A STABILITY CONJECTURE FOR DISCRETE SYSTEMS WITH UNILATERAL CONTACT AND DRY FRICTION

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Abstract

We introduce a new notion of stability specially adapted to discrete systems involving unilateral contact and Coulomb friction. This notion deals with perturbations of the forces. Using the terminology of ordinary differential equations this means perturbations of the right-hand side instead of classical stability notions which deal with perturbations of the initial data. It appears as a consequence of the graph of the non-regularized Coulomb's friction law. In this context we give a conjecture that the present paper aims at justifying.

Key words

Coulomb friction, unilateral contact, nonsmooth dynamics, mass-spring systems, stability

1 Introduction

We study the stability of systems of particles that are forced to remain on one side of an obstacle so that the trajectories involve shocks together with friction conditions. We recall that these conditions imply that the equation of the dynamics should be understood in the sense of measures and we give some results concerning the set of equilibrium states, the smoothness of the trajectories and their approximation. Next we focuss on the stability analysis. Since unilaterality means that the convenient framework for studying the dynamics is not a vector space and since friction implies dissipativity, classical stability theorems no longer apply. Moreover, analyzing the trajectories issued from an initial perturbation in a classical phase space in order to get stability results, although already carried out [3] is not really satisfactory. The purpose of the present work is to introduce a new notion of stability which deals with perturbations of the external forces, and not of the initial data, and to back up a corresponding conjecture.

2 The mathematical framework

2.1 The exact mathematical formulation

For a system having a single mass such as the simple model given by Klarbring [8], the exact mathematical formulation of the dynamical problem reads [9]:

$$\begin{cases} \text{Find a function } U(t) \in \mathcal{MMA}([0,T];\mathrm{R}^n) \\ 1) \quad U(0)=U_0; \quad \dot{U}^+(0)=V_0 \\ 2) \quad \ddot{U}+K\cdot U = F+R, \qquad in\ [0,T] \\ 3) \quad U_n \le 0, \quad R_n \le 0, \quad U_nR_n = 0 \\ 4) \quad \displaystyle\int_{[0,T]} \left[R_t\cdot(V-\dot{U}_t^+) - \mu R_n(|V|-|\dot{U}_t^+|)\right] \ge 0, \\ \qquad \forall\, V \in\ C^0([0,T];\mathrm{R}^{n-1}) \\ 5) \quad U_n(t)=0 \implies \dot{U}_n^+(t) = -e\dot{U}_n^-(t),\ in\ ]0,T]. \end{cases} \tag{1}$$

Let us just add some comments:
Line 1) gives the initial data,
line 2) is the equation of the dynamics where R is the reaction of the obstacle, F the external forces and K the stiffness matrix of the springs,
line 3) gives the unilateral contact conditions which means in particular that only the negative half-space is allowed for the motion of the particle,
line 4) is a variational form of the Coulomb's friction law,
line 5) is the impact law with $e \in [0,1]$.

$\mathcal{MMA}([0,T];\mathrm{R}^n)$ stands for motions with measure acceleration, which means that the motion $U(t)$ is such that its first derivative is a function of bounded variation and its second derivative is a measure. We recall that this framework, which follows from the possible occurence of shocks and from the positivity condition of the reaction is the exact framework which gives in particular an adequate meaning to the initial data and to the impact law, since a function of bounded variation has everywhere a left and a right limit. Moreover line 2) being understood in the sense of measures, it can always be integrated over an interval, which will be the basis of a time stepping method.

2.2 Some basic results

We just recall without proof three important results concerning problem (1), which have benn given in particular in [3] and in [4].

1. Given initial data compatible with the obstacle, the existence of a solution to problem (1) is ensured as soon as F is an integrable function of time.
2. In general problem (1) has several solutions, and uniqueness is ensured only if F is an analytical function of time.
3. In the case where one has uniqueness, the time stepping type algorithm NonSmooth Contact Dynamics (NSCD, [6]) has been shown to converge.

3 Stability of the equilibrium states

3.1 On a classical stability analysis

From problem (1), we can get the whole set of equilibrium states under a given constant force of components $\{F_n, F_t\}$ as soon as the parameters of the problem, coefficients of K, friction coefficent μ, have been given. Depending on these parameters, the set of equilibrium solutions may consist of a single point, or of a set of infinitely many points, this set being either bounded or unbounded [5].

