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UPPER BOUNDS FOR FREQUENCY OF PERIODIC REGIMES IN MANY-DIMENSIONAL AND INFINITE DIMENSIONAL PHASE SYNCHRONIZATION SYSTEMS

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Abstract

A lot of control systems arising in electrical engineering, electronics, mechanics and telecommunications may be modeled as interconnection of a linear plant, described by differential or integro–differential equations and a periodic nonlinear feedback. These mathematical models are often referred to as phase synchronization systems (PSS). Typically such systems are featured by the *gradient–like behavior*, i.e. any solution of the system converges to one of equilibrium points. If a PSS in not gradient-like it may have periodic regimes which are undesirable for most systems. In the present paper, we address the problem of lack or existence of periodic regimes for phase synchronization systems with lumped and with distributed parameters. New effective frequency–algebraic estimates for the frequency of possible periodic regimes are obtained by means of Fourier expansions and the tool of Popov functionals destined specially for periodic nonlinearities.

Key words

Phase synchronization systems, frequency–response methods, periodic solutions, infinite dimensional systems, Fourier series.

1 Introduction

Many systems, arising in electrical and electronic engineering, mechanics and telecommunications are based on controlled phase synchronization of several periodic processes. Such systems are often referred to as phase synchronization systems (PSS) [Yang and Huang, 2007], being the subject matter of phase synchronization theory [Leonov, 2006, Lindsey,1972, Leonov and Kuznetsov, 2014]. This theory gives the opportunity to examine the asymptotic behavior of phase–locked loops (PLL) [Best, 2007, Leonov, Kuznetsov, Yuldashev, and Yuldashev, 2011, Kudrewicz and Wasowicz, 2007], self–synchronization systems [Blekhman, 2000, Blekhman, 2012, Pena-Ramirez, Fey, and Nijmeijer, 2012], electrical and mechanical machines [Leonov and Kondrat'eva, 2008, Stoker, 1950].

Phase synchronization systems have as a rule an infinite denumerable set of equilibrium points which correspond to synchronous regimes. So the basic problem most of the papers considering PSS deal with is convergence of any trajectory to an equilibrium point. This property is called gradient–like behavior.

The problem of gradient–like behavior has been explored in many published works, see [Leonov, 2006] and reference therein for details. For PSS with lumped parameters Lyapunov direct method turned out to be rather fruitful. But as soon as the two traditional Lyapunov functions, i.e. "quadratic form" and "quadratic form plus integral of the nonlinearity" proved to be of no use here, efficient criteria of gradient–like behavior has been obtained by means of several new types of Lyapunov functions [Leonov, 2006, Leonov, Ponomarenko, and Smirnova, 1996].

In particular, periodic Lyapunov functions have been exploited. In the pioneering paper [Bakaev and Guzh, 1965] a periodic Lyapunov function for a third order PSS was considered. Then in [Gelig, Leonov, and Yakubovich, 1978] the results of [Bakaev and Guzh, 1965] were extended to a multidimensional PSS, in [Leonov, Ponomarenko, and Smirnova, 1996] a further periodic function was introduced and in [Perkin, Shepeljavyi, and Smirnova, 2009] a generalized periodic function was offered.

With the help of Kalman–Yakubovich–Popov (KYP) lemma the necessary and sufficient conditions for the existence of a periodic Lyapunov function took the form of frequency–algebraic inequalities with a number of varying parameters. The Popov method of a priori integral indices [Popov, 1973] gave the opportunity to extend the frequency–algebraic criteria of gradient–like behavior to PSS with distributed parameters [Leonov, Ponomarenko, and Smirnova, 1996, Perkin, Shepeljavyi, and Smirnova, 2012].

If a PSS is not gradient–like it may have periodic regimes which regimes are undesirable for most systems. So the problem arises if a PSS has a periodic regime of a certain frequency. The problem has been investigated both by approximate calculus [Shakhil'dyan and Lyakhovkin, 1972] and by analytical methods [Evtyanov and Snedkova, 1968].

In paper [Leonov and Speranskaya, 1985] with the help of Fourier series it was shown that the frequency–algebraic conditions of gradient–like behavior can be used to guarantee that multidimensional PSS has no periodic regimes of certain frequencies. The results of [Leonov and Speranskaya, 1985] were extended then to infinite dimensional PSS [Leonov, Ponomarenko, and Smirnova, 1996] and to discrete ones [Leonov and Fyodorov, 2011]. On the other hand in paper [Perkin, Shepeljavyi, Smirnova, and Utina, 2014] the results of [Leonov and Speranskaya, 1985] were generalized by means of Lyapunov function borrowed from [Perkin, Shepeljavyi, and Smirnova, 2009].

