STRANGE ATTRACTORS IN SIMPLEST MODELS OF THE BIOLOGICAL POPULATIONS NUMBER DYNAMICS

Efim Ya. Frisman
Complex Analysis of Regional Problems Institute
Russia
frisman@mail.ru

Oksana L. Zdanova
Institute for Automation and Control Processes
Russia
axanka@iacp.dvo.ru

Abstract
Our work is devoted to the theoretical analysis and discussing of the nature of biological populations' number fluctuations by the examples of simplest mathematical models, which consider the density dependent limitation of the population number. The model for population with two age classes has been analyzed in our work. We have shown that presence of density depended ecological factors is capable to result in highly complicated behavior of population number, on condition that population number changes discretely in fixed breeding seasons.

Key words
Biological population number, dynamics, mathematical model.

1 Introduction

The simplest population dynamics models consider changes of whole population number only, assuming that different generations of the population don't overlap. But such conception is not correct, when lifetime of each generation is essentially longer then time between breeding seasons is. In this case each local population consists of the individual from different age groups during breeding season. So, it is natural to consider the number of each separated age group as a model's variable. The treatment for clustering the population is defined by the characteristics of biological species.

2 The model of population with two age classes

Let's consider the model with age structure, which may be presented by the set of two age classes, there are junior and elder. The junior age class consists of impuberal individuals and elder one consists of individuals participating in reproduction process.

Let's define x_n – the number of junior age class in n-th reproduction season, and y_n - the number of reproductive part of population. The reproductive season comes to end by the appearing of newborn individuals of next generation. Let's assume that during the time between two reproductive periods the individuals from junior age class reach the age of elder, and newborns (or larvae) - to junior. Also let's assume that fitness and reproductive potential of reproductive individuals are not depended from their age. These assumptions are correct for organisms with not great lifetime consisting of two or three reproductive periods; such as many insect, fishes, small mammals, biennials or triennials and others.

Let's assume that limitation of junior age class's number is realized by linear rule; so we can receive the dynamics equations which describe the numbers of discussed age classes in neighboring generations:

$$\begin{cases} x_{n+1} = ay_n \\ y_{n+1} = x_n(1-x_n) + cy_n \end{cases} \qquad (1)$$

Here a is reproductive potential of elders and c is their fitness.

The set of acceptable values for parameters a and c ($a > 0$, $0 < c < 1$) is separated on the tree regions:

1. $a + c < 1$ - system (1) hasn't nontrivial equilibrium here and so stationary solution $\bar{x} = 0; \bar{y} = 0$ is globally stable;
2. $a + c > 1$; $a + 2c < 3$ - the nontrivial equilibrium exists here and it is stable solution;

3. $a + 2c > 3$ - there are not any stable stationary solutions here; also both of trivial and nontrivial equilibrium exist but both of them are unstable.

Then numerical investigation of model's (1) limit paths have been done with crossing bound $a + 2c = 3$ by the system's parameters as well as in region $a + 2c > 3$.

2.1 Strange attractors in the model of population which consists of two age classes

Let's consider received results for case of $c = 0{,}15$. In this case stationary solution loses it's stability wile a overlaps the value 2,7; at that the invariant limit curve appears and it looks like circle in the beginning. The "circle" becomes deformed more and more with increasing of parameter a; but limit set has shape of closed curve until parameter a doesn't overlap bound $a = 3{,}0$. Zooming of various parts of phase-plane portrait allows us to illustrate this conclusion. The invariant curve "reduces" to limit cycle of finite length with some values $a \in (2{,}09;3{,}05)$ however. With further increasing of a the attractor appears (fig. 1a) which has been named as "scroll" by us.

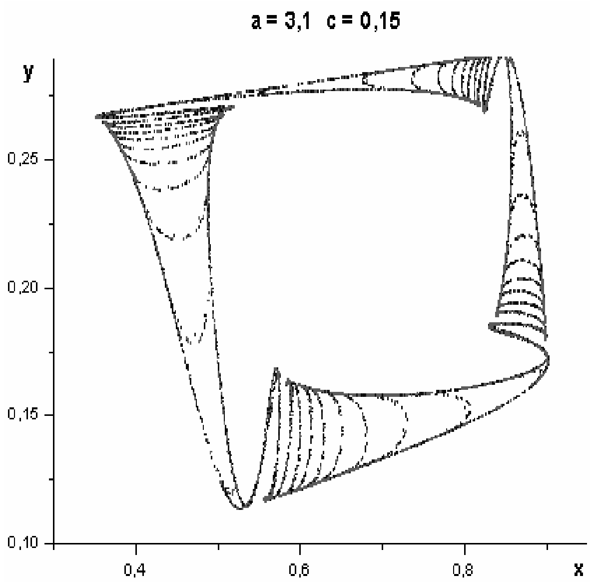

Figure 1a. "Scroll". Attractor of system (1)