Unilateral contact and Coulomb friction make it impossible to use classical stability theorems to deal with the stability of these equilibria. Nevertheless given any of the equilibria, we can choose an initial data that belongs to a neighbourhood of this equilibria in a classical phase space. The previous basic results then allow us to compute the trajectory starting from this initial data and conclude about the stability using only elementary definitions of stability. This has been carried out for the whole sets of equilibria (see [3]) for the one-mass case. But this is in fact not really satisfactory in view of the graph of the Coulomb's law. As a matter of fact, perturbing a given strictly stuck equilibrium by a tangential velocity may require a large perturbation of the external forces no matter how small the velocity is. This means in turn that it is quite possible that an equilibrium defined by $(U = U^{eq}, \dot{U} = 0)$ is not modified by adding any relatively small external forces.

3.2 A stability conjecture

As a counterpart to the previous remarks about stability analyzis given in section 3.1, we now give the following definition:

Definition *An equilibrium state will be said to be stable if there exists a ball in a convenient functional framework such that the system remains at equilibrium for any external force that remains inside this ball.*

And we add the following remark:

Remark *Analytical calculations and numerical experiments show that when an equilibrium state for which the reaction is on the border of the cone is set into motion by a change of the external force, the trajectory can lead to a new equilibrium where the reaction is strictly inside the cone.*

This leads to the following conjecture:

Conjecture *Let a discrete system with any finite number of degrees of freedom be submitted to unilateral contact and Coulomb friction. Assume the data are such that there exists an equilibrium state in which some reactions are strictly inside the Coulomb cone while the other reactions are in imminent sliding. Then the trajectory produced by any sufficiently small perturbation of the data leads to a new equilibrium where the number of reactions strictly inside the cone is larger than before the perturbation.*

The question is now: how to back up this conjecture? This will be done, partially, in four steps.

3.2.1 Step 1

The first step is quite elementary. We simply observe that the statement of the conjecture is trivial if the equilibrium state we are dealing with is such that all the reactions are strictly inside the cone.

3.2.2 Step 2

Let a discrete system be made of a single mass such as Klarbring's model. Then we can establish by closed-form calculations that the statement given by the conjecture is satisfied for any nongrazing equilibrium. In particular, if the equilibrium state is in imminent sliding and we change the forces exerted on the system by a small enough constant or by an oscillat-

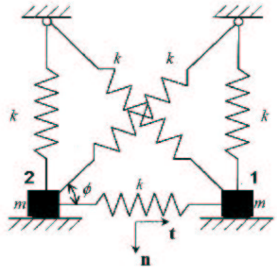

Figure 1. The simple two-mass system

ing perturbation, then the system is set into motion and the trajectory leads to a strictly stuck equilibrium after a jump of the tangential component of the reaction.

3.2.3 Step 3 We can prove that the conjecture is in agreement with the dynamics of a mass-spring system composed of two masses and moving in an half-plane. This will be done in the next section.

3.2.4 Step 4 The last step consists in numerical experiments. We shall give an example in the last section.

4 A slightly more complex system

We are now studying a two-mass system which have been given in [1] or [10]. This system is represented on Figure 1.

4.1 The basic equations

We first give the equations of the dynamics, particularized to this two-mass system. All the equations of this system should be understood in the sense of measures and bounded variation functions as recalled previously. The equations of the dynamics read, with obvious notations except s and c which stand for $\cos\phi$ and $sin\phi$ and which give the coefficients of the stiffness matrix if the stiffness of the springs is equal to 1:

$$\begin{cases} \ddot{u}_{1t} + (1+c^2)u_{1t} + csu_{1n} - u_{2t} = F_{1t} + R_{1t}, \\ \ddot{u}_{1n} + csu_{1t} + (1+s^2)u_{1n} = F_{1n} + R_{1n}, \\ \ddot{u}_{2t} - u_{1t} + (1+c^2)u_{2t} - csu_{2n} = F_{2t} + R_{2t}, \\ \ddot{u}_{2n} - csu_{2t} + (1+s^2)u_{2n} = F_{2n} + R_{2n}. \end{cases} \tag{2}$$

The initial conditions:

$$\begin{cases} u_{1t}(0) = u_{1t0}, \dot{u}_{1t}(0) = v_{1t0}, \\ u_{1n}(0) = u_{1n0}, \dot{u}_{1n}(0) = v_{1n0}, \\ u_{2t}(0) = u_{2t0}, \dot{u}_{2t}(0) = v_{2t0}, \\ u_{2n}(0) = u_{2n0}, \dot{u}_{2n}(0) = v_{2n0}. \end{cases} \tag{3}$$

