OPEN PROBLEMS ON HURWITZ POLYNOMIALS

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Abstract
The study of the stability of a linear system of differential equations is carried out by means of analysis of the characteristic polynomial associated with the system. If such polynomial has the property that all its roots have negative real part then the source will be a balancing point asymptotically stable. This class of polynomials receive the name of Hurwitz polynomials. It is interesting to determine whether a polynomial is or is not Hurwitz without calculating their roots. In this paper we present some open problems about Hurwitz polynomials.

Key words
Hurwitz polynomials, stable systems, principal diagonal minors.

1 Introduction

The importance of studying criteria for deciding if a polynomial has all its roots with negative real part is due to the stability of a linear system is determined by the fact that the characteristic polynomial has this property. There are several criteria to verify this fact, among which we can mention the Routh-Hurwitz criterion [Hurwitz, 1895], the Hermite-Biehler's theorem [Hermite, 1856], Liènard- Chipart conditions [Liènard and Chipart, 1914], the stability test [Battacharayya, Chapellat and Keel, 1995], the Leonhard-Mihailov's theorem [Loredo, 2005] or Routh's algorithm [Routh, 1877]. Without doubt the Routh-Hurwitz criterion is the most known. It is named by that was shown in an independent manner by Hurwitz and Routh.

The problem was first raised by Maxwel in 1868 ([Maxwell, 1868]). Routh knew of the problem and resolved it. On the other hand Hurwitz knew of the problem through the Austrian engineer A. Stodola.

A substantial amount of information about these polynomials and issues can be found in [Battacharayya, Chapellat and Keel, 1995], [Gantmacher, 1959], [Lancaster and Tismenetsky, 1985], [Loredo, 2005] and [Uspensky, 1990]. For some applications can be found references [Barnett and Cameron, 1985], [Loredo, 2005], and [Zabczyk, 1992]. Recent information about Hurwitz polynomials can be consulted in [Rahman and Schmeiser, 2002] and [Fisk, 2008]. In this paper we will discuss some of these criteria with the intent to develop some open research problems.

2 Routh-Hurwitz theorem

We begin this section by setting the main definition.

Definition. A polynomial with real coefficients $f(t) = b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ is Hurwitz, if all its roots have negative real part.

Example 1. The polynomial $g(t) = t^2 + 5t + 6$ is a polynomial Hurwitz because its roots are $t = -2, -3$.

Example 2. The polynomial $h(t) = t^2 + 25$ is not a Hurwitz polynomial because its roots are $t = -5i, 5i$.

Next we set up the Routh-Hurwitz criterion . There are several tests of this theorem. To see some of them please refer to the references [Battacharayya, Chapellat and Keel, 1995], [Lancaster and Tismenetsky, 1985] and [Loredo, 2005].

Theorem (Routh-Hurwitz criterion). Given a polynomial with real coefficients $f(t) = b_0t^n + b_1t^{n-1} +$

$\cdots + b_{n-1}t + b_n$ we define the Hurwitz matrix associated with this polynomial

$$H(f) = \begin{bmatrix} b_1 & b_0 & 0 & \cdots & 0 \\ b_3 & b_2 & b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{2n-3} & b_{2n-4} & b_{2n-5} & \cdots & b_{n-2} \\ b_{2n-1} & b_{2n-2} & b_{n-3} & \cdots & b_n \end{bmatrix}$$

where $b_k = 0$ if $k > n$.
to such polynomial has all of his roots with negative real part , it is necessary and sufficient that it satisfies

$$b_0\Delta_1 > 0, \Delta_2 > 0, b_0\Delta_3 > 0, \Delta_4 > 0, \\ \ldots, \begin{cases} b_0\Delta_n > 0, & \text{if } n \text{ is odd} \\ \Delta_n > 0, & \text{if } n \text{ is even} \end{cases}$$

where Δ_i are principal diagonal minors of the matrix of Hurwitz, i.e.

$$\begin{aligned} \Delta_1 &= \det(b_1), \\ \Delta_2 &= \det \begin{pmatrix} b_1 & b_0 \\ b_3 & b_2 \end{pmatrix}, \\ &\vdots \\ \Delta_n &= \det H(f) \end{aligned}$$

In case $b_0 = 1$, the condition simply said that the lower main diagonals must be positive, i.e. $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \ldots, \Delta_n > 0$.

Example 3. Consider the polynomial $p(t) = t^3 + 5t^2 + 3t + 7$. The Hurwitz matrix corresponding to the polynomial p is the matrix

$$H(p) = \begin{pmatrix} 5 & 1 & 0 \\ 7 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix}$$

so we have to the lower main diagonals are $\Delta_1 = 5 > 0$, $\Delta_2 = 8 > 0$, $\Delta_3 = 56 > 0$, then we can claim that the polynomial $p(t)$ is a Hurwitz polynomial.

