PROPERTIES OF THE LAPLACIAN SPECTRA OF CERTAIN BASIC AND HIERARCHICAL GRAPHS

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Abstract

The study of spectra of Laplacian matrices is important in decentralized optimization and multi-agent control problems. Namely, the largest and the smallest nonzero eigenvalues significantly affect both the stability of decentralized algorithms and their convergence rate. In this paper, we study the Laplacian spectra of some basic graphs and hierarchical graphs obtained from them. Explicit expressions for the eigenvalues of interest are given and analyzed.

Key words

undirected graph, Laplacian matrix, Laplacian spectrum, multi-agent systems, decentralized optimization

1 Introduction

Decentralized approaches to control and optimization in complex systems consisting of autonomous subsystems (or agents) are an active research topic and have many practical applications. The idea of interactions of this kind, alternative to the classical "centralized" approach, is to achieve a global goal through local interactions, see [Proskurnikov and Fradkov, 2016], [Chen et al., 2019], [Bullo, 2022] and references therein. In a centralized system, one central agent that has access to all information about the system makes decisions for all agents. In this way, the decision to cooperate in the system is obtained directly and all interactions are optimally taken into account. The basic features of multi-agent systems are autonomy of agents, local interactions between them without the use of global information about the system as a whole, and decentralization, i.e., the absence of a central regulator or module producing common solutions for all agents. Multi-agent control protocols find their application in many tasks of cyber-physical systems that bring together sensing, computation, control, and networking. In particular, such decentralized controllers are used in the problems of formation control and vehicle platooning.

The flexibility and cost-effectiveness of decentralized solutions compared to the classical centralized ones have led to the widespread application of multi-agent systems in various fields of science and industry, as well as to the rapid development of the corresponding mathematical theory.

In a multi-agent system, each agent interacts only with a limited number of neighbors, so it is convenient to describe the pattern of such interactions by means of a graph. In both decentralized optimization problems and decentralized control problems, such basic properties of the system as its stability and convergence rate depend on the spectrum of the Laplacian matrix of the graph (see, e.g., [Borrelli and Keviczky, 2008], [Nedić et al., 2018], [Granichin et al., 2020], [Zhu et al., 2022], [Gorbunov et al., 2022], [Erofeeva and Kizhaeva, 2023], [Uzhva and Granichin, 2021], [Amelin and Ershov, 2021]). For a particular graph, it is not difficult to find the spectrum, but if we consider graphs constructed in some regular way, for example, by repeating the same fragment, the problem of exact computation or localization of spectra turns out to be nontrivial. In this direction, there are studies devoted to obtaining expressions for the Laplacian spectra of undirected topologies, including various lattices [Pozrikidis, 2014], hierarchical small-world networks [Liu et al., 2015], products [Kammerdiner et al., 2017] and coronas of graphs [Barik and Sahoo, 2017], and many others. In the paper, we find explicit expressions and analyze the behavior of the largest and smallest nonzero eigenvalues of the Laplacian matrices of certain undirected hierarchical graphs.

The paper is organized as follows. The preliminary information and problem statement are given in Section 2. Basic graph topologies are described and studied in 3, followed by hierarchical ones given in Section 4. In Section 5, we analyze and discuss the results obtained in terms of stability and convergence rate of network dynamics problems. Section 6 concludes the paper.

2 Preliminaries and Problem Statement

In this paper, we study certain basic and hierarchical networks that have scalable structure and Laplacian spectra of the corresponding graphs. After introducing the terminology, we formulate the problem.

Throughout the paper, we consider finite unweighted graphs without multiple edges and loops. A graph is denoted by $\mathcal{G}_N = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ stands for the node set and $\mathcal{E}$ for the set of edges.

The formal definition of the Laplacian matrix of an unweighted graph $\mathcal{G}_N$ is given below.

Definition 1. *The Laplacian matrix $\mathcal{L}_N \in \mathbb{R}^{N\times N}$ of $\mathcal{G}_N$ is the matrix with entries l_{ij} given by*

$$l_{ij} = \begin{cases} -1 & \text{if } (i,j) \in \mathcal{E}, \\ d_i & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

where d_i is the degree of the i^{th} vertex.

Denote the eigenvalues of Laplacian matrix $\mathcal{L}_N$ by $\lambda_1, \dots, \lambda_N$. It is known that if graph $\mathcal{G}_N$ is connected, then $\lambda_{\min} = 0$ and all other eigenvalues of $\mathcal{L}$ are in the open right half of the complex plane (see, e.g., [Lewis et al., 2013]). We define an eigenvalue with the maximum value as $\lambda_{\max}$, and the smallest non-zero eigenvalue as $\lambda^{+}_{\min}$.