In this paper we show that the frequency conditions obtained in [Perkin, Shepeljavyi, Smirnova, and Utina, 2014] can be extended to infinite dimensional PSS, described by integro–differential Volterra equations. Integro–differential equations can be used in particular for mathematical description of PLL with time–delays. The most wide spread example here is the PLL with delay in the loop [Biswas, Banerjee, and Bhattaharia, 1977, Belustina, 1992]. Networks of mutually coupled PLLs with transmission delays constitute another significant example [Pollakis, Wetzel, Jorg, Rave, Fettweis, and Julicher, 2014]. In this paper estimates for the regions of the absence of periodic regimes for a concrete PLL with a proportional integrating filter are presented.

2 Frequency–algebraic Conditions for the Absence of Periodic Solutions in Many-dimensional Phase Synchronization Systems

In this section we consider a PSS with lumped parameters. Its mathematical description is presented by a system of differential equations of the form

$$\begin{aligned} \frac{dz(t)}{dt} &= Az(t) + Bf(\sigma(t)), \\ \frac{d\sigma(t)}{dt} &= C^* z(t) + Rf(\sigma(t)) \;\; (z \in \mathbf{R}^m, \sigma \in \mathbf{R}^l). \end{aligned} \tag{1}$$

Here $\sigma(t) = (\sigma_1(t), ..., \sigma_l(t))^T$ and $f(\sigma)$ is an input–output decoupled MIMO nonlinearity: $f(\sigma) = (\varphi_1(\sigma_1), ..., \varphi_l(\sigma_l))^T$. A, B, C, R are respectively $m \times m$, $m \times l$, $m \times l$ and $l \times l$ real matrices and * is used for Hermitian conjugation.

Matrix A is assumed to be a Hurvitz one, the pair (A,B) is controllable and the pair (A,C) is observable. The transfer matrix of linear part of system (1) from the input f to the output $(-\dot{\sigma})$ has the form

$$K_L(p) = -R + C^*(A - pE_m)^{-1}B \;\; (p \in \mathbf{C}), \tag{2}$$

where E_m is $m \times m$ identity matrix. Its real part is defined as follows

$$ReK_L(p) = \frac{1}{2}(K_L(p) + K_L^*(p)). \tag{3}$$

Each map φ_j is assumed to be $\mathbf{C}^1$–smooth and Δ_j–periodic, with simple and isolated roots. Let

$$\begin{aligned} \mu_{1j} &= \inf_{\xi \in [0,\Delta_j)} \varphi'(\xi), \\ \mu_{2j} &= \sup_{\xi \in [0,\Delta_j)} \varphi'(\xi) \;\; (j = 1, ..., l). \end{aligned} \tag{4}$$

It is clear that $\mu_{1j}\mu_{2j} < 0$. Define the matrices $M_1 = diag\{\mu_{11}, \mu_{12}, ..., \mu_{1l}\}$, $M_2 = diag\{\mu_{21}, \mu_{22}, ..., \mu_{2l}\}$ and introduce the functions

$$\Phi_j(\xi) = \sqrt{(1 - \mu_{1j}^{-1}\varphi_j'(\sigma))(1 - \mu_{2j}^{-1}\varphi_j'(\sigma))}. \tag{5}$$

$$\nu_j = \frac{\int_0^{\Delta_j} \varphi_j(\xi)\, d\xi}{\int_0^{\Delta_j} |\varphi_j(\xi)|\, d\xi}, \tag{6}$$

$$\nu_{0j} = \frac{\int_0^{\Delta_j} \varphi_j(\xi)\, d\xi}{\int_0^{\Delta_j} |\varphi_j(\xi)|\Phi_j(\xi)\, d\xi}, \tag{7}$$

$$\nu_{1j}(x,y) = \frac{\int_0^{\Delta_j} \varphi_j(\xi)\, d\xi}{\int_0^{\Delta_j} |\varphi_j(\xi)|\sqrt{1 + \frac{x}{y}\Phi_j^2(\xi)}\, d\xi}. \tag{8}$$

A phase system (1) has an infinite denumerable set of equilibriums $\Lambda = \{(z_{eq};\sigma_{eq}) : \quad z_{eq} = 0, \sigma_{eq} = (\sigma_{eq,1}, ..., \sigma_{eq,l})^T, \ \varphi_j(\sigma_{eq,j}) = 0\}$. If every solution of system (1) converges to a certain equilibrium the system (1) is called gradient–like. In case the phase system (1) is not gradient–like it may have periodic solutions which are divided into two classes: periodic solutions of the first kind and periodic solutions of the second kind.