The unilateral contact conditions:

$$\text{For } i = 1,2 \;\; R_{in} \leq 0, \; u_{in} \leq 0, \; R_{in}u_{in} = 0. \tag{4}$$

The Coulomb's friction law:

$$\text{For } i = 1,2 \;\; |R_{it}| \leq -\mu R_{in},$$

$$\begin{cases} |R_{it}| < -\mu R_{in} \implies \dot{u} = 0, \\ |R_{it}| = -\mu R_{in} \implies \exists \lambda \geq 0 \text{ such that } \dot{u} = -\lambda R_{it}. \end{cases} \tag{5}$$

The impact law:

$$\text{For } i = 1,2 \;\text{ when }\; \mathrm{u_{in}(t)} = 0, \qquad \dot{u}_{in}(t^+) = -e\dot{u}_{in}(t^-) \;\text{ with }\; \mathrm{e} \in [0,1]. \tag{6}$$

4.2 The equilibrium states

In the same way as for the one-mass system, we can explore the whole set of equilibria, given the parameters of the system and the external forces F_1 and F_2. Nevertheless, compared to the one-mass case, the set of equilibria is extremely intricated due to a lot of possiblities where either both masses are not in contact, or one is in contact while the other is not, or both are in contact. In the latter case, a large number of situations may occur with a set of equilibria which again can be bounded or unbounded. When both masses are in contact the space of the reactions is R^4, and Figure 2 presents an example of an unbounded set of equilibria observed in the $\{R_{1n}, R_{2n}\}$ plane.

4.3 The stability result

Proposition *Let an equilibrium state of the two-mass system be such that both masses are in nongrazing contact.*
Then any small enough perturbation of the forces leads to a new equilibrium where all the reactions are strictly inside the Coulomb cone. Moreover, if the perturbing forces do not depend on time, then the final equilibrium is reached in finite time.

Sketch of the proof The proof is divided into three points.
Point 1 We first observe that the result is trivial if the equilibrium state is strictly stuck before perturbation.
Point 2 One mass is strictly stuck while the other is in imminent sliding. In this case, the proof is relatively

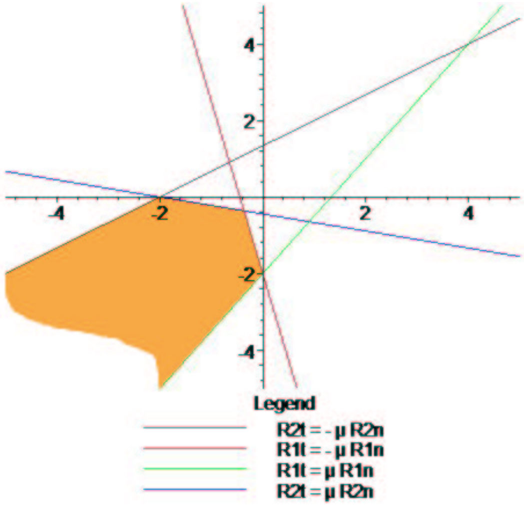

Figure 2. An example of the set of equilibria.

simple, and close to that of the one-mass case, except the fact that one has to show that when the sliding mass stops it cannot be set into motion again and its reaction always jumps towards a value strictly inside the cone. This is done by closed-form calculations

Point 3 Both masses are in imminent sliding. Proving the result in this case is by far longer and more technical. Nevertheless, the proof can be carried out using only simple mathematical tools. It is divided into five steps which we just enumerate as a hint for the proof.

Step i) Calculating the motion of both masses, by closed-form calculations.

Step ii) One of the masses generically stops first. After this stop, there exists a time interval during which this mass stays motionless.

Step iii) The reaction of the mass which is motionless jumps to a value strictly inside the cone.

Step iv) The coupling of this reaction with the one of the mass which goes on sliding cannot set the motionless mass into motion again if the perturbation of the forces is small enough.

Step v) The second mass stops and its reation jumps to a value strictly inside the cone. Then both masses are strictly stuck, and the system is not set into motion again if the perturbation of the forces is small enough, whether this perturbation continues to be applied or not.

