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# STABILIZATION OF NONLINEAR LIPSCHITZ SYSTEMS WITH INPUT DELAY AND DISTURBANCES USING STATE AND DISTURBANCE PREDICTORS

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**Abstract**

This paper proposes an algorithm for stabilizing nonlinear Lipschitz systems with a known constant input delay and an unknown bounded disturbance. The control law is designed based on both a state predictor and a disturbance predictor. The stability of the closed-loop system is established using the Lyapunov-Krasovskii functional method, which provides sufficient conditions in the form of a feasible linear matrix inequality (LMI). The ultimate boundedness of all system signals is formally proven. Furthermore, it is shown that the derived LMI is influenced by system parameters, sector bounds of the nonlinearity, and the delay, allowing for the determination of their limit values while ensuring system stability. The effectiveness of the proposed approach is validated through numerical simulations in MATLAB/Simulink.

**Key words**

nonlinear lipschitz system, input delay, state predictor, disturbance predictor, Lyapunov-Krasovskii functional method, LMI.

## 1 Introduction

The control of nonlinear systems in the presence of external disturbances presents a significant challenge with broad applications across various practical domains, particularly where delayed input signals are involved. These nonlinear systems serve as the foundation for numerous technologies and processes, including automation, robotics, energy systems, and other industrial sectors. Advancing control methodologies for nonlinear systems remains a critical focus in control theory.

In the field of control with delay, a common synthesis approach involves designing a predictor block that forecasts the system's future state. This approach gained popularity after the publication of [Smith, 1959], where O. Smith proposed a compensator based on a predictor for stable linear systems with a known delay. Various modifications and applications of the Smith predictor are discussed in [Palmor, 1996; Furtat and Tsykunov, 2005]. In [Manitius and Olbrot, 1979], A. Manitius and A. Olbrot introduced a proportional-integral predictor for unstable systems, constructed by solving the system's equations. The predictor scheme presented in [Manitius and Olbrot, 1979] was employed in [Krstic, 2009; Mazenc et al., 2012] to attenuate bounded disturbances, due to the resulting distribution of the closed-loop eigenvalues. However, subsequent studies [Van Assche et al., 1999; Engelborghs et al., 2001; Mondié et al., 2002; Furtat et al., 2018; Furtat, 2012; Furtat, 2014; Margun and Furtat, 2015] indicated that the numerical implementation of the predictor proposed in [Manitius and Olbrot, 1979] is effective only for a specific class of unstable systems with delay.

A new predictor for controller synthesis in unstable systems was introduced in [Dugard and Verriet, 1997]. Unlike the predictor in [Manitius and Olbrot, 1979], the approach in [Dugard and Verriet, 1997] omits an integral component, simplifying both technical implementation and parameter calculation. Additionally, in [Najafi et al., 2013], a subpredictor for the controlled variable was developed based on the predictor from [Dugard and Verriet, 1997]. This subpredictor enhances control performance for systems with longer delay times compared to the method in [Dugard and Verriet, 1997]. The application of the predictor and subpredictor from [Dugard and Verriet, 1997; Najafi et al., 2013] to systems with significant external disturbances was further explored in [Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b]. However, the results in [Dugard and Verriet, 1997; Najafi et al., 2013; Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b] are limited to linear systems.

The objective of this paper is to extend the results from [Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b] by synthesizing a control law for nonlinear systems with input delays under external disturbances. Specifically, a new state predictor is developed based on the predictor in [Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b], and a disturbance predictor is introduced for nonlinear systems. The paper is structured as follows: Section 2 formulates the problem of stabilizing an unstable nonlinear system with delayed input signals in the presence of disturbances. Section 3 presents methods for synthesizing state and disturbance predictors, along with sufficient conditions for closed-loop system stability. These conditions, expressed in terms of LMI solvability, depend on system parameters and delay values. Section 4 provides computational results and simulation studies for a specific nonlinear control system with given parameters.

*Notation:* $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space with the vector norm $|\cdot|$; $\mathbb{R}^{n\times m}$ represents the set of all real matrices of size $n \times m$; $col\{\cdot\}$ denotes a column vector; $I$, 0, and $diag\{\cdot\}$ denote the identity matrix, zero matrix, and diagonal matrix (of the corresponding dimensions), respectively; $blkdiag\{A, B, \dots, F\}$ denotes the block diagonal matrix of matrices $A, B, \dots, F$; $\delta(s) = \mathcal{O}(s)$ means $\lim\limits_{s\to 0}\left|\frac{\delta(s)}{s}\right| < \infty$; $p = \frac{d}{dt}$ denotes the differentiation operator; symmetric elements of a symmetric matrix are denoted by $\star$.

## 2 Problem Statement

We consider a nonlinear system with an input delay, represented by the following equation:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + G\varphi(x(t)) + Bu(t-h) + Bf(t),\\ &t \geq 0,\ u(s) = 0,\ s < 0.\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the measurable state vector, $u(t) \in \mathbb{R}^m$ is the control signal, $f(t) \in \mathbb{R}^m$ is an unknown external bounded disturbance with bounded $(r+3)$ derivatives, $h > 0$ is the known time delay, $\varphi(\cdot)$ is a known nonlinearity function, $A$, $B$, and $G$ are known matrices of appropriate dimensions. The pair $(A, B)$ is controllable, and the condition $B^+B = I$ holds, where $B^+$ denotes the pseudoinverse of matrix $B$.