Definition 1. We say that a solution $\{z(t), \sigma(t)\}$ of (1) is a periodic solution if there exist a number $T > 0$ and integers I_j $(j = 1, ..., l)$ such that

$$\begin{aligned} &z(T) = z(0), \\ &\sigma_j(T) = \sigma_j(0) + I_j\Delta_j \quad (j = 1, ..., l). \end{aligned} \tag{9}$$

If all $I_j = 0$ $(j = 1, ..., l)$ the solution $\{z(t), \sigma(t)\}$ is called a periodic solution of the first kind. If $I_1^2 + ... + I_l^2 \neq 0$ it is called a periodic solution of the second kind. The number T is the period and the number $\omega = \frac{2\Pi}{T}$ is the frequency of a periodic solution.

Theorem 1. Suppose there exist an $\omega_0 > 0$, diagonal matrices $\varkappa = diag\{\varkappa_1, ..., \varkappa_l\}$, $\tau = diag\{\tau_1, ..., \tau_l\}$, $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\delta = diag\{\delta_1, ..., \delta_l\}$ with positive $\varepsilon_j, \delta_j, \tau_j$ and the numbers $a_j \in [0; 1]$ $(j = 1, ..., l)$, such that the following conditions hold:
1) for $\omega = 0$ and all $\omega \geq \omega_0$ the inequality

$$\begin{aligned} &\Re e\Big\{æK_L(i\omega) - \\ &-(K_L(i\omega) + i\omega M_1^{-1})^*\tau(K_L(i\omega) + i\omega M_2^{-1}) - \\ &-K_L^*(i\omega)\varepsilon K_L(i\omega)\Big\} - \delta \geq 0; \quad (i^2 = -1) \end{aligned} \tag{10}$$

is true;
2) the quadratic forms

$$\begin{aligned} &Q_j(\xi, \eta, \zeta) = \varepsilon_j\xi^2 + \delta_j\eta^2 + \tau_j\zeta^2 + \varkappa_j a_j \nu_j \xi\eta + \\ &+\varkappa_j(1 - a_j)\nu_{0j}\eta\zeta \quad (j = 1, ..., l) \end{aligned} \tag{11}$$

are positive definite.
Then system (1) has no periodic solutions with frequency $\omega \geq \omega_0$.

Theorem 2. Suppose there exist an $\omega_0 > 0$, diagonal matrices $\varkappa = diag\{\varkappa_1, ..., \varkappa_l\}$, $\tau = diag\{\tau_1, ..., \tau_l\}$, $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\delta = diag\{\delta_1, ..., \delta_l\}$ with positive $\varepsilon_j, \delta_j, \tau_j$ such that the following conditions hold:
1) the inequality (10) is true for $\omega = 0$ and all $\omega \geq \omega_0 > 0$;
2)

$$4\delta_j\varepsilon_j > \varkappa_j^2\nu_{1j}^2(\varepsilon_j, \tau_j) \quad (j = 1, ..., l). \tag{12}$$

Then system (1) has no periodic solutions with frequency $\omega \geq \omega_0$.

Theorems 1 and 2 are proved in [Perkin, Shepeljavyi, Smirnova, and Utina, 2014]. The argument combines the idea of Fourier expansions for periodic solution borrowed from [Garber, 1967] and the tool of periodic Lyapunov functions and corresponding Popov functionals [Leonov and Speranskaya, 1985].

It should be pointed out that if the frequency–domain inequality (10) is true for all ω and either condition 2) of Theorem 1 holds or condition 2) of Theorem 2 holds then system (1) is gradient–like [Perkin, Shepeljavyi, and Smirnova, 2009].

3 Estimates for Frequency of Periodic Solutions for Infinite Dimensional Phase Synchronization Systems

In this section we extend the results of the previous one to infinite dimensional PSS which are described by a system of integro–differential Volterra equations

$$\begin{aligned} &\tfrac{d\sigma(t)}{dt} = \alpha(t) + Rf(\sigma(t-h)) - \\ &-\int_0^t \gamma(t-\tau)f(\sigma(\tau))\,d\tau \quad (t \geq 0), \end{aligned} \tag{13}$$

where $\sigma(t) = (\sigma_1(t), ..., \sigma_l(t))^T$. Nonlinearity $f(\sigma)$ is defined in the previous section. It preserves all its characteristics here. The matrix $R \in \mathbf{R}^{l\times l}$, delay $h \geq 0$, the function $\alpha : [0, +\infty) \to \mathbf{R}^l$, and kernel map $\gamma : [0, +\infty) \to \mathbf{R}^{l\times l}$ in (13) are known, and $\alpha(\cdot)$ is continuous. The solution of (13) is defined by specifying initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{14}$$

Assume that $\sigma^0(\cdot)$ is continuous and $\sigma(0 + 0) = \sigma^0(0)$.

