = \overline{1,2}, \quad (18)$$

$$\frac{\partial J}{\partial a^i_{kj}} = -\int_0^T \int_{M^i_{T,u^i}} \psi^i_k x_j \, \mathrm{d}x_t \mathrm{d}t, \\ i \neq j, k, j = \overline{1,2}, \quad (19)$$

$$\frac{\partial J}{\partial c^i_k} = -\int_0^T \int_{M^i_{T,u^i}} \psi^i_k \, \mathrm{d}x_t \mathrm{d}t, \\ k = \overline{1,2}, \quad (20)$$

$$\frac{\partial J}{\partial T} = -\sum_{i=1}^{N} \int_{M^i_{T,u^i}} [(a^i_{11}x_1 + a^i_{12}x_2 + c^i_1)\psi^i_1 + \\ +(a^i_{21}x_1 + a^i_{22}x_2 + c^i_2)\psi^i_2 + \\ +\lambda^i(a^i_{11} + a^i_{22})]\, \mathrm{d}x_T. \tag{21}$$

Equations (10) and (11) have the following form

$$\frac{d\psi^i}{dt} = -[A^i + E(a^i_{11} + a^i_{22})]^*\psi^i, \tag{22}$$

$$\frac{d\lambda^i}{dt} = -(a^i_{11} + a^i_{22})\lambda^i \tag{23}$$

with final conditions

$$\psi^{i^*}(T, x^i(T)) = -2(\rho^i(T, x^i(T)) - \hat{\rho}(x^i(T)))\times \\ \times(\frac{\partial\rho^i(T, x^i(T)}{\partial x} - \frac{\partial\hat{\rho}(x^i(T))}{\partial x}), \tag{24}$$

$$\lambda^i(T, x^i(T)) = \rho^i(x^i(T))^2 - \hat{\rho}(x^i(T))^2. \tag{25}$$

In formulas (18) - (25) i changes from 1 to N ($i = \overline{1,N}$).

In the case of an optical flow, we have $\mathrm{div}_x\, f(t, x, u^i) = 0$, therefore, within the linear model the equality $a^i_{11} = -a^i_{22}$ has to be fulfilled for all i.

Then the equation (18) will be considered only for $k = 1$. The equations (19) and (20) remain unchanged, while the equation (21) is reduced to the following

$$\frac{\partial J}{\partial T} = -\sum_{i=1}^{N} \int_{M^i_{T,u^i}} [(a^i_{11}x_1 + a^i_{12}x_2 + c^i_1)\psi^i_1 + \\ +(a^i_{21}x_1 - a^i_{11}x_2 + c^i_2)\psi^i_2]\, \mathrm{d}x_T.$$

Meanwhile the equation (22) takes a form

$$\frac{d\psi^i}{dt} = -A^{i^*}\psi^i$$

with the final condition (24).

In this case it is following from the equation (23), that λ^i is equal to a constant and can be calculated using the formula (25). Thus, when considering only the optical flow one can obtain simpler calculation formulas and a number of unknown parameters is reduced.

5 Conclusion

The approach proposed in the article gives us new opportunities to construct the velocity field for the case of an optical flow, as well as for the case of a non-optical flow. Within the proposed approach the set M_0 is split to subsets. It is due to difficulties of determining the velocity field given by a single system in the whole area, since the given velocity field may be essentially nonlinear. Splitting the set to subsets we have an ability to obtain the velocity field determined by a separate system for every subset, that can essentially simplify a solution process for the given task. In particular, an ability to use the linear model more effectively appears. Splitting to subsets allows one to point out significant image areas and to search the corresponding velocity field for each of them.

Note, that in this work the difference between systems (1) consists only in the vector u^i selection. However, systems (1) can be different for different subsets, i.e., not only parameter vector selection can be different, but also a form of function f. Then instead of the $f(t, x, u^i)$ function one should consider the $f^i(t, x, u^i)$ function in all corresponding formulas.

An application of the described algorithm can be useful for different image processing fields, such as movement detection and its correction, tracing movement trajectory, constructing contours on an image, analyzing images and so on. Given approach is of a particular interest for processing radionuclide research data [Ovsyannikov, Kotina and Shirokolobov, 2013; Ploskikh and Kotina, 2021; Kotina, Ploskikh and Shirokolobov, 2022], for obtaining visual as well as quantitative information when diagnostic images are analyzed.

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