**Assumption 1.** *The function $\varphi(x)$ is globally Lipschitz, meaning there exists a constant $L > 0$ such that for any $x_1, x_2 \in \mathbb{R}^n$, the following condition holds:*

$$|\varphi(x_1) - \varphi(x_2)| \leq L|x_1 - x_2|. \tag{2}$$

It follows that if $\varphi(x)$ is globally Lipschitz, it also satisfies the sector constraint:

$$|\varphi(x)| \leq L|x|, \quad \forall x \in \mathbb{R}^n. \tag{3}$$

The objective of the control algorithm is to ensure that the system state satisfies the following condition:

$$\overline{\lim_{t\to\infty}}\, |x(t)| \leq \delta, \tag{4}$$

where:

$$\delta = \mathcal{O}\left(\overline{\lim_{t\to\infty}}\, h^{r+1}\left|f^{(r+1)}(t)\right|\right). \tag{5}$$

**Remark 1.** *From* (5)*, it can be observed that the regulation accuracy depends on the magnitude of $\overline{\lim\limits_{t\to\infty}}\, h^{r+1}\left|f^{(r+1)}(t)\right|$. This indicates that the proposed algorithm is capable of compensating for disturbances with larger amplitudes. In contrast, for the known algorithms [Manitius and Olbrot, 1979; Krstic, 2009; Mazenc et al., 2012], the steady-state regulation error $\delta$ depends only on the magnitude of $\overline{\lim\limits_{t\to\infty}}\, |f(t)|$.*

## 3 Solution Method. Main Results

To ensure the stability of the closed-loop system, we employ the control signal $u_1(t)$, while disturbance compensation is achieved using the control signal $u_2(t)$. The control law for the system is given by:

$$u(t) = u_1(t) + u_2(t). \tag{6}$$

Since direct state feedback control is not feasible due to the unavailability of $x(t+h)$, we propose a state predictor for the variable $x(t)$ in Subsection 3.1 to facilitate the synthesis of the control signal $u_1(t)$ for nonlinear systems. The disturbance is then estimated and approximated using a derivative estimation algorithm.

To compensate for the disturbance, we introduce a predictor that estimates $f(t)$, from which the compensation signal $u_2(t)$ is generated. This predictor is referred to as the “disturbance predictor”.

In Subsection 3.2, using the Lyapunov-Krasovskii method, we derive sufficient conditions for the stability of the closed-loop system, formulated as a solvable LMI.

### 3.1 Synthesis of the State and Disturbance Predictors

Following [Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b], we introduce a predictor for the nonlinear system (1) as follows:

$$\dot{\bar{x}}(t) = A\bar{x}(t) + De(t) + G\varphi(\bar{x}(t)) + Bu_1(t), \tag{7}$$

where $e(t) \triangleq x(t) - \bar{x}(t-h)$ denotes the state prediction error, and the matrix $D \in \mathbb{R}^{n\times n}$ is chosen such that the following linear time-delay system is exponentially stable:

$$\dot{\omega}(t) = A\omega(t) - D\omega(t-h), \tag{8}$$

where $\omega(t) \in \mathbb{R}^n$.

The selection of the matrix $D$ is formulated as a feasibility condition for a LMI, presented in Proposition 1.

**Proposition 1.** *If, for a given matrix $D$, there exist positive definite matrices $P = P^{\mathrm{T}}, S = S^{\mathrm{T}}, R = R^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and matrices $P_2,\ P_3 \in \mathbb{R}^{n\times n}$ such that the following LMI is feasible:*

$$\begin{bmatrix} \Pi_1 & P - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 & P_2^{\mathrm{T}}D + R \\ \star & -P_3 - P_3^{\mathrm{T}} + h^2R & -P_3^{\mathrm{T}}D \\ \star & \star & -S - R \end{bmatrix} < 0, \tag{9}$$

*where $\Pi_1 = A^{\mathrm{T}}P_2 + P_2^{\mathrm{T}}A + S - R$, then $\lim_{t\to\infty} \omega(t) = 0$.*

*Proof:* The proof of Proposition 1 can be found in Sec. 3.6.2 of [Fridman, 2014].

**Remark 2.** *Finding a matrix $D$ that satisfies the condition in Proposition 1 can be challenging in practice, particularly for high-dimensional matrices $A$. Instead of explicitly defining $D$ and verifying the feasibility of LMI* (9)*, an alternative approach is to fix $P_2$ and $P_3$ as known matrices, for example, $P_2 = P_3 = I$, and solve* (9) *for $D, P, S,$ and $R$. The obtained matrix $D$ is then used in the predictor* (7)*.*

The control signal $u_1(t)$ is chosen in the form

$$u_1(t) = -K\bar{x}(t), \tag{10}$$

where $K \in \mathbb{R}^{m\times n}$ is selected such that $A - BK$ is Hurwitz.