This attractor has complicated cross structure which becomes more evident by zooming. "Scroll" exists with $a = 3{,}10; 3{,}11; 3{,}12$, and with some $a \in (3{,}12; 3{,}13)$ it "collapses" because with $a = 3{,}13$ the limit cycle – 8 appears; this cycle bifurcates to cycle – 16 with $a = 3{,}15$; than one can see next series of bifurcations which followed by the appearance of new structure (fig. 1b) named by us as a "propeller".

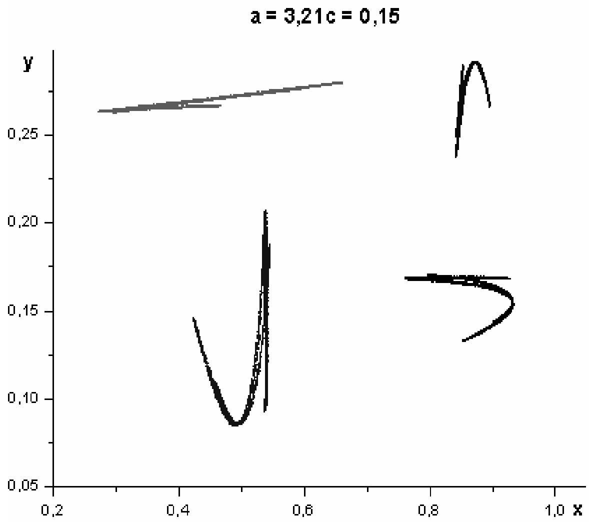

Figure 1b. "Propeller". Attractor of system (1).

It is evident (on figure 1b already) that "propeller" has highly complicated cross structure (terminology of M. Hennon (1981)). This structure became more visible with consecutive zooming of attractor's separate parts (fig. 1c). The procedure of cascade zooming (by analogy of those from [Hennon, 1981]) confirms that each visible "curve" of propeller im kleinen consists of great number of quasi parallel lines.

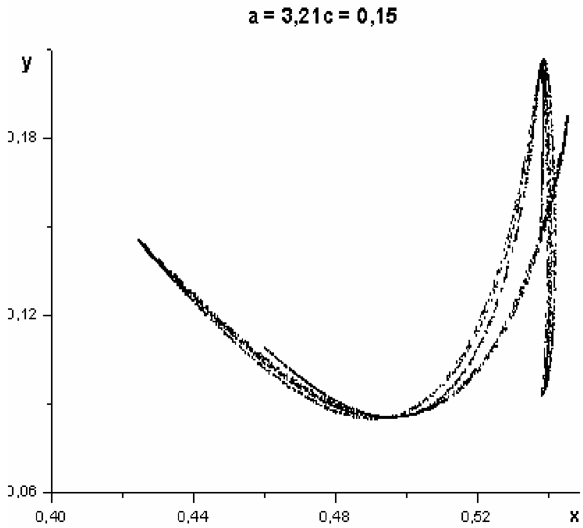

Figure 1c. Zoomed fragment of "propeller". Attractor of system (1).

Further increasing of a value results in thickening of all "propeller's" lines and progressively set of trajectory points fills out some region of phase space more or less densely (fig 1d).

Such system's behavior let us to assume the existence of complicated series of attractors' bifurcations possibly with changing of its' dimension.