We suppose also that the following restrictions are valid:

$$\begin{aligned} &\alpha(t) \to 0 \text{ as } t \to +\infty; \\ &|\alpha(t)| + |\gamma(t)| \in L_1[0, +\infty) \cap L_2[0, +\infty). \end{aligned} \tag{15}$$

The transfer function of the linear part of (13) is as follows

$$K(p) = -Re^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \ \ (p \in \mathbf{C}). \tag{16}$$

Note that if a vector–function $\sigma(t)$ is a solution of system (13) then a function $(\sigma_1(t) + I_1\Delta_1, ..., \sigma_l(t) + I_l\Delta_l)^T$ with $I_j \in \mathbf{Z}$ $(j = 1, ..., l)$ is also a solution of system (13). So system (13) has an infinite set of equilibriums. The system may be gradient–like. Sufficient frequency–algebraic conditions for its gradient–like behavior are proved in [Perkin, Shepeljavyi, and Smirnova, 2012] On the other hand the system may have periodic solutions.

Definition 2. We say that a solution $\sigma(t)$ of (13) is a periodic solution if there exist a number $T > 0$ and integers I_j $(j = 1, ..., l)$ such that

$$\sigma_j(t+T) = \sigma_j(t)+I_j\Delta_j \quad \forall t \geq 0 \;(j = 1, ..., l). \quad (17)$$

In this section we demonstrate the analogies of Theorem 1 and Theorem 2. The conditions for the absence of periodic solutions with certain frequencies for system (13) are alike those, presented in Section 2 for system (1).

We shall need some preliminaries [Leonov and Speranskaya, 1985, Leonov, Ponomarenko, and Smirnova, 1996]. Suppose that $\sigma(t)$ is a T–periodic solution of system (13). Then $f(\sigma(t))$ is a T–periodic function. Indeed it follows from (17) that

$$\varphi_j(\sigma_j(t+T)) = \varphi_j(\sigma_j(t) + I_j\Delta_j) = \varphi_j(\sigma_j(t)). \quad (18)$$

Then

$$f(\sigma(t)) = \sum_{k=-\infty}^{+\infty} B_k e^{i\omega k t} \quad (i^2 = -1), \quad (19)$$

where B_k are l–vectors.

By substituting (19) in (13) we have

$$\dot{\sigma}(t) = \alpha(t) + \beta(t) - \sum_{k=-\infty}^{+\infty} K(i\omega k)B_k e^{i\omega k t}, \quad (20)$$

where

$$\beta(t) = \int_t^{+\infty} \gamma(\tau) f(\sigma(t-\tau))\,d\tau. \quad (21)$$

It follows from the restrictions (15) that $\alpha(t) + \beta(t) \to 0$ as $t \to +\infty$. But since $\dot{\sigma}(t)$ is T–periodic it follows that $\alpha(t) + \beta(t) = 0$ and

$$\dot{\sigma}(t) = -\sum_{k=-\infty}^{+\infty} K(i\omega k)B_k e^{i\omega k t}. \quad (22)$$

Theorem 3. Suppose there exist $\omega_0 > 0$, matrix $\varkappa = diag\{\varkappa_1, ..., \varkappa_l\}$, positive definite matrices $\tau = diag\{\tau_1, ..., \tau_l\}$, $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\delta = diag\{\delta_1, ..., \delta_l\}$ and numbers $a_j \in [0; 1]$ $(j = 1, ..., l)$, such that the following conditions are valid:
1) for $\omega = 0$ and all $\omega \geq \omega_0$ the inequality

$$\begin{aligned}\Omega(\omega) := Re\Big\{&\varkappa K(i\omega) - \\ &-(K(i\omega) + M_1^{-1} i\omega)^* \tau (K(i\omega) + M_2^{-1} i\omega) - \\ &-K^*(i\omega)\varepsilon K(i\omega)\Big\} - \delta > 0;\end{aligned} \quad (23)$$

is true;
2) the quadratic forms

$$\begin{aligned}Q_j(\xi, \eta, \zeta) = &\varepsilon_j\xi^2 + \delta_j\eta^2 + \tau_j\zeta^2 + \varkappa_j a_j \nu_j \xi\eta + \\ &+\varkappa_j(1 - a_j)\nu_{0j}\eta\zeta \quad (j = 1, ..., l)\end{aligned} \quad (24)$$

are positive definite.

Then system (13) has no periodic solutions of the frequency $\omega \geq \omega_0$.