Taking the derivative of the error $e(t)$ using (1) and (7), we obtain

$$\begin{aligned} \dot{e}(t) = {} & Ae(t) - De(t-h) + Bu_2(t-h) \\ & + G[\varphi(x(t)) - \varphi(\bar{x}(t-h))] + Bf(t). \end{aligned} \tag{11}$$

From (11), it is evident that if $u_2(t) = -f(t+h)$, the disturbance is completely compensated. However, since $f(t)$ is unmeasured, it is necessary to estimate and predict the disturbance.

To achieve this, we introduce an auxiliary loop

$$\begin{aligned} \dot{e}_a(t) = {} & Ae_a(t) - De_a(t-h) \\ & + G\big[\varphi\big(x(t)\big) - \varphi\big(\bar{x}(t-h)\big)\big] \\ & + Bu_2(t-h). \end{aligned} \tag{12}$$

where $e_a(t) \in \mathbb{R}^n$.

Defining $\zeta(t) \triangleq e(t) - e_a(t)$ and substituting (11) and (12), we obtain

$$\dot{\zeta}(t) = A\zeta(t) - D\zeta(t-h) + Bf(t). \tag{13}$$

Hence, the disturbance $f(t)$ can be estimated as [Furtat and Gushchin, 2019a; Furtat and Gushchin, 2019b]

$$\hat{f}(t) = B^+\left(\hat{\dot{\zeta}}(t) - A\zeta(t) + D\zeta(t-h)\right), \tag{14}$$

where the signal $\hat{\dot{\zeta}}(t)$ is defined by

$$\hat{\dot{\zeta}}_i(t) = \frac{p}{\mu p + 1}\zeta_i(t), \quad i = 1, \ldots, n. \tag{15}$$

Here, $\zeta_i, \hat{\zeta}_i$ denote the $i$-th elements of the vectors $\zeta(t)$ and $\hat{\zeta}(t)$, respectively, $\mu$ is selected as a small positive constant satisfying $\mu \in (0,1)$.

To predict the disturbance $\hat{f}(t+h)$, we apply the method described in [Furtat et al., 2018]:

$$\begin{aligned} \hat{f}(t+h) = {} & \sum_{j=1}^{r+1}(-1)^{j-1}C_{r+1}^{j}\,\hat{f}\big(t - h(j-1)\big) \\ & + R(t). \end{aligned} \tag{16}$$

where $C_{r+1}^{j} = \frac{(r+1)!}{(r+1-j)!j!}$, and

$$R(t) = h^{r+1}\hat{f}^{(r+1)}\left(t - \left[(r+1)\theta - 1\right]h\right),$$

represents the remaining term in the expansion, with $\theta \in (0,1)$.

Since $R(t)$ is not available for measurement, the control signal $u_2(t)$ is chosen as

$$u_2(t) = -\tilde{f}(t+h), \tag{17}$$

where

$$\begin{aligned} \tilde{f}(t+h) & \triangleq \hat{f}(t+h) - R(t) \\ & = \sum_{j=1}^{r+1}(-1)^{j-1}C_{r+1}^{j}\hat{f}(t - h(j-1)). \end{aligned} \tag{18}$$

Thus, the proposed algorithm consists of the state predictor (7), the auxiliary loop (12), the disturbance predictor (18), and the control laws (10) and (17). In the next subsection, we analyze the closed-loop system and present the main results of the proposed approach.

### 3.2 Closed-Loop Stability Analysis

We introduce the variables $\xi \triangleq \dot{\zeta} - \hat{\dot{\zeta}}$, $\eta \triangleq \xi^{(r+1)}$, and $g \triangleq \zeta^{(r+3)}$. Defining $\lambda(t) = u_2(t-h) + f(t)$ as the disturbance compensation error, we obtain the following relation:

$$\begin{aligned}\lambda(t) &= \hat{f}(t) - \tilde{f}(t) + f(t) - \hat{f}(t) \\ &= R(t-h) + B^{+}\xi(t) \\ &= h^{r+1}\Big[f^{(r+1)}\big(t-(r+1)\theta h\big) \\ &\quad - B^{+}\,\eta\big(t-(r+1)\theta h\big)\Big] + B^{+}\xi(t).\end{aligned} \tag{19}$$

Since $f^{(r+1)}(t)$, $\eta(t)$, and $\xi(t)$ are bounded signals, it follows that $\lambda(t)$ is also bounded (for a detailed proof, see Sec. 4 in [Furtat and Gushchin, 2019b]).

Taking into account (19), one can rewrite (11) as

$$\begin{aligned}\dot{e}(t) &= Ae(t) - De(t-h) \\ &\quad + G\big[\varphi\big(x(t)\big) - \varphi\big(\bar{x}(t-h)\big)\big] + B\lambda(t).\end{aligned} \tag{20}$$

Next, we introduce a new variable $x_e(t) \triangleq \bar{x}(t-h) = x(t) - e(t)$. Taking the derivative of $x_e(t)$ using (7) and (10), we obtain

$$\dot{x}_e(t) = (A - BK)x_e(t) + De(t-h) + G\varphi(x_e(t)). \tag{21}$$

As a result, the closed-loop system is described by (20) and (21).