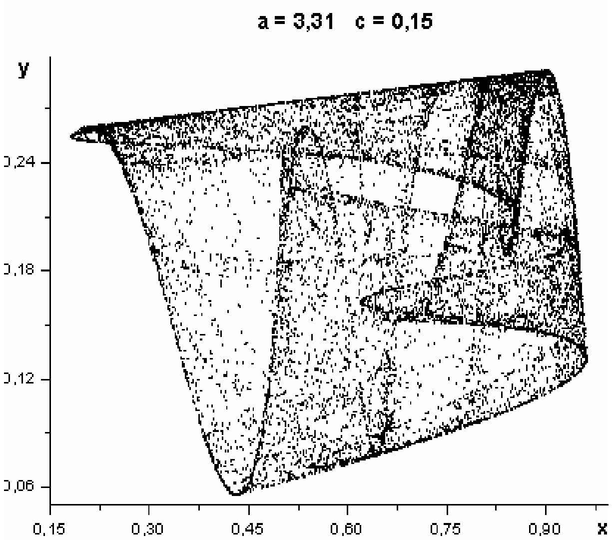

Figure 1d. Dense structure. Attractor of system (1).

Increasing of parameter a (with others values of parameter c) results in analogous changes of limit dynamics mode of model (1). Overlapping the bound $3-2c$ by the parameter a usually follows by appearance of invariant curve, which becomes more and more curved with a increasing; then this curve collapses and gives place to some complicated figure.

Also the "scrolls" or "propellers" appearance is not necessary (for any values of c); and we can see them not always. So when $c = 0{,}326$ with a increasing the invariant curve (fig. 2a) reduces to limit cycle – 9, the each point of this cycle gives place to separate closed graph (fig. 2b) generated by points of 9-th iteration of transformation (1). When $a = 2{,}725$ each of these closed graphs reduces to "local limit cycle" – 6, i.e. it is a cycle – 54. Further increasing of parameter a results in appearance of closed graph around each point of cycle – 54 (fig. 2c: there are 54 quasi ellipses!). It is possible that such process is continuing but it is difficult for us to proceed with illustrations. When a becomes more, the strange attractor appears; this attractor consists of nine separate structures having sufficiently dense structure and original shape ("pyramids", fig. 6d). These structures grow and diffuse occupying increasingly more spice and finally them flow together in common structure (fig. 6e).

3 Conclusion

So we have shown that presence of density depended ecological factors is capable to result in highly complicated behavior of population number, on condition that population number changes discretely in fixed breeding seasons. The influence of density dependant factors capable particularly to make for establishment of stable fluctuations of number of certainly one-species biological systems.

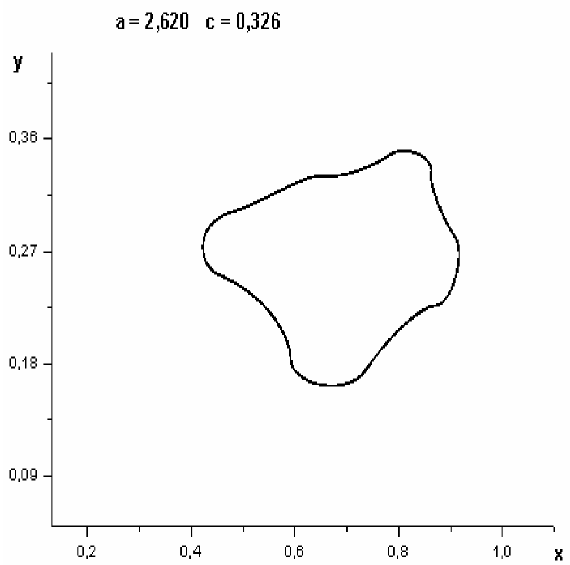

Figure 2a. Invariant curve. Attractor of system (1).

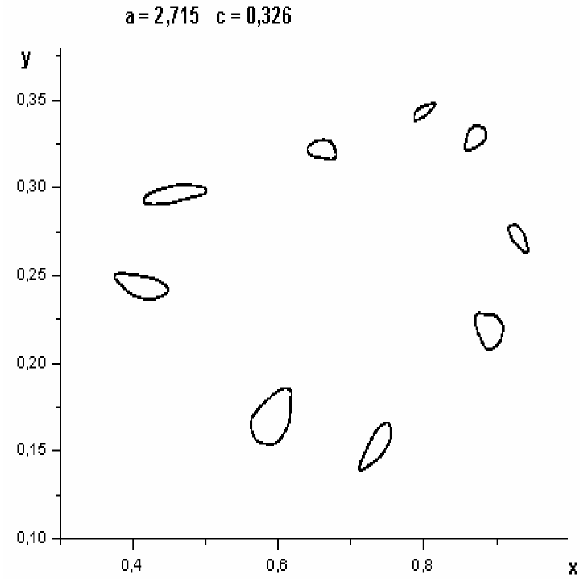

Figure 2b. 9 quasi ellipses. Attractor of system (1).