Proof. Let us introduce the functions

$$F_j(\zeta) = \varphi_j(\zeta) - \nu_j|\varphi_j(\zeta)|, \quad (25)$$

$$\Psi_j(\zeta) = \varphi_j(\zeta) - \nu_{0j}\Phi_j(\zeta)|\varphi_j(\zeta)| \quad (26)$$

and vector functions $F(\sigma) = (F_1(\sigma_1), ..., F_l(\sigma_l))^T$, $\Psi(\sigma) = (\Psi_1(\sigma_1), ..., \Psi_l(\sigma_l))^T$. It is obvious that

$$\int_0^{\Delta_j} F_j(\zeta)\,d\zeta = \int_0^{\Delta_j} \Psi_j(\zeta)\,d\zeta = 0. \quad (27)$$

Introduce also the matrices $A = diag\{a_1, ..., a_l\}$, $A_0 = diag\{1 - a_1, ..., 1 - a_l\}$. Define a function

$$\begin{aligned}G(t) = &\dot{\sigma}^*(t)\varepsilon\dot{\sigma}(t) + \dot{\sigma}^*(t)\varkappa f(\sigma(t)) + \\ &+f^*(\sigma(t))\delta f(\sigma(t)) - F^*(\sigma(t))A\varkappa\dot{\sigma}(t) - \\ &-\Psi^*(\sigma(t))A_0\varkappa\dot{\sigma}(t) + \\ &+(\dot{\sigma}(t) - M_1^{-1}\dot{f}(\sigma(t)))^*\tau(\dot{\sigma}(t) - M_2^{-1}\dot{f}(\sigma(t)))\end{aligned} \quad (28)$$

and consider a set of functionals

$$J(\Theta) = \int_0^{\Theta} G(t)\,dt \quad (\Theta > 0). \quad (29)$$

Suppose $\sigma(t)$ is a T–periodic solution of (13). Let us transform the integral of $J(T)$ using (26) and (25):

$$\begin{aligned}J(T) = &\int_0^T \sum_{j=1}^l \Big\{\varepsilon_j\dot{\sigma}_j^2(t) + \varkappa_j\varphi_j(\sigma_j(t))\dot{\sigma}_j(t) - \\ &-a_j\varkappa_j F_j(\sigma_j(t))\dot{\sigma}_j(t) + \delta_j\varphi_j^2(\sigma_j(t)) - \\ &-(1 - a_j)\varkappa_j\Psi_j(\sigma_j(t))\dot{\sigma}_j(t) + \\ &+\tau_j(\dot{\sigma}_j(t) - \mu_{1j}^{-1}\dot{\varphi}_j(\sigma_j(t)))\cdot \\ &\cdot(\dot{\sigma}_j(t) - \mu_{2j}^{-1}\dot{\varphi}_j(\sigma_j(t)))\Big\}\,dt = \\ = &\int_0^T \sum_{j=1}^l \Big\{\varepsilon_j\dot{\sigma}_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \\ &+\tau_j\dot{\sigma}_j^2(t)\Phi_j^2(\sigma_j(t)) + \varkappa_j a_j\nu_j|\varphi(\sigma_j(t))|\dot{\sigma}_j(t) + \\ &+\varkappa_j(1 - a_j)\nu_{0j}|\varphi(\sigma_j(t))|\dot{\sigma}_j(t)\Phi_j(\sigma_j(t))\Big\}\,dt = \\ = &\int_0^T \sum_{j=1}^l Q_j(\dot{\sigma}_j(t), |\varphi(\sigma_j(t))|, \dot{\sigma}_j(t)\Phi_j(\sigma_j(t))).\end{aligned} \quad (30)$$

In virtue of condition 2) of the theorem

$$J(T) > 0. \quad (31)$$

Suppose now that $\sigma(t)$ has the frequency $\omega \geq \omega_0$. Let us transform the functional $J(T)$ using expansions (19) and (22) under the following obvious equalities:

$$B_{-k} = \bar{B}_k \quad (k \in \mathbf{Z}), \tag{32}$$

where the symbol $-$ is used for complex conjugation;

$$\int_0^T e^{i\omega kt} e^{i\omega mt}\, dt = \begin{cases} 0, \text{ if } k \neq -m, \\ T, \text{ if } k = -m, \end{cases} \quad (k, m \in \mathbf{Z}). \tag{33}$$

Notice that in virtue of Definition 2 the following equalities are valid:

$$\int_0^T F_j(\sigma_j(t))\dot{\sigma}_j(t)\, dt = \int_{\sigma_j(0)}^{\sigma_j(T)} F_j(\zeta)\, d\zeta = 0, \tag{34}$$

$$\int_0^T \Psi_j(\sigma_j(t))\dot{\sigma}_j(t)\, dt = 0. \tag{35}$$

We have

$$J(T) = \sum_{k=1}^{4} J_k(T), \tag{36}$$

where

$$\begin{aligned} &J_1(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\varkappa f(\sigma(t))\, dt, \\ &J_2(T) = \textstyle\int_0^T f^*(\sigma(t))\delta f(\sigma(t))\, dt, \\ &J_3(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\varepsilon\dot{\sigma}(t)\, dt, \\ &J_4(T) = \textstyle\int_0^T (\dot{\sigma}(t) - M_1^{-1}\dot{f}(\sigma(t)))^*\tau \cdot \\ &\cdot(\dot{\sigma}(t) - M_2^{-1}\dot{f}(\sigma(t)))\, dt. \end{aligned} \tag{37}$$