Before analyzing the input-to-state stability (ISS) of the closed-loop system, we define the following vectors and matrices:

$$\begin{aligned}&x_p = col\{x_e, e\}, \\ &\psi(t) = col\{\varphi(x_e(t)), \varphi(x(t)) - \varphi(\bar{x}(t-h))\}, \\ &A_p = blkdiag\{A - BK, A\},\ B_p = blkdiag\{0, B\}, \\ &C_p = blkdiag\{G, G\},\ D_p = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}.\end{aligned}$$

Thus, (20) and (21) can be rewritten as

$$\begin{aligned}\dot{x}_p(t) &= A_p x_p(t) + D_p x_p(t-h) \\ &\quad + B_p\lambda(t) + C_p\psi(t).\end{aligned} \tag{22}$$

Using the Newton-Leibniz formula, we further express (22) as

$$\begin{aligned}\dot{x}_p(t) &= (A_p + D_p)x_p(t) - D_p\int_{t-h}^{t}\dot{x}_p(s)ds \\ &\quad + B_p\lambda(t) + C_p\psi(t).\end{aligned} \tag{23}$$

**Theorem 1.** *Consider the nonlinear system* (1)*, the state predictor* (7)*, the disturbance predictor* (18)*, and the control laws* (10) *and* (17)*. If, for a given $\alpha > 0$ and matrices $K$ and $D_p$, there exist coefficients $\beta > 0$, $\gamma > 0$, and positive-definite matrices $P, Q, S \in \mathbb{R}^{n\times n}$ and matrices $P_2, P_3 \in \mathbb{R}^{n\times n}$ such that the following LMI is feasible:*

$$\Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} & 0 & \Psi_{14} & \Psi_{15} & \Psi_{16} \\ \star & \Psi_{22} & 0 & \Psi_{24} & \Psi_{25} & \Psi_{26} \\ \star & \star & \Psi_{33} & 0 & 0 & 0 \\ \star & \star & \star & -hS & 0 & 0 \\ \star & \star & \star & \star & -\beta I & 0 \\ \star & \star & \star & \star & \star & -\gamma I \end{bmatrix} \leq 0, \tag{24}$$

*where*

$$\begin{aligned}\Psi_{11} &= P_2^T(A_p + D_p) + (A_p + D_p)^T P_2 \\ &\quad + 2\alpha P + Q + \beta L^2 I, \\ \Psi_{12} &= P - P_2^{\mathrm{T}} + (A_p + D_p)^{\mathrm{T}} P_3, \\ \Psi_{14} &= -hP_2^{\mathrm{T}} D_p,\ \Psi_{15} = P_2^{\mathrm{T}} B_p,\ \Psi_{16} = P_2^{\mathrm{T}} C_p, \\ \Psi_{22} &= -P_3 - P_3^{\mathrm{T}} + hS, \\ \Psi_{24} &= -hP_3^{\mathrm{T}} D_p,\ \Psi_{25} = P_3^{\mathrm{T}} B_p,\ \Psi_{26} = P_3^{\mathrm{T}} C_p, \\ \Psi_{33} &= -e^{-2\alpha h}Q.\end{aligned}$$

*then the closed-loop system defined by* (1), (6), (7), (10), (12), (17)*, and* (18) *is ultimately bounded. Moreover, the tracking objective* (4) *holds with*

$$\delta = \mathcal{O}\left(\overline{\lim_{t\to\infty}}\, h^{r+1}\left|f^{(r+1)}(t)\right|\right). \tag{25}$$

*Proof.* Consider the Lyapunov-Krasovskii candidate function in the form

$$V = V_1 + V_2 + V_3, \tag{26}$$

where

$$\begin{aligned}&V_1 = x_p^T P x_p,\ V_2 = \int_{t-h}^{t} e^{2\alpha(\sigma - t)}\, x_p^T(\sigma)\, Q\, x_p(\sigma)\, d\sigma, \\ &V_3 = \int_{-h}^{0}\int_{t+\tau}^{t} e^{2\alpha(\sigma - t)}\, \dot{x}_p^T(\sigma)\, S\, \dot{x}_p(\sigma)\, d\sigma\, d\tau.\end{aligned}$$