3 Lienard-Chipart's conditions

In verifying whether a polynomial of degree n is a polynomial Hurwitz or not using the Routh-Hurwitz criterion, we see that we will have to compute n determinants and check if they have positive sign. If the degree is so large then we will have to make a good amount of operations. For this reason it is desirable that one could work with criteria to reduce operations. This objective is satisfied by following theorem, which may be considered as an improvement Criteria Routh-Hurwitz.

Theorem (Lienard-Chipart's conditions). The polynomial $f(t) = b_0t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ ($b_0 > 0$) is Hurwitz if and only if satisfies any of the following conditions:
1) $b_n > 0, b_{n-2} > 0, b_{n-4} > 0, ...$;
$\Delta_1 > 0, \Delta_3 > 0, \Delta_5 > 0, ...$

2) $b_n > 0, b_{n-2} > 0, b_{n-4} > 0, ...$;
$\Delta_2 > 0, \Delta_4 > 0, \Delta_6 > 0, ...$

3) $b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...$;
$\Delta_1 > 0, \Delta_3 > 0, \Delta_5 > 0, ...$

4) $b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...$;
$\Delta_2 > 0, \Delta_4 > 0, \Delta_6 > 0, ...$

Example 4. Consider the polynomial $q(t) = t^3 + 8t^2 + 2t + 9$. The Hurwitz matrix corresponding to the polynomial q is the array

$$H(q) = \begin{pmatrix} 8 & 1 & 0 \\ 9 & 2 & 8 \\ 0 & 0 & 9 \end{pmatrix}$$

First let's look at that all the coefficients of the polynomial are positive , then we can use the 2) or 3) of the conditions of Lienard-Chipart: as $\Delta_2 = 7 > 0$ then $q(t)$ is a Hurwitz polynomial.

Problem 1. Does Lienard-Chipart's criterion have the lowest number of principal minors needed or could be improved?

Remark 1. The answer to this question could offer a computational advantage.

4 Phase theorem

The following is the theorem of the phase, also known as the theorem of Leonhard-Mihailov .

Theorem (Leonhard-Mihailov). The real polynomial $p(t) = a_0t^n + a_1t^{n-1} + ... + a_n$ is Hurwitz if and only if the argument of $p(i\omega)$, $\arg(p(i\omega))$, is a function of ω and strictly increasing continuous on $(-\infty, \infty)$. In addition, the net increase of the argument of $-\infty$ to ∞ is $n\pi$, i.e.

$$\arg[p(i\infty)] - \arg[p(-i\infty)] = n\pi \tag{1}$$

Problem 2. What will be the roots of $P(t)$ if the curve $p(i\omega)$ not intersects and what is the meaning into applications?

Remark 2. It would be interesting the meaning of this property in electrical circuits or mechanical systems where the Hurwitz polynomials appear.

Problem 3. Do a description of the analytical functions $f(t)$ that satify $\arg(f(i\omega))$ are strictly increasing functions?

Remark 3. This is a problem of mathematical interest.

5 Hermite-Biehler's Theorem

The theorem of Hermite-Biehler is one of the most useful criteria to determine the stability of a real polynomial. Spell it out for the following definitions are necessary, for more details see [Gantmacher, 1959]. Consider the real polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n$$

We can write p the following way

$$p(x) = (a_0 + a_2 x^2 + \cdots) + x(a_1 + a_3 x + \cdots) \quad (2)$$

Evaluanting in $i\omega$, we have

$$p(i\omega) = (a_0 - a_2\omega + \cdots) + i\omega(a_1 - a_3\omega^2 + \cdots) \quad (3)$$

We define

$$p^{even}(x) = a_0 + a_2x^2 + a_4x^4 + \cdots \quad (4)$$
$$p^{odd}(x) = a_1 + a_3x^2 + a_5x^4 + \cdots \quad (5)$$
$$p^{e}(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \cdots \quad (6)$$
$$p^{o}(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 - \cdots \quad (7)$$

A pair of polynomials u, v are said to be a couple positive coefficients, if the principal of u and v have the same sign and the roots μ_i u and ν_i of v are various, real and negative and satisfies any of the following two properties of interlaced:

$$\nu_m < \mu_m < \nu_{m-1} < \cdots < \nu_1 < \mu_1 < 0, \ (m = k) \quad (8)$$
$$\mu_m < \nu_{m-1} < \mu_{m-1} \cdots < \nu_1 < \mu_1 < 0, \ (m = k+1) \quad (9)$$

where $m = \deg(u)$ y $k = \deg(v)$. Any polynomial that satisfies one of these partnerships has only real zeros. Note that the polynomials given in (6) and (7) form a couple positive and that the expression (2) can be written and the way

$$p(x) = f(x^2) + xg(x^2)$$

where f and g are like (4) and (5).