Now we may transform each of the integrals $J_j(T)$ using the formulas (19) and (22). We obtain

$$\begin{aligned} &J_1(T) = -\textstyle\int_0^T \Big\{ (\sum_{k=-\infty}^{+\infty} B_k^* K^*(i\omega k) e^{-i\omega kt})\varkappa \cdot \\ &\cdot(\sum_{r=-\infty}^{+\infty} B_r e^{i\omega rt}) \Big\}\, dt = -T\Big\{ B_0^* K^*(0)\varkappa B_0 + \\ &+ \textstyle\sum_{k=1}^{+\infty} \Big(B_k^* K^*(i\omega k)\varkappa B_k + \\ &+ B_{-k}^* K^*(-i\omega k)\varkappa B_{-k} \Big) \Big\}. \end{aligned} \tag{38}$$

Since $K(-i\omega k) = \bar{K}(i\omega k)$ we have from (32) that

$$\begin{aligned} &J_1(T) = -T\Big\{ B_0^* K^*(0)\varkappa B_0 + \\ &+2\textstyle\sum_{k=1}^{+\infty} \Big(B_k^* Re(\varkappa K(i\omega k)) B_k \Big) \Big\}. \end{aligned} \tag{39}$$

Further

$$\begin{aligned} &J_2(T) = \textstyle\int_0^T \Big\{ (\sum_{k=-\infty}^{+\infty} B_k^* e^{-i\omega kt})\delta \cdot \\ &\cdot(\sum_{r=-\infty}^{+\infty} B_r e^{i\omega rt}) \Big\}\, dt = \\ &= T\{ B_0^*\delta B_0 + 2\textstyle\sum_{k=1}^{+\infty} B_k^*\delta B_k \}. \end{aligned} \tag{40}$$

$$\begin{aligned} &J_3(T) = \textstyle\int_0^T \Big\{ (\sum_{k=-\infty}^{+\infty} B_k^* K^*(i\omega k) e^{-i\omega kt})\varepsilon \cdot \\ &\cdot(\sum_{r=-\infty}^{+\infty} K^*(i\omega r) B_r e^{i\omega rt}) \Big\}\, dt = \\ &= T\{ B_0^* K^*(0)\varepsilon K(0) B_0 + \\ &+2\textstyle\sum_{k=1}^{+\infty} B_k^* K^*(i\omega k)\varepsilon K(i\omega k B_k \}. \end{aligned} \tag{41}$$

For integral $J_4(T)$ the following representation is true

$$\begin{aligned} &J_4(T) = \textstyle\int_0^T \dot{\sigma}^*(t)\tau\dot{\sigma}(t)\, dt - \\ &- \textstyle\int_0^T \dot{f}^*(\sigma(t)) M_1^{-1}\tau\dot{\sigma}(t)\, dt - \\ &- \textstyle\int_0^T \dot{\sigma}^*(t)\tau M_2^{-1}\dot{f}(\sigma(t))\, dt + \\ &+ \textstyle\int_0^T \dot{f}^*(\sigma(t)) M_1^{-1}\tau M_2^{-1}\dot{f}(\sigma(t))\, dt. \end{aligned} \tag{42}$$

We get from (19) that

$$\dot{f}(\sigma(t)) = \textstyle\sum_{k=-\infty}^{+\infty} i\omega k B_k e^{i\omega kt}. \tag{43}$$

Then the following equalities are true:

$$\begin{aligned} &\textstyle\int_0^T \dot{f}^*(\sigma(t)) M_1^{-1}\tau\dot{\sigma}(t)\, dt = \\ &= \textstyle\int_0^T \Big\{ (\sum_{k=-\infty}^{+\infty}(-i\omega k) B_k^* e^{-i\omega kt}) M_1^{-1}\tau \cdot \\ &\cdot(-\textstyle\sum_{r=-\infty}^{+\infty} K(i\omega r) B_r e^{i\omega rt}) \Big\}\, dt = \\ &= T\textstyle\sum_{k=1}^{+\infty}(B_k^* M_1^{-1}\tau(i\omega k K(i\omega k)) B_k + \\ &+ B_{-k}^* M_1^{-1}\tau(-i\omega k K(-i\omega k)) B_{-k}) = \\ &= T\textstyle\sum_{k=1}^{+\infty}(B_k^* M_1^{-1}\tau(i\omega k K(i\omega k)) B_k + \\ &+ B_k^T M_1^{-1}\tau(-i\omega k \bar{K}(i\omega k))\bar{B}_k) = \\ &= T\textstyle\sum_{k=1}^{+\infty}(B_k^* M_1^{-1}\tau(i\omega k K(i\omega k)) B_k + \\ &+ B_k^*(-i\omega k K^*(i\omega k))\tau M_1^{-1} B_k) = \\ &= 2T\textstyle\sum_{k=1}^{+\infty} B_k^* Re(M_1^{-1}\tau(i\omega k K(i\omega k))) B_k; \end{aligned} \tag{44}$$