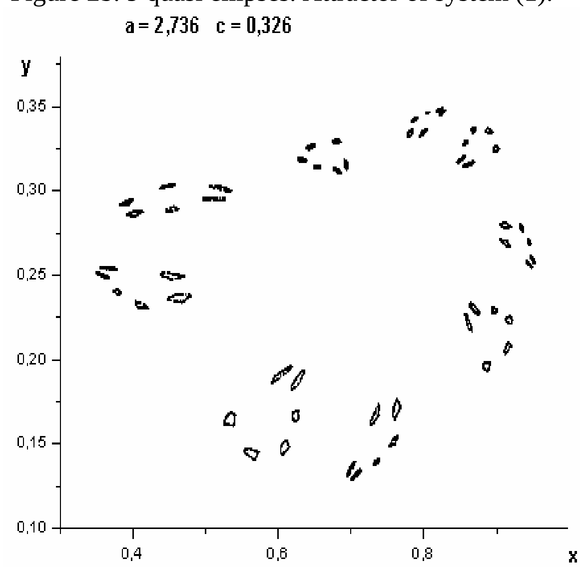

Figure 2c. 54 quasi ellipses. Attractor of system (1).

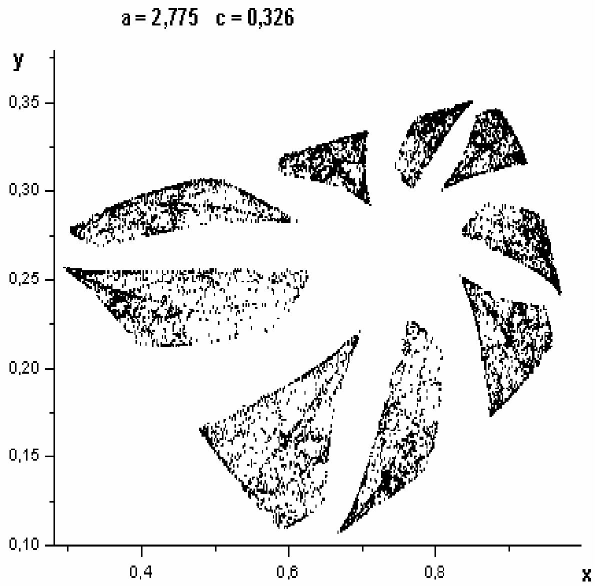

Figure 2d. Nine “pyramids”. Attractor of system (1).

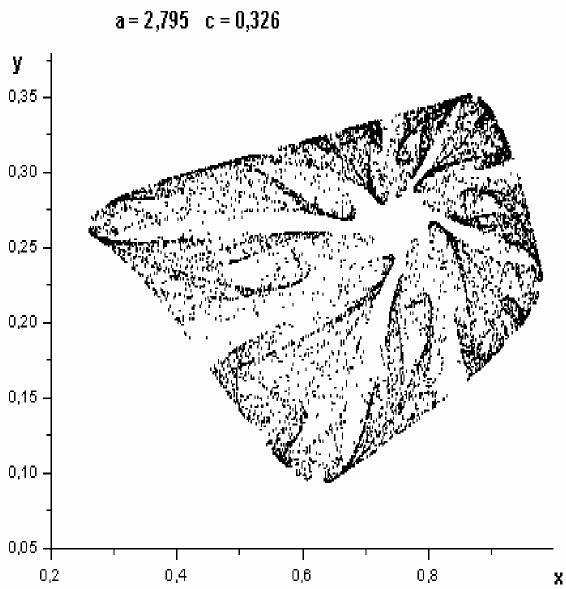

Figure 2e. The fusion of “pyramids” into dense structure. Attractor of system (1).

At this case fluctuations of number caused by particularly internal processes of population and these fluctuations are not connected with fluctuations of any external factors or influence of predators, parasites or feeding factors.

Moreover even simplest models of number’s dynamics can lead to irregular chaotic regimes of dynamic behavior. Such regimes are typical for stochastic models describing the change of population number under influence of random factors. Now it is clear that cyclic and chaotic dynamic behavior can be found in particularly deterministic models provided that these models consist of recurrent equations.