Note that the component $V_3$ is essential in deriving the stability condition for the closed-loop system, explicitly incorporating the delay (i.e., a delay-dependent condition [Fridman, 2014]). Using equation (23), we construct the following expressions:

$$\begin{aligned}\dot{V}_1 + 2\alpha V_1 &= x_p^T P\dot{x}_p + 2\alpha x_p^T P x_p \\ &\quad + 2\left(x_p^T P_2^T + \dot{x}_p^T P_3^T\right)\Big[\,(A_p + D_p)\,x_p(t) \\ &\quad - D\int_{t-h}^{t}\dot{x}_p(s)ds + B_p\lambda(t) \\ &\quad + C_p\psi(t) - \dot{x}_p(t)\Big], \\ \dot{V}_2 + 2\alpha V_2 &= x_p^T(t)Qx_p(t) \\ &\quad - e^{-2\alpha h}x_p^T(t-h)Qx_p(t-h), \\ \dot{V}_3 + 2\alpha V_3 &= h\dot{x}_p^T S\dot{x}_p \\ &\quad - \int_{t-h}^{t} e^{2\alpha(\sigma - t)}\dot{x}_p^T(\sigma)S\dot{x}_p(\sigma)d\sigma.\end{aligned} \tag{27}$$

In deriving the first expression in (27), the descriptor method [Fridman, 2014] was employed. By applying Jensen's inequality [Fridman, 2014], the last expression in (27) can be estimated as follows

$$\dot{V}_3 + 2\alpha V_3 \leqslant h\dot{x}_p^{\mathrm{T}} S\dot{x}_p - \frac{e^{-2\alpha h}}{h}\varpi^{\mathrm{T}} S\varpi. \qquad (28)$$

where $\varpi(t) = \int_{t-h}^{t} \dot{x}_p(\sigma)d\sigma$.

Using the Lipschitz property of the function $\varphi(\cdot)$, we obtain:

$$\begin{aligned}\psi^T\psi &\leq [\varphi(x_e(t))]^2 + \big[\varphi(x(t)) - \varphi(\bar{x}(t-h))\big]^2 \\ &\leq L^2\big[x_e^2(t) + \big(x(t) - \bar{x}(t-h)\big)^2\big] \\ &= L^2\big[x_e^2(t) + e^2(t)\big] = L^2\, x_p^T\, x_p.\end{aligned} \qquad (29)$$

Defining $y(t) = col\{x_p(t), \dot{x}_p(t), x_p(t-h), \frac{1}{h}\varpi(t), \lambda(t), \psi(t)\}$, we establish the following inequality to study ISS property:

$$\dot{V} + 2\alpha V + \beta\left(L_1^2 x_p^{\mathrm{T}} x_p - \psi^{\mathrm{T}}\psi\right) - \gamma\lambda^{\mathrm{T}}\lambda \leqslant y^{\mathrm{T}}\Psi y. \quad (30)$$

If condition (30) is satisfied, then it follows that $y^{\mathrm{T}}\Psi y \leq 0$, which implies

$$\dot{V} + 2\alpha V \leq \gamma\lambda^{\mathrm{T}}\lambda$$

for all $x_p \in \mathbb{R}^{2n\times 2n}$, provided that the function $\varphi(\cdot)$ satisfies the Lipschitz condition. Therefore, $x_p(t)$ is ultimately bounded.

Next, consider the control objective (4). From (6) and (26), we rewrite (1) as

$$\dot{x}(t) = Ax(t) + G\varphi(x(t)) + Bu_1(t-h) + B\lambda(t).$$

If $\lambda \equiv 0$, then according to (30), the closed-loop system is exponentially stable. Defining

$$\Delta(\mu) = \overline{\lim_{t\to\infty}}\, |\lambda(t)|,$$

and using (19), we obtain

$$\lim_{\mu\to 0} \Delta(\mu) = \overline{\lim_{t\to\infty}}\, h^{r+1}\left|f^{(r+1)}(t)\right|.$$

Thus, the proposed algorithm ensures the control objective (4), achieving the accuracy

$$\delta = \mathcal{O}\left(\overline{\lim_{t\to\infty}}\, h^{r+1}\left|f^{(r+1)}(t)\right|\right). \qquad (31)$$

Theorem 1 is proved.

**Remark 3.** *We establish the boundedness of all signals in the closed-loop system. Since $x_p(t)$ is bounded, it follows that $x_e(t)$ and $e(t)$ are also bounded. The relation $x_e(t) = x(t) - e(t)$ implies the boundedness of $x(t)$. Consequently, $\bar{x}(t)$ is bounded, ensuring that $u_1(t)$ remains bounded. Furthermore, given that $\tilde{f}(t)$ is bounded, it follows that $u_2(t)$ is also bounded. Therefore, all signals in the closed-loop system are bounded.*

## 4 Example

Consider a single-link robot with a flexible joint rotating in a vertical plane, described by the following parameters [Alessandri, 2004]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_1} & 0 & \frac{k}{J_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_2} & 0 & -\frac{k}{J_2} & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0\,0\,0 \\ -\frac{mgl}{J_1} & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \end{bmatrix},$$

$$\varphi(x(t)) = \sin x(t),\, x_0 = \begin{bmatrix} 0.1\ 0\ 0.1\ 0 \end{bmatrix}^{\mathrm{T}},$$

where $x = col\{x_1, x_2, x_3, x_4\} \in \mathbb{R}^4$, $\sin x = col\{\sin x_1, \sin x_2, \sin x_3, \sin x_4\}$, with $x_1$ and $x_2$ representing the link displacement and velocity, and $x_3$ and $x_4$ representing the rotor displacement and velocity, respectively. The parameters $J_1, J_2, k, l, g$ denote the link inertia, rotor inertia, elastic constant, position of the center of mass, and gravitational acceleration, respectively. The system parameters are given as:

$$J_1 = J_2 = 30\,\mathrm{Kg/m^2},\ \ k = 1\,\mathrm{N/m},\ \ l = 1\,\mathrm{m},\ \ g = 10\,\mathrm{m/s^2}.$$

The Lipschitz constant is equal to 1.