$$\int_0^T \dot\sigma^*(t)\tau M_2^{-1}\dot f(\sigma(t))\,dt = \\ = -2T\sum_{k=1}^{+\infty} B_k^* Re(i\omega k K^*(i\omega k)\tau M_2^{-1})B_k; \quad (45)$$

$$\int_0^T \dot f^*(\sigma(t))M_1^{-1}\tau M_2^{-1}\dot f(\sigma(t))\,dt = \\ = \int_0^T\{\sum_{k=-\infty}^{+\infty} B_k^*(-i\omega k e^{-i\omega kt})\cdot \\ \cdot M_1^{-1}\tau M_2^{-1}\sum_{r=-\infty}^{+\infty}(i\omega k e^{i\omega kt})B_r\}\,dt = \\ = 2T\sum_{k=1}^{+\infty} k^2\omega^2 B_k^* M_1^{-1}\tau M_2^{-1}B_k. \quad (46)$$

From (42)–(46) it follows that

$$J_4(T) = TB_0^*K^*(0)\tau K(0)B_0+ \\ +2T\sum_{k=1}^{+\infty} B_k^* Re\{(K(i\omega k)+M_1^{-1}i\omega k)^*\cdot \\ \cdot\tau(K(i\omega k)+M_2^{-1}i\omega k)\}B_k. \quad (47)$$

From (36)–(41) and (47) we get that

$$J(T) = -TB_0^*\{\varkappa K(0) - K^*(0)(\varepsilon+\tau)K(0)- \\ -\delta\}B_0 - 2T\sum_{k=1}^{+\infty} B_k^*\{Re(\varkappa K(i\omega k)- \\ -(K(i\omega k)+i\omega k M_1^{-1})^*\tau(K(i\omega k)+i\omega k M_2^{-1}))- \\ -\delta - K^*(i\omega k)\varepsilon K(i\omega k)\}B_k. \quad (48)$$

Condition 1) of the Theorem guarantees that all the terms $B_k^*\Omega(\omega k)B_k$ $(k = 0, 1, 2, ...)$ in (48) are non-negative and consequently

$$J(T) \le 0. \quad (49)$$

This inequality contradicts with (31). The contradiction means that our assumption is wrong and the system (13) has no periodic solutions of the frequency $\omega \ge \omega_0$. Theorem 3 is proved.

Theorem 4. Suppose there exist $\omega_0 > 0$, matrix $\varkappa = diag\{\varkappa_1, ..., \varkappa_l\}$, positive definite matrices $\tau = diag\{\tau_1, ..., \tau_l\}$, $\varepsilon = diag\{\varepsilon_1, ..., \varepsilon_l\}$, $\delta = diag\{\delta_1, ..., \delta_l\}$, such that condition 1) of Theorem 3 is true and the inequalities

$$4\delta_j\varepsilon_j > \varkappa_j^2\nu_{1j}^2(\varepsilon_j,\tau_j) \quad (j = 1, ..., l) \quad (50)$$

are valid. Then the system (13) has no periodic solutions of the frequency $\omega \ge \omega_0$.

Proof. Introduce the functions

$$P_j(\xi) = \sqrt{1+\tfrac{\tau_j}{\varepsilon_j}\Phi_j^2(\xi)}, \\ Y_j(\xi) = \varphi_j(\xi) - \nu_{1j}|\varphi_j(\xi)|P_j(\xi) \quad (j = 1, ..., l). \quad (51)$$

Let

$$Y(\xi) = (Y_1(\xi), ..., Y_l(\xi))^T. \quad (52)$$

It is obvious that

$$\int_0^{\Delta_j} Y_j(\xi)\,d\xi = 0. \quad (53)$$

Determine the function

$$G_0(t) = \dot\sigma^*(t)\varepsilon\dot\sigma(t) + f^*(\sigma(t))\varkappa\dot\sigma(t)+ \\ +f^*(\sigma(t))\delta f(\sigma(t))+ \\ +(\dot\sigma(t) - M_1^{-1}\dot f(\sigma(t)))^*\tau(\dot\sigma(t) - M_2^{-1}\dot f(\sigma(t)))- \\ -Y^*(\sigma(t))\varkappa\dot\sigma(t) \quad (54)$$