Definition 2. *The circulant matrix $\mathcal{C}_N$ is a Toeplitz matrix having the form*

$$\mathcal{C}_N = \begin{bmatrix} c_1 & c_2 & \cdots & c_N \\ c_N & c_1 & \cdots & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix} \quad (1)$$

with the rows formed by the vector $\mathbf{c} = [c_1, c_2, \dots, c_N]$ and its $N-1$ circular permutations.

We define such matrices as $\mathrm{circ}(\cdot)$; i.e.,

$$\mathcal{C}_N := \mathrm{circ}(\mathbf{c}).$$

2.1 Motivation

The spectrum of Laplacian matrices is important in estimating the convergence rate of decentralized optimization methods, as well as for the analysis of consensus reachability and convergence rate to consensus in multi-agent systems. In the following, we provide the main motivations for studying Laplacian spectra from the perspectives of decentralized optimization and control.

2.1.1 Decentralized Optimization. Consider a network system of N nodes connected through an undirected graph $\mathcal{G}_N$ referred to as the network topology. Each node $i \in \mathcal{V}$ has a local objective function $f_i(\mathbf{x})$ that depends on the optimization variable $\mathbf{x}$. The objective is to find a minimizer $\mathbf{x}^\star$ of the *global* objective function $F(\mathbf{x})$ defined as

$$F(\bar{\mathbf{x}}) = \sum_{i=1}^{N} f_i(\mathbf{x}_i), \quad \text{s.t. } \mathbf{x}_1 = \ldots = \mathbf{x}_N, \quad (2)$$

where $\bar{\mathbf{x}} = [\mathbf{x}_1, \dots, \mathbf{x}_N]^\top$.

We consider a common decentralized optimization workflow: Each node maintains its own optimization variable, which is repeatedly updated and aligned with its neighbors to obtain a solution that minimizes the global objective function. Specifically, at iteration $t = 0, 1, 2, \ldots$, node i updates its optimization variable as follows:

$$\mathbf{x}_i^{(t+1)} = \mathbf{x}_i^{(t)} - \text{Optimize}(\mathbf{x}_i^{(t)}) - \text{Consensus}(\mathcal{W}, \mathbf{x}_i^{(t)}). \quad (3)$$

Here, $\text{Optimize}(\mathbf{x}_i^{(t)})$ is an optimization step, i.e., gradient descent iteration at the point $\mathbf{x}_i^{(t)}$; $\text{Consensus}(\mathcal{W}, \mathbf{x}_i^{(t)})$ is a procedure that maintains the equality constraint in (2). The consensus involves the information exchange with the neighboring nodes, i.e. $\mathcal{V}_i = \{k \in \mathcal{V} : (k,i) \in \mathcal{E}\}$, through the network topology with the corresponding communication matrix $\mathcal{W}$ represented by a mixing matrix or Laplacian matrix (see [Gorbunov et al., 2022] for more details).

As presented in [Gorbunov et al., 2022], the convergence depends on the spectral properties of the underlying communication matrix:

- when $\mathcal{W}$ is a mixing matrix, consensus scheme $\bar{\mathbf{x}}^{(t+1)} = \mathcal{W}\bar{\mathbf{x}}^{(t)}$ requires $\mathcal{O}\left(\frac{1}{1-\lambda_2(\mathcal{W})}\log(\frac{1}{\varepsilon})\right)$ iterations to achieve accuracy ε;
- when $\mathcal{W}$ is a Laplacian matrix, the consensus scheme achieved through the optimization of the objective function $\bar{\mathbf{x}}^\top\mathcal{W}\bar{\mathbf{x}}$, e.g., $\bar{\mathbf{x}}^{(t+1)} = \bar{\mathbf{x}}^{(t)} - \frac{1}{\lambda_{\max}}\mathcal{W}\bar{\mathbf{x}}^{(t)}$, requires $\mathcal{O}\left(\frac{\lambda_{\max}(\mathcal{W})}{\lambda^{+}_{\min}(\mathcal{W})}\log(\frac{1}{\varepsilon})\right)$ iterations to achieve accuracy ε.

In this paper, we analyse how the iteration complexity scales with the number of nodes N through the properties of the communication matrix $\mathcal{W}$ represented by the *condition number* $\chi = \frac{\lambda_{\max}(\mathcal{W})}{\lambda^{+}_{\min}(\mathcal{W})}$.

2.1.2 Multi-Agent Systems. The eigenvalue $\lambda^{+}_{\min}$ determines the exponential rate of convergence in first-order linear multi-agent systems with dynamics of the form

$$\dot{\mathbf{x}} = -\mathcal{L}_N\mathbf{x}, \quad (4)$$

where $\mathbf{x} = [x_1, x_2, \dots, x_N]^\top$, $x_i(t) \in \mathbb{R}$ is the position of each agent at time $t \geq 0$, $i \in \{1, \dots, N\}$. The matrix $\mathcal{L}_N$ is the Laplacian of the corresponding graph of communications. Thus, the equation of dynamics of each agent has the form of a first-order equation (single integrator), the right-hand side of which is a feedback in terms of deviations from the states of its neighboring agents.