Let us consider the disturbance with the following form:

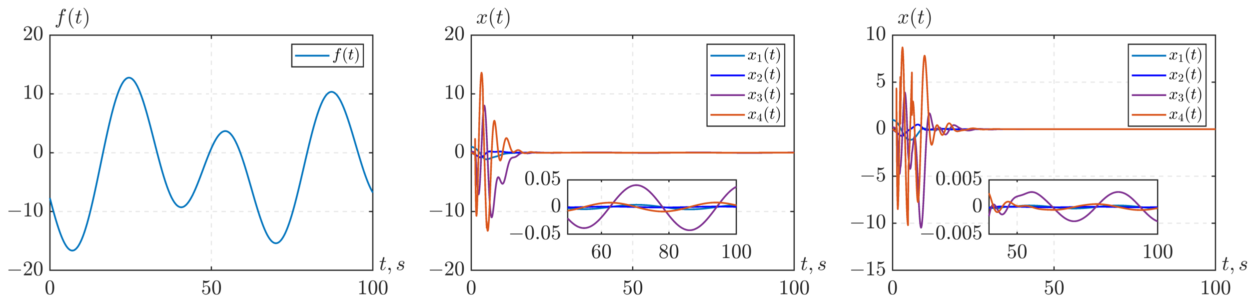
$$\begin{aligned} f(t) &= 2 + 10\sin(0.2t) \\ &\quad + 5\cos(0.1t) + \sin\left(0.15t + \frac{\pi}{4}\right) \\ &\quad + \frac{100}{(5p+1)^7}\,\mathrm{sat}\left(\frac{d(t)}{10}\right), \end{aligned} \qquad (32)$$

where $\mathrm{sat}(\cdot)$ denotes the saturation function, and $d(t)$ is a noise signal modeled in MATLAB Simulink using the "Band-Limited White Noise" block with a noise power of 0.1 and a sample time of 0.1.

To determine the matrix $D$, we set $P_2 = P_3 = I \in \mathbb{R}^8$. It is then verified that the LMI (9) is solvable for the variables $D, P, R, S$ when $h \leq 1.2$ seconds. For $h_{\max} = 1.2$ seconds, we obtain

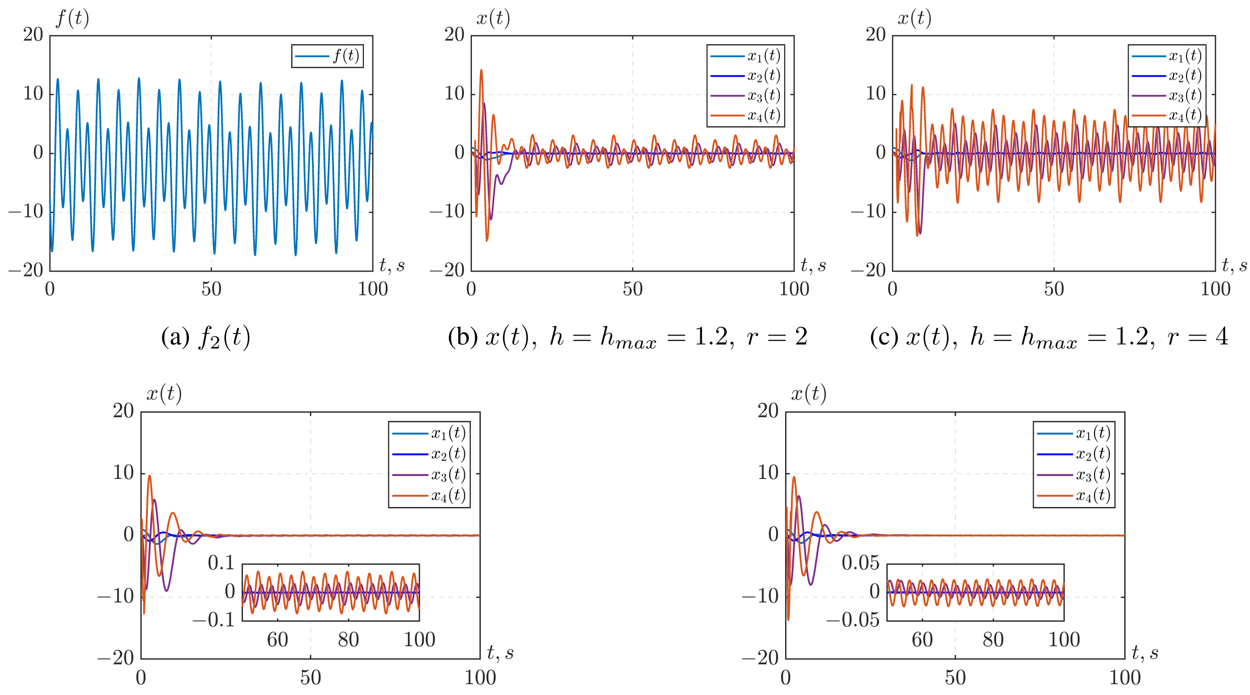
$$D_{\max} = \begin{bmatrix} 0.4896 & 0.5744 & 0.0085 & -0.0101 \\ -0.4052 & 0.4583 & 0.0227 & 0.0098 \\ 0.0085 & -0.0101 & 0.4896 & 0.5744 \\ 0.0227 & 0.0098 & -0.4052 & 0.4583 \end{bmatrix}.$$