5 Numerical computations

In fact, the conjecture essentially follows from numerical experiments. A great number of compuations have been performed with a large number of particles such as collections of rigid bodies in a box. A system with 200 bodies is represented on Figure 4, but calculations with several thousands of bodies have been performed. The idea is to study the trajectories starting from a given equilibrium of the bodies in the box after a pertubation produced by shaking the box. The calculations were performed using the NSCD software (cf. [6] and [9]) which results from a time stepping discretization of problem (1). We compute the reactions at each contact point after the final equilibrium state is reached and, as a post-processing, we choose a real number $\chi \in [0,1]$ and calculate the number of reactions for which the ratio $\frac{|R_t|}{\mu|R_n|}$ is larger than χ. Examples of results are plotted on Figure 4 in the $\left\{\chi, \frac{|R_t|}{\mu|R_n|}\right\}$ plane. Each equilibrium state corresponds to a curve in this plane. If a curve C_1 is under another curve C_2 then that means that the number of reactions for which $\frac{|R_t|}{\mu|R_n|}$ is in $[\chi, 1]$ (that means closer to the edge of the cone or on the edge of the cone) is smaller for C_1 than for C_2. And we observe that after each shaking of the box, the curve corresponding to the final state is lower than the previous one.

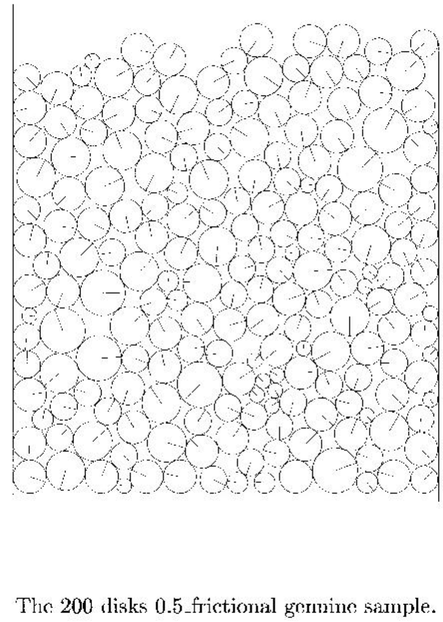

Figure 3. A system with 200 disks in a box.

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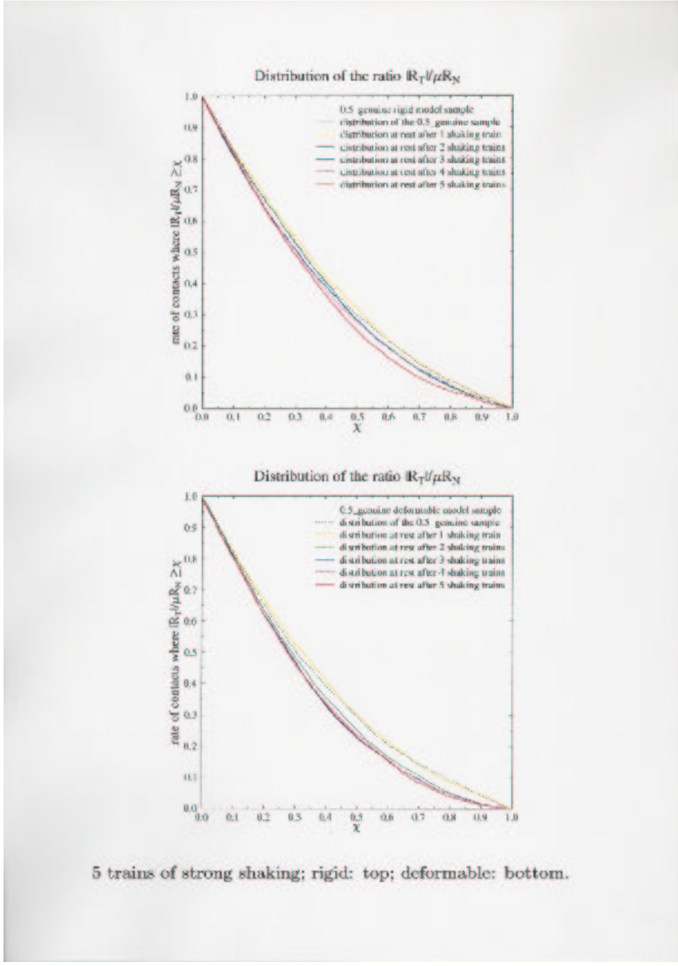

Figure 4. The results for the 200 disks systems for two cases of data.

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