When the agent models are of order two or higher, it is not enough to achieve consensus for the graph to be connected or for the zero eigenvalue of the Laplacian matrix to be simple. In such cases, it is checked whether all non-zero eigenvalues of the Laplacian matrix belong to the so-called consensus region or not, see [Polyak and Tsypkin, 1996], [Hara et al., 2014], [Lewis et al., 2013], [Proskurnikov and Fradkov, 2016], [Li and Duan, 2017]. Since in the case of undirected graphs we deal with real spectra, it is important for us to find the endpoints $\lambda^+_{\min}$, $\lambda_{\max}$ and check if the segment $[\lambda^+_{\min},\ \lambda_{\max}]$ lies in the consensus region or not. Hereafter in the text, $\Delta = \lambda_{\max} - \lambda^+_{\min}$ denotes the length of such a segment.

In case of higher-order agents (e.g., in formation control problems), the following situation may arise: If the number of agents in the platoon increases, the spectrum of the new Laplacian matrix is no longer located in the consensus region. This phenomenon is referred to as eventual instability [Stüdli et al., 2017]. An example of such stability loss is shown in [Parsegov et al., 2023].

2.2 Problem Statement

The goal of this paper is to study the behavior of the largest and smallest nonzero eigenvalues of the Laplacian matrices of specific undirected unweighted graphs, as well as a comparative analysis of the behavior of $\lambda^+_{\min}$ and $\lambda_{\max}$.

The following connected *basic* graphs are considered in this paper: cycle graphs, path graphs, rectangular lattices, stars and wheels. Each of such graphs has the scalability property, i.e., increasing the number of nodes does not change the overall graph nature. In addition, based on the basic graphs, we introduce the hierarchy and construct the corresponding graphs.

By *hierarchical* network we define a two-layer network involving interactions both between the agents within a subgroup and between the groups. The corresponding graph obtained as the Cartesian product of the subgroup graph and the subgroup interaction graph we also refer to as *hierarchical*.

3 Basic Topologies

Let us introduce some additional definitions necessary for further explanation.

Definition 3.

1. *The cycle graph is a graph containing a single cycle through all nodes. It can be also defined as a connected graph that is regular of degree 2.*
2. *The path graph on N vertices is a graph obtained from the cycle graph by removing an edge.*
3. *The graph obtained from the cycle graph on $N-1$ vertices by connecting each vertex to a new "hub" vertex is the wheel on N vertices.*
4. *The star graph of order N is a tree on N nodes with one node (the central vertex or "hub") having vertex degree $N-1$ and the other $N-1$ having vertex degree 1.*

Let us show next how the graphs and the corresponding Laplacian matrices look like.

Path graph, $N \geq 2$

The path graph on four vertices is shown in Fig. 1a. The general Laplacian matrix is a tridiagonal matrix of the form

$$\mathcal{L}_N^{\text{path}} = \begin{bmatrix} 1 & -1 & 0 \cdots & 0 \\ -1 & 2 & -1 \cdots & 0 \\ 0 & -1 & 2 \cdots & 0 \\ \vdots & \vdots & \ddots \quad \ddots & \vdots \\ 0 \cdots & & 0 \ -1 & 1 \end{bmatrix}.$$

The Laplacian spectrum of the path graph is of the form [Bullo, 2022]

$$\lambda_k^{\text{path}} = 2 - 2\cos\frac{\pi(k-1)}{N}, k \in \{1, \dots, N\}.$$

It follows from the properties of the cosine function that $\lambda_{\min}^{\text{path}\,+} = 2 - 2\cos\frac{\pi}{N}$, $\quad \lambda_{\max}^{\text{path}} = 2 - 2\cos\frac{\pi(N-1)}{N}$.

Evidently, $\lambda_{\min}^{\text{path}\,+} \to 0$ and $\lambda_{\max}^{\text{path}} \to 4$, as $N \to \infty$.