Since LMI (9) is convex with respect to $h$, the system (1) remains stable for $h \in [0,\ 1.2]$ when using the computed matrix $D_{\max}$ [Fridman, 2014].




(a) $f(t)$ (b) $x(t),\ h = h_{max} = 1.2,\ r = 2$ (c) $x(t),\ h = h_{max} = 1.2,\ r = 4$

Figure 1: Graphs of: (a) $f(t)$, (b) $x(t)$ for $r = 2$, (c) $x(t)$ for $r = 4$.




(a) $f_2(t)$ (b) $x(t),\ h = h_{max} = 1.2,\ r = 2$ (c) $x(t),\ h = h_{max} = 1.2,\ r = 4$

(d) $x(t),\ h = 0.3,\ r = 2$ (e) $x(t),\ h = 0.3,\ r = 4$

Figure 2: Graphs of: (a) $f_2(t)$, (b) $x(t)$ for $h = 1.2,\ r = 2$, (c) $x(t)$ for $h = 1.2,\ r = 4$, (d) $x(t)$ for $h = 0.3,\ r = 2$, (e) $x(t)$ for $h = 0.3,\ r = 4$.

Using the *acker* command in MATLAB, we compute the gain matrix $K$ such that the eigenvalues of the closed-loop system are $\{-0.8, -0.8, -0.8, -0.8\}$, yielding

$$K = [255.4,\ 1747.2,\ 113.2,\ 96].$$

For the derivative estimator (15), we set $\mu = 0.001$. Checking the feasibility of LMI (24) for the given $D$ and $K$, we verify that (24) is feasible for $h \leq h_{\max} = 1.2$ seconds.

Let us consider the simulation results.

Fig. 1a shows the graph of the disturbance $f(t)$. Fig. 1b and Fig. 1c depict the transient response of $x(t)$ for a delay $h = h_{\max} = 1.2$ seconds, respectively. As can be observed, the given open-loop nonlinear system is unstable, and the disturbance $f(t)$ has a large magnitude. However, the proposed algorithm ensures the boundedness of all controlled variables in the closed-loop system (see Fig. 1a and Fig. 1b). Furthermore, when increasing the value of $r$ in the disturbance predictor, the ultimate bound $\delta$ decreases. As the simulation results indicate, when $r$ is doubled, the value of $\delta$ reduces by a factor of 10. This can be explained by the fact that although increasing $r$ causes the value of $h^{r+1}$ to rise, the magnitude $\overline{\lim_{t\to\infty}} \left|f^{(r+1)}(t)\right|$ decreases faster, leading to a reduction in the value of $\overline{\lim_{t\to\infty}}\ h^{r+1}\left|f^{(r+1)}(t)\right|$. Consequently, the value of $\delta$ decreases.

Thus, we can see that the proposed algorithm is highly effective for disturbances with large magnitudes that satisfy the condition $\overline{\lim_{t\to\infty}}\ h^{r+1}\left|f^{(r+1)}(t)\right| \ll \overline{\lim_{t\to\infty}} |f(t)|$.

Now, consider the case where the disturbance $f(t)$ has an amplitude $\overline{\lim\limits_{t\to\infty}} \left|f^{(r+1)}(t)\right|$ that increases as $r$ increases. For example, consider the signal $f(t)$ in (32), where the frequency of each harmonic component increases by a factor of 10 (see Fig. 2a) as

$$\begin{aligned} f(t) = f_2(t) &= 2 + 10\sin(2t) \\ &+ 5\cos(t) + \sin\left(1.5t + \frac{\pi}{4}\right) \\ &+ \frac{100}{(5p+1)^7}\,\mathrm{sat}\left(\frac{d(t)}{10}\right). \end{aligned}$$

As shown in Fig. 2b and Fig. 2c, the bound $\delta$ has larger values compared to the case in Fig. 1. Furthermore, $\delta$ increases significantly as $r$ increases.

The value of $\delta$ can be significantly reduced when the time delay value $h$ is small ($h \ll 1$, see Fig. 2d and Fig. 2e for the case $h = 0.3$ seconds). In this case, $h^{r+1}$ can decrease faster than the increase in $\overline{\lim\limits_{t\to\infty}} \left|f^{(r+1)}(t)\right|$ as $r$ increases, leading to a reduction in the ultimate bound $\delta$.