Theorem (Hermite-Biehler). A polynomial $p(x) = f(x^2) + xg(x^2)$ with real coefficients is Hurwitz if and only if f and g are a couple positive.

Hermite-Biehler's theorem can be also found in [Battacharayya, Chapellat and Keel, 1995].

6 Stability test

This procedure is to determine if a polynomial is Hurwitz, verifying if a polynomial of degree less than it is too.

Definition. Given the polynomial $P(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, if $a_{n-1} \neq 0$, we define

$$Q(t) = a_{n-1}t^{n-1} + \left(a_{n-2} - \frac{a_n}{a_{n-1}}a_{n-3}\right)t^{n-2} + a_{n-3}t^{n-3} + \left(a_{n-4} - \frac{a_n}{a_{n-1}}a_{n-5}\right)t^{n-4} + \cdots \quad (10)$$

Theorem. If $P(t)$ has all of its positive coefficients, then $P(t)$ is Hurwitz if and only if $Q(t)$ is Hurwitz.

The previous theorem shows how to check if a polynomial P(t) is Hurwitz through successive reduction of your grade. This result allows us to provide an algorithm to check if a polynomial is Hurwitz or not.

Algorithm.

1) Do $P^{(0)}(t) = P(t)$.
2) Verify that all coefficients of $P^{(i)}(t)$ are positive.
3) Build $P^{(i+1)}(t) = Q(t)$ using equation 1
4) Return to (2). If the polynomial does not satisfy (2) stop the process and then $P(t)$ is not Hurwitz. In another case to continue the process until reach $P^{(n-2)}(t)$ which is grade 2 and then $P(t)$ is Hurwitz.

Example 6. We verify if the polynomial

$$q(t) = t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1$$

is Hurwitz. We take

$$P^{(0)}(t) = t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1$$

then we build $P^{(1)}(t)$:

$$\begin{aligned} P^{(1)}(t) &= 5t^4 + \left(10 - \frac{1}{5}10\right)t^3 + 10t^2 \\ &\quad + \left(5 - \frac{1}{5}1\right)t + 1 \\ &= 5t^4 + 8t^3 + 10t^2 + \frac{24}{5}t + 1 \end{aligned}$$

Observe that the coefficients of $P^{(1)}(t)$ are positive, so the step 2) is hold, then we build $P^{(2)}(t)$:

$$\begin{aligned} P^{(2)}(t) &= 8t^3 + \left(10 - \frac{5}{8}\left(\frac{24}{5}\right)\right)t^2 + \frac{24}{5}t \\ &\quad + \left(1 - \frac{5}{8}0\right) \\ &= 8t^3 + 7t^2 + \frac{24}{5}t + 1 \end{aligned}$$

We see that $P^{(2)}(t)$ satisfies the step 2) of the algorithm, then following we build $P^{(3)}(t)$

$$\begin{aligned} P^{(3)}(t) &= 7t^2 + \left(\frac{24}{5} - \frac{8}{7}(1)\right)t + 1 \\ &= 7t^2 + \frac{128}{75}t + 1 \end{aligned}$$

As $P^{(3)}(x)$ is of degree 2 and all its coefficients are positive then, by the step 4) of the algorithm, we can conclude that $q(x)$ is Hurwitz.

Problem 4. What of the mentioned criteria will be the most efficient from a computational point of view?

Remark 4. Obviously this problem has a computational interest.

Problem 5. Describe the functions that can be approximate by Hurwitz polynomials.

Remark 5. This problem is in the domain of the Mathematical Analysis.

7 Neccesary condition

The following theorem establishes a necessary condition for a polynomial is a Hurwitz polynomial.

Theorem. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a Hurwitz polynomial and $\xi \in \mathcal{C}$ then one of the following conditions is hold:

a) If $\mathrm{Re}(\xi) > 0$, $|p(\xi)| > |p(-\xi)|$.
b) If $\mathrm{Re}(\xi) = 0$, $|p(\xi)| = |p(-\xi)|$.
c) If $\mathrm{Re}(\xi) < 0$, $|p(\xi)| < |p(-\xi)|$.

Problem 6. An open problem is to determine which other assumptions must be added to the previous theorem to have necessary and sufficient conditions.

8 Conclusions

In this paper we presented some criteria for deciding if a polynomial is a Hurwitz polynomial and we showed some open problems about these polynomials, in order to motivate the researcher continue with the study of them.

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