Cycle graph, $N \geq 3$

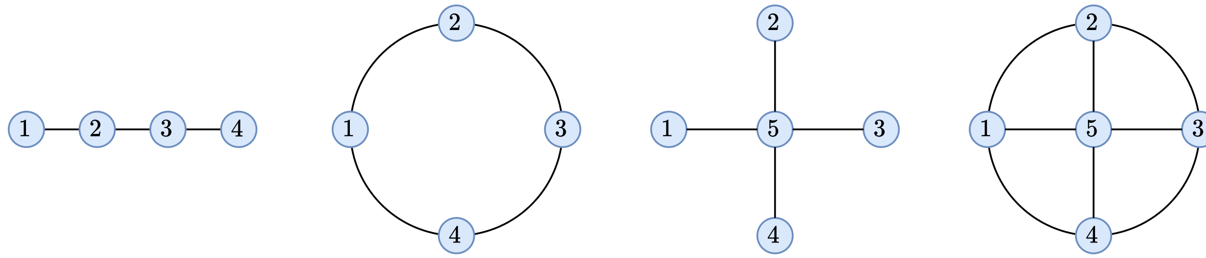
The cycle graph on four vertices is shown in Fig. 1b. Its Laplacian matrix is a circulant matrix given below:

$$\mathcal{L}_N^{\text{cycle}} = \text{circ}(2, -1, 0, \dots, 0, -1) = \begin{bmatrix} 2 & -1 & 0 \cdots & -1 \\ -1 & 2 & -1 \cdots & 0 \\ 0 & -1 & 2 \cdots & 0 \\ \vdots & \vdots & \ddots \quad \ddots & \vdots \\ -1 \cdots & & 0 \ -1 & 2 \end{bmatrix}.$$

The Laplacian spectrum of the cycle graph is given by [Bullo, 2022]

$$\lambda_k^{\text{cycle}} = 2 - 2\cos\frac{2\pi(k-1)}{N}, k \in \{1, \dots, N\}.$$

It follows from the properties of the cosine function that $\lambda_{\min}^{\text{cycle}\,+} = 2 - 2\cos\frac{2\pi}{N}$, $\quad \lambda_{\max}^{\text{cycle}} = 4$ (N is even), or $\lambda_{\max}^{\text{cycle}} = 2 - 2\cos\frac{\pi(N-1)}{N}$ (N is odd). Evidently, $\lambda_{\min}^{\text{cycle}\,+} \to 0$ and $\lambda_{\max}^{\text{cycle}}$ is equal or tends to 4, as $N \to \infty$.



(a) Path graph (b) Cycle graph (c) Star graph (d) Wheel

Figure 1. Basic graphs on four (a), (b) and five (c), (d) vertices.

Star graph, $N \geq 4$

The star graph on five nodes is shown in Fig. 1c. The Laplacian matrix has the following form

$$\mathcal{L}_N^{\text{star}} = \left[\begin{array}{c|c} N-1 & -\mathbf{1}_{N-1}^\top \\ \hline -\mathbf{1}_{N-1} & I_{N-1} \end{array}\right], \tag{5}$$

where $\mathbf{1}_{N-1} = [1, 1, \ldots, 1]^\top \in \mathbb{R}^{N-1}$ and $I_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ is the identity matrix. The eigenvalues of this matrix are [Bullo, 2022]

$$\lambda_1^{\text{star}} = 0,\ \lambda_k^{\text{star}} = 1, k \in \{2, \ldots, N-1\},\ \lambda_N^{\text{star}} = N.$$

Thus, the Laplacian matrix of the star graph has algebraic connectivity always constant and equal to unity $\lambda_{\min}^{\text{star}\,+} = 1$, and the largest eigenvalue $\lambda_{\max}^{\text{star}} = N$.

Wheel graph, $N \geq 4$

The wheel graph on five nodes is shown in Fig. 1d. The Laplacian matrix of the wheel is of the form [Ipsen and Mallik, 2023]

$$\mathcal{L}_N^{\text{wheel}} = \left[\begin{array}{c|c} N-1 & -\mathbf{1}_{N-1}^\top \\ \hline -\mathbf{1}_{N-1} & \mathcal{B}_{N-1} \end{array}\right], \tag{6}$$

where $\mathbf{1}_{N-1} = [1, 1, \ldots, 1]^\top \in \mathbb{R}^{N-1}$ and $\mathcal{B}_{N-1} = \text{circ}(3, -1, 0, \ldots, 0, -1) \in \mathbb{R}^{(N-1)\times(N-1)}$ is the circulant matrix. The eigenvalues of the Laplacian matrix are [Alotaibi et al., 2023] $\lambda_1^{\text{wheel}} = 0$, $\lambda_k^{\text{wheel}} = 3 - 2\cos\frac{\pi(k-1)}{N-1}, k \in \{2, \ldots, N-1\}, \lambda_N^{\text{wheel}} = N$.

Here, $\lambda_{\min}^{\text{wheel}\,+} = 3 - 2\cos\frac{\pi}{N-1} \to 1$, as $N \to \infty$. The largest eigenvalue is $\lambda_{\max}^{\text{wheel}} = N$.