## 5 Conclusion

In this work, we synthesized control algorithms for nonlinear systems with delayed input signals. The proposed approach employs predictors for both the controlled variable and the disturbance. By applying the Lyapunov-Krasovskii method, we derived sufficient stability conditions for the closed-loop system in the form of LMI solvability.

Simulation results demonstrate that the algorithm is highly effective for disturbances with large magnitudes that satisfy the condition $\overline{\lim}_{t\to\infty} h^{r+1} \left|f^{(r+1)}(t)\right| \ll \overline{\lim}_{t\to\infty} |f(t)|$. Moreover, for disturbances whose amplitude $\overline{\lim}_{t\to\infty} \left|f^{(r+1)}(t)\right|$ increases with $r$, the control system designer can select a smaller value of $r$ if $h > 1$, and a larger value of $r$ if $h \ll 1$.

Future research would be devoted to applying the proposed method to the study of control for real-world systems, taking into account nonlinearity under input delay conditions, such as stepper motor control [Furtat et al., 2023], vibration stand system [Tomchina, 2023; Zaitceva et al., 2023], etc.

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## References

Smith, J. M. (1959). Closer control of loops with dead time. *Chem. Eng. Prog.*, **53**, pp. 2217–2219.

Palmor, Z. J. (1996). Time-delay compensation Smith predictor and its modifications. *The Control Handbook*, **1**, 224–229.

Furtat, I. B. and Tsykunov, A. M. (2005). Adaptive plant control with output delay. *Izvestiya VUZov. Priborostroenie*, **7**, 15–19.

Manitius, A. Z. and Olbrot, A. W. (1979). Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, **AC-24**(4), 541–553.

Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhäuser, Basel.

Mazenc, F., Niculescu, S.-I., and Krstić, M. (2012). Lyapunov–Krasovskii functionals and application to input delay compensation for linear time-invariant systems. *Automatica*, **48**(6), 1317–1323.

Van Assche, V., Dambrine, M., Lafay, J. F. and Richard, J. P. (1999). Some problems arising in the implementation of distributed-delay control laws. In *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, 2243–2248.

Engelborghs, K., Dambrine, M. and Rose, D. (2001). Limitations of a class of stabilization methods for delay systems. *IEEE Transactions on Automatic Control*, **46**(2), 336–339.

Mondié, S., Dambrine, M. and Santos, O. (2002). Approximation of control laws with distributed delays: a necessary condition for stability. *Kybernetika*, **38**(5), 541–551.

Furtat, I. B., Fridman, E. and Fradkov, A. (2018). Disturbance compensation with finite spectrum assignment for plants with input delay. *IEEE Transactions on Automatic Control*, **63**(1), 298–305.

Furtat, I. B. (2012). Adaptive control of an object with a delay in control without the use of predictive devices. *Control of Large Systems*, **40**, 144–163.

Furtat, I. B. (2014). Adaptive predictor-free control of a plant with delayed input signal. *Automation and Remote Control*, **75**(1), 144–163.

Margun, A. and Furtat, I. (2015). Robust control of linear MIMO systems in conditions of parametric uncertainties, external disturbances and signal quantization. In *Proceedings of the 20th International Conference on Methods and Models in Automation and Robotics (MMAR)*, Miedzyzdroje, Poland, 341–346.

Dugard, L. and Verriet, E. (1997). *Stability and Control of Time-delay Systems*. Springer, London.

Najafi, M., Hosseinnia, S., Sheikholeslam, F. and Karimadini, M. (2013). Closed-loop control of dead time systems via sequential sub-predictors. *International Journal of Control*, **86**(4), 599–609.

Furtat, I. B. and Gushchin, P. A. (2019a). A control algorithm for an object with delayed input signal based on subpredictors of the controlled variable and disturbance. *Automation and Remote Control*, **80**(2).

Furtat, I. B. and Gushchin, P. (2019b). Tracking control algorithms for plants with input time-delays based on state and disturbance predictors and sub-predictors. *Journal of the Franklin Institute*, **356**, 4496–4512.

Fridman, E. (2014). *Introduction to Time-Delay Systems: Analysis and Control*. Birkhäuser, Basel.

Alessandri, A. (2004). Design of observers for Lipschitz nonlinear systems using LMI. *IFAC Proceedings Volumes*, **37**(13), 459–464.

Furtat, I. B., Zhukov, Y. A., and Matveev, S. A. (2023). Design of optimal-robust control of step motors with partially unknown parameters and disturbances. *Cybernetics and Physics*, **12**(1), 23–27.

Tomchina, O. P. (2023). Digital control of the synchronous modes of the two-rotor vibration set-up. *Cybernetics and Physics*, **12**(4), 282–288.

Zaitceva, I., Andrievsky, B., and Sivachenko, L. A. (2023). Enhancing functionality of two-rotor vibration machine by automatic control. *Cybernetics and Physics*, **12**(4), 289–295.