and consider the integral

$$J_0(\Theta) = \int_0^\Theta G_0(t)\,dt \quad (\Theta > 0). \quad (55)$$

Suppose that $\sigma(t)$ is a T–periodic solution of (13). Then

$$J_0(T) = \int_0^T\Big\{\sum_{j=1}^l\Big(\varepsilon_j\dot\sigma_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t))+ \\ +\tau_j\dot\sigma_j^2(t)\Phi_j^2(\sigma_j(t))+ \\ +\varkappa_j\nu_{1j}|\varphi_j(\sigma_j(t))|P_j(\sigma_j(t))\dot\sigma_j(t)\Big)\Big\}\,dt = \\ = \int_0^T\Big\{\sum_{j=1}^l\Big(\varepsilon_j(\dot\sigma_j P_j(\sigma_j(t)))^2 + \delta_j\varphi_j^2(\sigma_j(t))+ \\ +\varkappa_j\nu_{1j}|\varphi_j(\sigma_j(t))|P_j(\sigma_j(t))\dot\sigma_j(t)\Big)\Big\}\,dt. \quad (56)$$

In virtue of (50) we have

$$J_0(T) > 0. \quad (57)$$

Let the T–periodic solution of (13) has the frequency $\omega \ge \omega_0$. Since

$$\int_0^T Y_j(\sigma_j(t))\dot\sigma_j(t)\,dt = \int_{\sigma_j(0)}^{\sigma_j(T)} Y_j(\xi)\,d\xi = 0, \quad (58)$$

we make a conclusion that $J_0(T) = J(T)$ with $J(\Theta)$ defined by formula (29). It has already been proved at the proof of Theorem 3 that if inequality (23) is true for all $\omega \ge \omega_0$ then

$$J_0(T) = J(T) \le 0, \quad (59)$$

which contradicts with (57). So the assumption that there exists a T–periodic solution of (13) with frequency $\frac{2\Pi}{T} \ge \omega_0$ is wrong. Theorem 4 is proved.

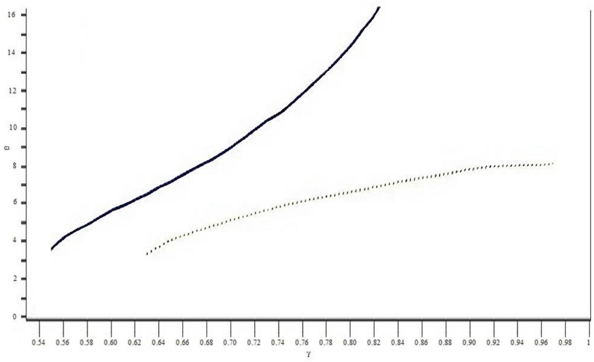

Figure 1. The region of the absence of periodic regimes for PLL with a proportional-integrating filter.

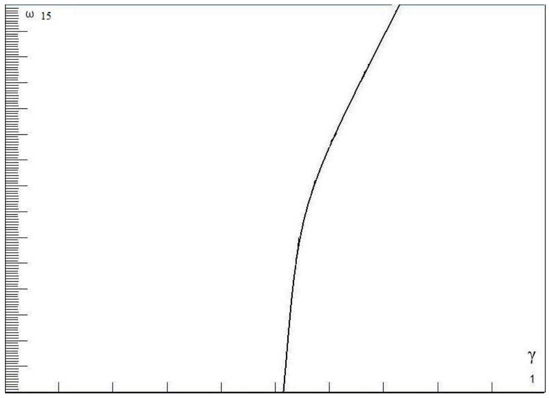

Figure 2. The region of the absence of periodic regimes for PLL with time–delay in the loop.

4 Example

Theorems 1,2 and 3 were applied to second order phase-locked loop (PLL) with a proportional–integrating filter, $l = 1$, and a sine-shaped characteristic of the phase detector: $l = 1$, $f(\sigma) = \varphi_1(\sigma_1) = \sin(\sigma_1) - \gamma, \;\; \gamma \in (0, 1)$, Transfer function for the PLL with lumped parameters has the form

$$K_L(p) = 100\frac{1 + 0.2p}{1 + p}. \qquad (60)$$

By means of Theorems 1 and 2 the estimate for the region of the absence of periodic regimes was obtained. The results are presented in Fig. 1. The region is situated to the left of the thick curve. The dotted curve gives the frequencies of oscillations obtained by an asymptotic method [Evtyanov and Snedkova, 1968].

The PLL with time–delay in the loop was also considered. Its transfer function is given by the formula

$$K(p) = 100\frac{1 + 0.2p}{1 + p}e^{-0.1p}. \qquad (61)$$

The estimate for the region of the absence of periodic regimes was obtained with the help of Theorem 3. It is presented in Fig. 2. The region is situated to the left of the curve.

5 Conclusion

The paper is devoted to the problem of periodic regimes in phase synchronization systems. PSS described by differential equations as well as PSS described by integro–differential equations are considered. The case of periodic nonlinearities is examined. The problem is investigated by means of Fourier extensions and Popov functional intended for periodic nonlinearities. New frequency–algebraic conditions for the absence of periodic regimes of certain frequencies are obtained.

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