4 Hierarchical Topologies

Hierarchies are often found in control problems of large-scale network dynamical systems with multi-layer structure, e.g., see [Williams et al., 2004], [Smith et al., 2005], [Hara et al., 2009], [Mukherjee and Ghose, 2016]. The two-layer hierarchical graphs considered below are obtained using the Cartesian product of certain basic graphs.

It is known that for two graphs $\mathcal{G}_n$ and $\mathcal{G}_m$ with the corresponding Laplacian matrices $\mathcal{L}_n$ and $\mathcal{L}_m$, the Cartesian product has the following property: the Laplacian matrix of the Cartesian product of both graphs is given by the Kronecker sum

$$\mathcal{L}_n \oplus \mathcal{L}_m = \mathcal{L}_n \otimes I_m + I_n \otimes \mathcal{L}_m.$$

Let us first formulate the following lemma necessary to analyze the spectra of Laplacian matrices of the obtained hierarchical graphs.

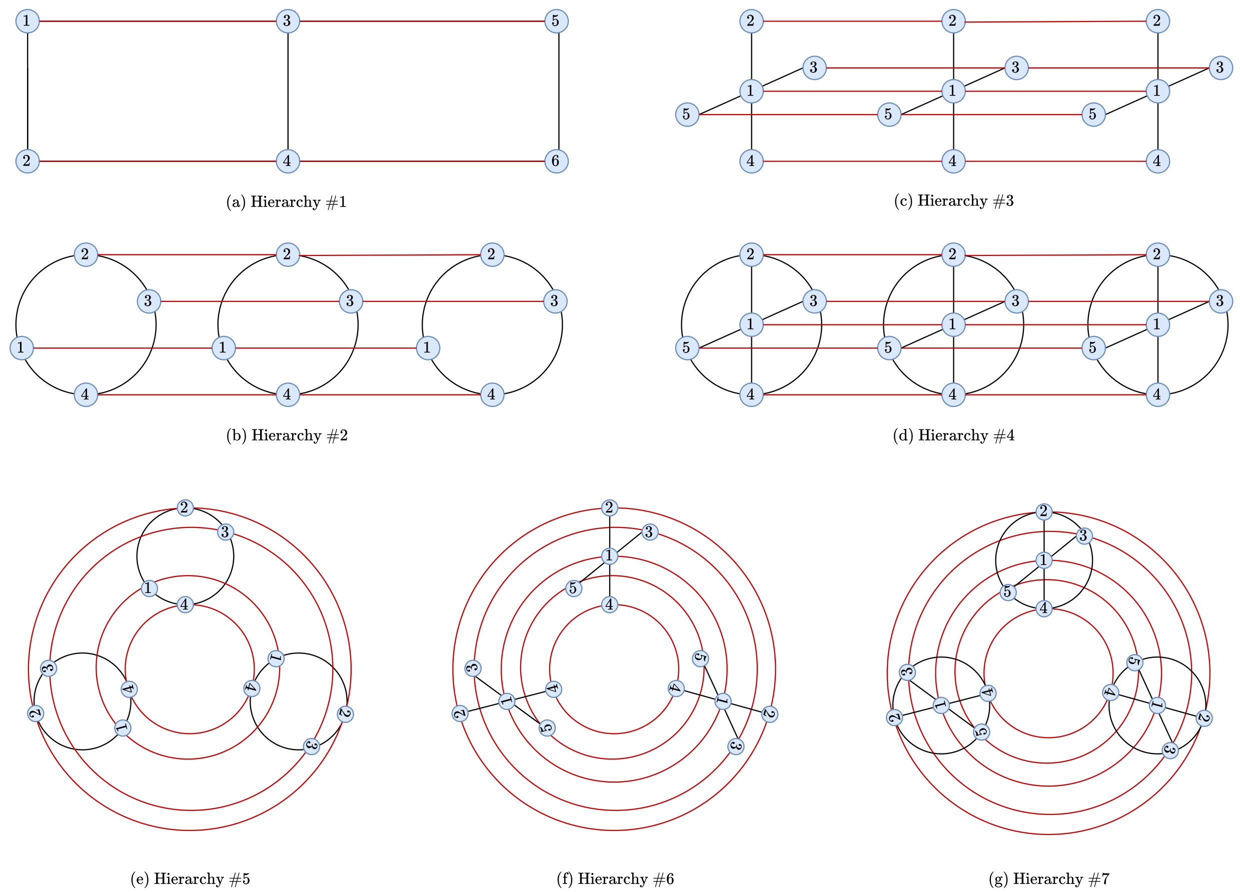
Lemma 1. *Let $\mathcal{L}_n \in \mathbb{R}^{n\times n}$, $\mathcal{L}_m \in \mathbb{R}^{m\times m}$ be two Laplacian matrices of connected undirected graphs, and λ_i, $i \in \{1, \ldots, n\}$, μ_j, $j \in \{1, \ldots, m\}$, be their eigenvalues, respectively. Then the minimum positive $\lambda_{\min}^{\text{KS}\,+}$ and maximum $\lambda_{\max}^{\text{KS}}$ eigenvalues of the Kronecker sum $\mathcal{L}_n \oplus \mathcal{L}_m$ satisfy*

$$\lambda_{\min}^{\text{KS}\,+} = \min\{\lambda_{\min}^+,\ \mu_{\min}^+\},\ \lambda_{\max}^{\text{KS}} = \lambda_{\max} + \mu_{\max}.$$