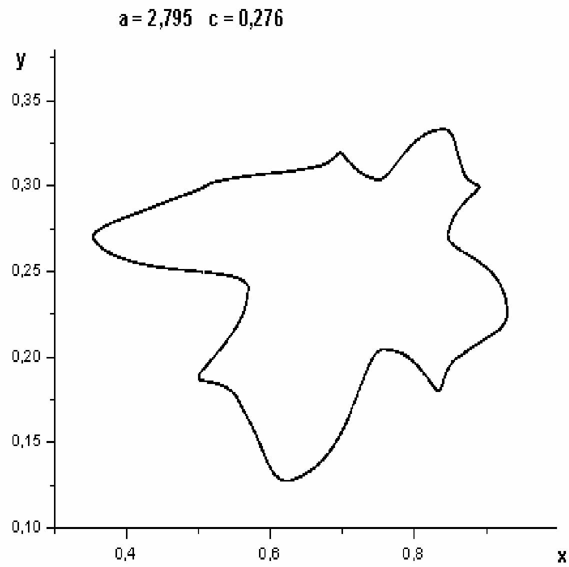

Figure 3a. “Pigeon”. Attractor of system (1).

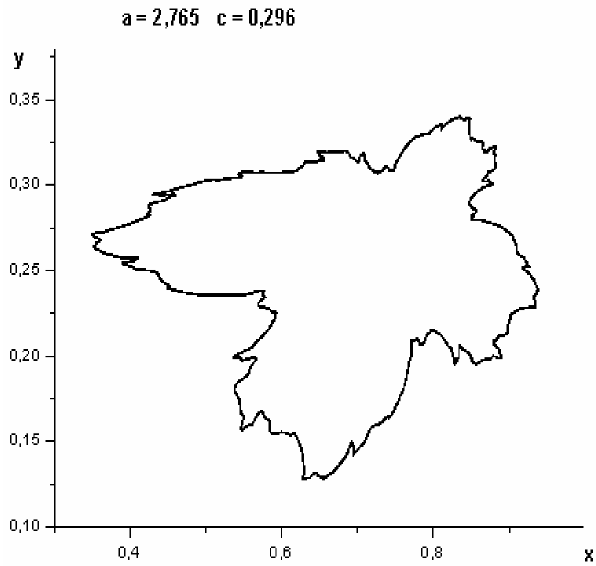

Figure 3b. “Maple leaf”. Attractor of system (1).

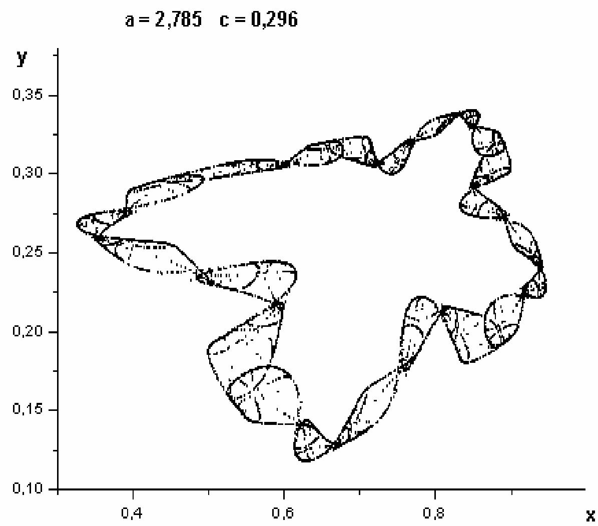

Figure 3a. “Ornament”. Attractor of system (1).

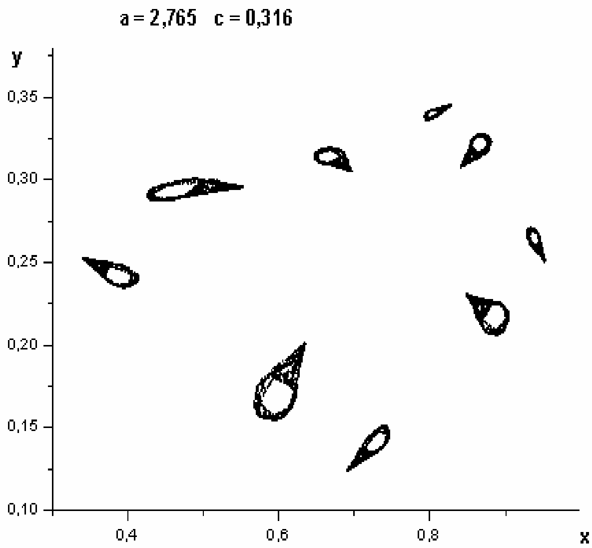

Figure 3b. 9 “drops”. Attractor of system (1).