Proof. The idea of the proof is based on the properties of the spectrum of the Kronecker sum of matrices [Laub, 2004] and the positive semidefiniteness of Laplacian matrices. First, by Kronecker sum properties, the eigenvalues of the resulting matrix will be pairwise sums $\lambda_i + \mu_j$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$. It follows that the largest eigenvalue of the new matrix will be the sum of the maximum eigenvalues of the matrices $\mathcal{L}_n$ and $\mathcal{L}_m$. Second, since the spectrum of any Laplacian matrix of a connected graph contains a simple zero eigenvalue, the smallest positive eigenvalue of their Kronecker sum is $\lambda_{\min}^{\text{KS}\,+} = \min\{\lambda_{\min}^+,\ \mu_{\min}^+\}$. This concludes the proof.

Next, we consider several hierarchical graphs on $N = m \cdot n$ nodes obtained as Cartesian products of some basic graphs. For each of these graphs, we present its graphical illustration and the form of corresponding Laplacian matrix, see Fig. 2 (inter-layer edges are in black, intra-layer edges are given in red). Then we formulate a theorem on the eigenvalues of the matrices.

First, let us consider the graph obtained as the Cartesian product of two path graphs.



(a) Hierarchy #1
(b) Hierarchy #2
(c) Hierarchy #3
(d) Hierarchy #4
(e) Hierarchy #5
(f) Hierarchy #6
(g) Hierarchy #7

Figure 2. Hierarchical topologies.

Hierarchy #1: Rectangular grid, $2 \leq n \leq m$

The rectangular lattice graph on six vertices is shown in Fig. 2a.The corresponding Laplacian matrix in the general form is given as the Kronecker sum of two Laplacian matrices:

$$\mathcal{L}_N^{\mathrm{rg}} = \mathcal{L}_n^{\mathrm{path}} \oplus \mathcal{L}_m^{\mathrm{path}}.$$

Next, several hierarchies obtained as Cartesian multiplication by path graph and cycle graph are presented; they can be visualized as cylindrical and toroidal in shape.

Hierarchy #2: Cartesian product of a cycle graph and a path graph, $m \geq 2, n \geq 3$

The graph is shown in Fig. 2b, $n = 4, m = 3$, the Laplacian matrix is as follows:

$$\mathcal{L}_N^{\mathrm{cp}} = \mathcal{L}_n^{\mathrm{cycle}} \oplus \mathcal{L}_m^{\mathrm{path}}.$$

Hierarchy #3: Cartesian product of a star graph and a path graph, $m \geq 2, n \geq 4$

The graph is presented in Fig. 2c, $n = 4, m = 3$, the matrix $\mathcal{L}_N^{\mathrm{sp}}$ is given by

$$\mathcal{L}_N^{\mathrm{sp}} = \mathcal{L}_n^{\mathrm{star}} \oplus \mathcal{L}_m^{\mathrm{path}}.$$

Hierarchy #4: Cartesian product of wheel and path graphs, $m \geq 2, n \geq 4$

Such a graph has already been considered in the literature, but from a different perspective [Joseph and Kureethara, 2023].

The resulting graph is shown in Fig. 2d, $n = 4, m = 3$. Its Laplacian matrix is as follows:

$$\mathcal{L}_N^{\mathrm{wp}} = \mathcal{L}_n^{\mathrm{wheel}} \oplus \mathcal{L}_m^{\mathrm{path}}.$$

Hierarchy #5: Cartesian product of two cycle graphs, $m \geq 3, n \geq 3$

The graph is depicted in Fig. 2e, $n = 4, m = 3$, the Laplacian matrix of this graph is as follows:

$$\mathcal{L}_N^{\mathrm{cp}} = \mathcal{L}_n^{\mathrm{cycle}} \oplus \mathcal{L}_m^{\mathrm{cycle}}.$$

Hierarchy #6: Cartesian product of a star graph and a cycle graph, $m \geq 3, n \geq 4$

The corresponding graph is shown in Fig. 2f, $n = 5, m = 3$, its Laplacian matrix $\mathcal{L}_N^{\mathrm{sc}}$ is of the form

$$\mathcal{L}_N^{\mathrm{sc}} = \mathcal{L}_n^{\mathrm{star}} \oplus \mathcal{L}_m^{\mathrm{cycle}}.$$

Hierarchy #7: Cartesian product of wheel and cycle graphs, $m \geq 3, n \geq 4$

The corresponding graph is shown in Fig. 2g, $n = 5, m = 3$, the Laplacian matrix is as follows:

$$\mathcal{L}_N^{\mathrm{wc}} = \mathcal{L}_n^{\mathrm{wheel}} \oplus \mathcal{L}_m^{\mathrm{cycle}}.$$

From the obtained expressions for the spectra of the basic graphs and applying Lemma 1 we can formulate the following theorem.