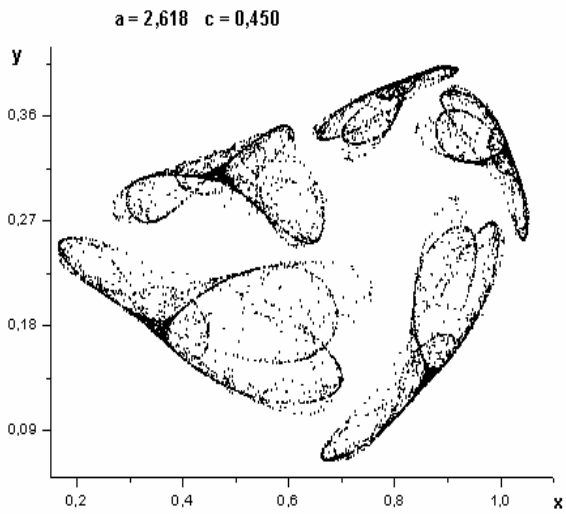

Figure 3c. 5 “heavy aeronauts”. Attractor of system (1).

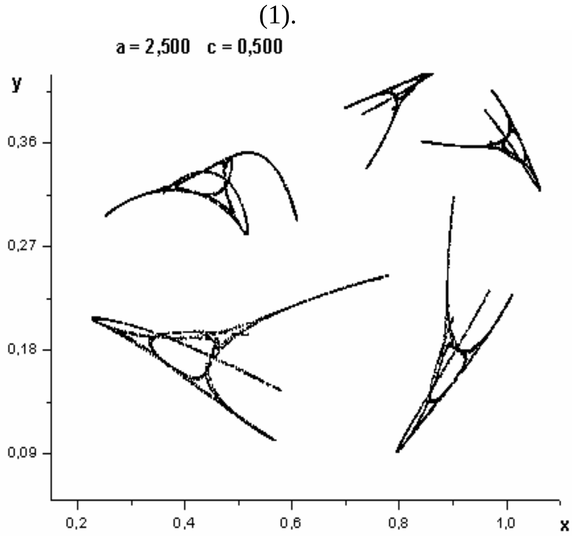

Figure 3d. 5 “swallows”. Attractor of system (1).

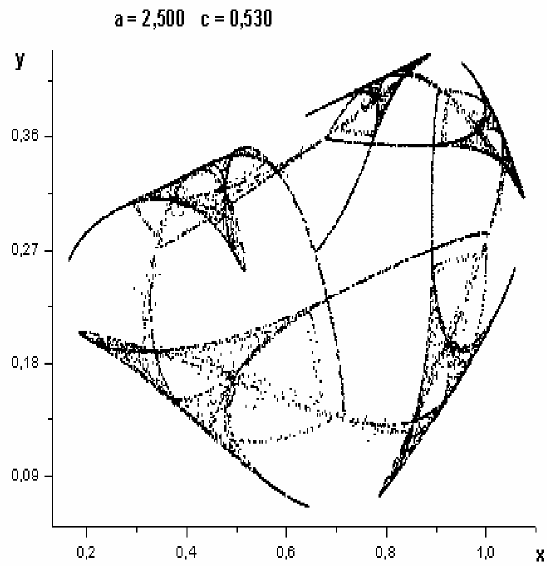

Figure 3e. The fusion of “swallows” into dense structure. Attractor of system (1).

Note that findings allows simple and at the same time unexpected the evolution interpretation. Let’s remind that we investigated changes in population’s number behavior regimes with increasing of a and c parameters, which describe the fertility and fitness of individuals. Such changes of these parameters may be caused by the evolution process under influence of natural selection. In essence, natural selection is the survival of individuals groups with maximum values of a and c parameters. This fact has been established by Charles Darwin and its mathematical form is known as Fisher’s fundamental theorem of natural selection [Fisher, 1930]. So, during the natural evolution of natural population with marked seasonality of its lifecycle it should be natural non-random transfer from stable regimes of number’s dynamics to fluctuations and chaos (pseudo stochastic behavior), i.e. transfer from equilibrium dynamic regime to non-equilibrium one.

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