Theorem 2. *For the hierarchies listed above, the values of the smallest positive and largest eigenvalues are of the form:*

1. $\lambda_{\mathrm{min}}^{\mathrm{rg}\,+} = 2-2\cos\frac{\pi}{m}$, $\lambda_{\mathrm{max}}^{\mathrm{rg}} = 4-2\cos\frac{\pi(m-1)}{m} - 2\cos\frac{\pi(n-1)}{n}$, $2 \le n \le m$;
2. $\lambda_{\mathrm{min}}^{\mathrm{cp}\,+} = 2-2\cos\frac{\pi}{m}$ *(for* $n \le 2m$*)*, $\lambda_{\mathrm{min}}^{\mathrm{cp}\,+} = 2-2\cos\frac{2\pi}{n}$ *(for* $n \ge 2m$*)*, $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 6-2\cos\frac{\pi(m-1)}{m}$ *(*n *is even), or* $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 4-2\cos\frac{\pi(m-1)}{m} - 2\cos\frac{\pi(n-1)}{n}$ *(*n *is odd);*
3. $\lambda_{\mathrm{min}}^{\mathrm{sp}\,+} = 1$, $2 \le m \le 3$, $\lambda_{\mathrm{min}}^{\mathrm{sp}\,+} = 2-2\cos\frac{\pi}{m}$, $m \ge 3$, $\lambda_{\mathrm{max}}^{\mathrm{sp}} = n+2-2\cos\frac{\pi(m-1)}{m}$;
4. $\lambda_{\mathrm{min}}^{\mathrm{wp}\,+} = 2-2\cos\frac{\pi}{m}$, $\lambda_{\mathrm{max}}^{\mathrm{wp}} = n+2-2\cos\frac{\pi(m-1)}{m}$;
5. $\lambda_{\mathrm{min}}^{\mathrm{cc}\,+} = 2-2\cos\frac{2\pi}{m}$, $(m \ge n)$, $\lambda_{\mathrm{min}}^{\mathrm{cc}\,+} = 2-2\cos\frac{2\pi}{n}$, $(m \le n)$, $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 8$ *(*n, m *are even),* $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 6-2\cos\frac{\pi(m-1)}{m}$ *(*n *is even,* m *is odd), or* $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 6-2\cos\frac{\pi(n-1)}{n}$ *(*n *is odd,* m *is even), or* $\lambda_{\mathrm{max}}^{\mathrm{cp}} = 4-2\cos\frac{\pi(m-1)}{m} - 2\cos\frac{\pi(n-1)}{n}$ *(*n, m *are odd);*
6. $\lambda_{\mathrm{min}}^{\mathrm{sc}\,+} = 1$, $3 \le m \le 6$, $\lambda_{\mathrm{min}}^{\mathrm{sc}\,+} = 2-2\cos\frac{2\pi}{m}$, $m \ge 6$ $\lambda_{\mathrm{max}}^{\mathrm{sc}} = n+4$ *(*m *is even),* $\lambda_{\mathrm{max}}^{\mathrm{sc}} = n+2-2\cos\frac{\pi(m-1)}{m}$ *(*m *is odd);*
7. $\lambda_{\mathrm{min}}^{\mathrm{wc}\,+} = 2-2\cos\frac{2\pi}{m}$, $\lambda_{\mathrm{max}}^{\mathrm{wc}} = n+4$ *(*m *is even), or* $\lambda_{\mathrm{max}}^{\mathrm{wc}} = n+2-2\cos\frac{\pi(m-1)}{m}$ *(*m *is odd).*

Proof. The proof follows from 1) Lemma 1, where the Laplacian matrices are the matrices of the basic graphs, and the smallest non-zero and largest eigenvalues are summarized in Table 1, as well as from the 2) property of the cosine function.

5 Discussion

In this section, we discuss the spectral properties of the Laplacian matrices of the basic and hierarchical graphs.

Basic graphs. The results of the analysis of Laplacian spectra of each of the basic graphs are summarized in Table 1. From the expressions presented in there, as well as from Fig. 3, one can conclude that $\lambda_{\mathrm{min}}^{+}$ and λ_{max} of certain basic graphs have similar asymptotics. Thus, it can be observed that for large values of N, there is not much difference between χ^{star}, Δ^{star} and χ^{wheel}, Δ^{wheel}, respectively, since $\lambda_{\mathrm{max}}^{\mathrm{star}} = \lambda_{\mathrm{max}}^{\mathrm{wheel}} = N$ and $\lambda_{\mathrm{min}}^{+}$ equals to or tends to 1. Thus, $\chi^{\mathrm{star}}, \chi^{\mathrm{wheel}} \sim N$. In the context of multi-agent systems, such interaction graphs mean that consensus will be reached for any value of N only for particular high-order agent models. In the context of multi-agent systems, such interaction graphs mean that consensus will be reached for any value of N only for particular high-order agent models. For example, if the agent dynamics has transfer function as in Eq. (10), see Example 2 in [Hara et al., 2014], then its corresponding consensus region includes only part of the real axis. Therefore, as N increases, it is necessary to change the agent parameters and/or the weights/type of the communication graph.

The properties of the path graph and the cycle graph are different from those of the star and wheel graphs: Both graphs have no central node (1), the condition number of the Laplacian matrix increases with the dimension as $\chi \sim N^2$ while $\Delta \approx 4$ for large N (2).

However, if we talk about first-order consensus dynamics in continuous time (4), then for $N > 4$, the convergence rate for the star and wheel graphs will be higher than for the path and cycle graphs.

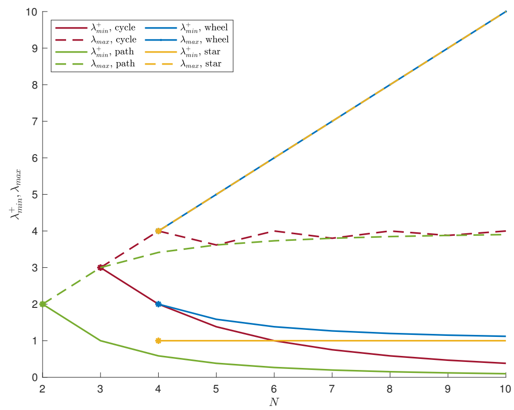


Figure 3. Behavior of $\lambda_{\mathrm{min}}^{+}$ and λ_{max} depending on the number of nodes N. The asterisks indicate the initial number of nodes.

Hierarchical graphs. For the case of a *square grid* $m = n$, one may notice that $\lambda_{\mathrm{min}}^{+} = 2-2\cos\frac{\pi}{n}$, $\lambda_{\mathrm{max}} = 4-4\cos\frac{\pi(n-1)}{n}$. Evidently, $\lambda_{\mathrm{min}}^{+} \to 0$ and $\lambda_{\mathrm{max}} \to 8$, as $n \to \infty$.

In the general case, for each of the obtained graphs the eigenvalues $\lambda_{\mathrm{min}}^{+}$, λ_{max} of their Laplacian matrices are functions of two variables m, n, and therefore their analysis is less illustrative.

6 Conclusion

Obtaining closed-form expressions for the eigenvalues of Laplacian matrices of special graphs helps to analyze decentralized dynamical networks for stability and convergence rate. In this study, we have obtained closed-form expressions for the smallest non-zero and largest eigenvalues of the Laplacian matrices of basic graphs as well as hierarchical graphs derived from basic graphs.

Graph	**Laplacian spectrum**, λ_k	$\lambda^+_{\min}$	$\lambda_{\max}$
Path graph	$2-2\cos\pi(k-1)/N, k\in\{1,\dots,N\}$	$2-2\cos\pi/N$	$2-2\cos\pi(N-1)/N$
Cycle graph	$2-2\cos 2\pi(k-1)/N, k\in\{1,\dots,N\}$	$2-2\cos 2\pi/N$	a) 4, (N is even), or b) $2-2\cos\pi(N-1)/N$, (N is odd)
Star graph	$\lambda_1=0, \lambda_k=1, k\in\{2,\dots,N-1\}, \lambda_N=N$	1	N
Wheel graph	$\lambda_1 = 0,\ \lambda_k = 3-2\cos\frac{\pi(k-1)}{N-1}, k \in \{2,\dots,N-1\}, \lambda_N=N$	$3-2\cos\frac{\pi}{N-1}$	N

Table 1. Laplacian spectra of the basic graphs.

Namely, for the basic graphs, the spectra have been analyzed in terms of the localization of the segment containing the Laplacian eigenvalues, and also in terms of the condition number, which is important for first and higher order consensus and decentralized optimization problems. For the proposed hierarchical graphs constructed as the Cartesian product of certain basic graphs, the corresponding theorem on the spectra has been formulated and the corresponding formulas have been derived.

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