LMI BASED ROBUST CONTROL DESIGN FOR MOTION SYSTEMS

Maria Hypiusova and Alena Kozakova
Institute of Automotive Mechatronics
Slovak University of Technology Bratislava, Slovakia
maria.hypiusova@stuba.sk

Abstract
The paper deals with state-feedback robust control design methods for uncertain SISO continuous-time systems modelled using parametric uncertainties. A software tool based on advanced optimization tools (YALMIP toolbox with LMI and SeDuMi solvers) is presented applicable for robust motion control design.
LMI-based control design to guarantee quadratic stability, H_2 and H_{inf} performance specification is considered. A practical application is provided illustrating application of the individual robust control design approaches for a laboratory plant – a Modular Servo System.

Key words
H_2 and H_{inf} controller, linear matrix inequality (LMI), motion control, quadratic stability, robust control

1 Introduction

The fundamental problem in designing control systems consists in accurately controlling outputs of plants whose dynamics contain significant uncertainties. Generally, uncertainties are caused by inherent modelling/identification inaccuracies in any model of a physical plant which often brings about poor closed-loop performance or even instability. Uncertainties can be characterized and modelled in a number of ways; those ones caused by plant parameters varying over some a priori known compact set are denoted as parametric uncertainties.

A possible way of coping with uncertainties is to apply robust-theoretical approach and related robust control design methods. Robust control can be defined as control of systems with uncertain dynamics using fixed-structure controllers to guarantee fulfillment of design specifications for a whole set of possible plant dynamics. A lot of robust control design methods are known from literature, e.g. [Ackerman, 1997; Bhattacharyya, Chapellat and Keel, 1995] in both the time and frequency domains. In the sequel, focus will be on the optimization-based approach according to which the robust control problem is reformulated to an optimization problem in form of linear matrix inequalities (LMI). LMIs are considered computationally tractable and the respective free solvers (SeDuMi) are available to solve them. However, LMI based solutions inherently include possibility to obtain ambiguous solutions when different solvers or computational parameters are chosen. This happens especially when solving a robust stability problem formulated as a feasibility one.

In this paper three LMI-based robust state-feedback controller design methods for plants with parametric uncertainties have been applied to control a Modular Servo System (MSS) with interval varying parameters. The first method considers quadratic stability and the respective LMI condition [Boyd, Ghaoui, Feron and Balakrishnam, 1994]; quadratic stability is solved without and with minimization of the trace of the Lyapunov function matrix P. The second and third methods are based on minimization of the H_2 and H_{inf} norms using LMIs, see [Henrion, 2009] and references therein. LMIs have been solved in MATLAB using the YALMIP Toolbox [Löfberg, 2004] with the LMI [Gahinet, Nemirovski, Laub and Chilali, 1995] and SeDuMi [Henrion and Lasserre, 2003] solvers to be able to compare obtained results. If states of a dynamic system are not available through measurements or some of them are impossible or too expensive to measure, an output-injection observer design has to be included [Lewis, 1992]. The three above-mentioned robust state feedback controller design methods, the observer design and a back calculation anti windup scheme proposed by [Fertik and Ross, 1967; Åström and Hägglund, 1995] have been incorporated in the created software tool for motion control system design.

The paper is organized as follows: in Section 2, a general structure of the motion control design tool is presented; its main modules are presented in Sections 3, 4 and 5 with the focus on the robust LMI-based state-feedback control design in Section 3. In Section 6 the motion control design software tool has been applied for a MSS laboratory plant. Conclusions are drawn at the end of the paper.

2 Motion control design software tool: structure

The created software for motion control design is an advanced optimization tool for the control engineering community (students, researches, teachers, practitioners, etc.) having the following modules:

1. Plant identification
2. Modelling of uncertainties
3. LMI-based robust state-feedback control design
4. Observer design
5. Anti-windup compensation

Module 1: Plant identification

To be able to model the uncertain plant, identification based on measured-data is performed in three different working points; continuous-time process models, or discrete-time polynomial models (ARX, ARMAX and IV4 model structures) can be selected. This module is not used if the plant transfer functions are already available.

Module 2: Modelling uncertainties

In this module, uncertainty model is calculated in form of interval or affine (polytopic) models [Vesely and Harsanyi, 2008].

Other parts of the program are explained in respective sections of the paper.

3 Robust state-feedback control design

In this section, basic notions and main results on robust stability based on Lyapunov stability approach are briefly recalled [Boyd, Ghaoui, Feron and Balakrishnam, 1994; Henrion, 2009].

Consider a linear continuous time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ are state, control and output vectors, respectively, *A*, *B* and *C* are known constant matrices of compatible dimensions. In what follows, a SISO plant will be considered, however the obtained results are valid for both SISO and MIMO plants.

Consider a state-feedback controller in the form

$$u(t) = -Kx(t) \tag{2}$$

The resulting closed loop respective to (1) and (2) is

$$\dot{x}(t) = (A - BK)x(t) = A_c x(t) \tag{3}$$

According to the Lyapunov stability theory the closed-loop system (3) is stable if and only if there exists a symmetric matrix *P* such that

$$A_c^T P + PA_c < 0 \tag{4}$$

Inequality (4) is a special form of a linear matrix inequality appropriate also for uncertain systems.

Consider the uncertain plant to be modelled using interval model in the form

$$\mathbf{G}(s) = \left\{ \frac{B(s)}{A(s)} : (B(s), A(s)) \in (\mathbf{N}(s), \mathbf{D}(s)) \right\} \tag{5}$$

where

$$\begin{aligned} \mathbf{N}(s) &= \left\{ B(s) : b_0 + \ldots + b_m s^m, b_i \in \langle \underline{b}_i, \overline{b}_i \rangle, i = 0,1,\ldots,m \right\} \\ \mathbf{D}(s) &= \left\{ A(s) : a_0 + \ldots + a_n s^n, a_i \in \langle \underline{a}_i, \overline{a}_i \rangle, i = 0,1,\ldots,n \right\} \end{aligned} \tag{6}$$

The interval model (5)-(6) can be rewritten in form of a state space model described by a polytope of linear dynamic systems defined by a list of its vertices:

$$\{(A_1, B, C_1), \ldots, (A_N, B, C_N)\} \tag{7}$$

where N is the number of vertices. Generally, such a polytopic description of the uncertain plant is beneficial in that it yields less conservative controller design results than if using other uncertainty descriptions [Boyd, Ghaoui, Feron and Balakrishnam, 1994].The design methods described in the following subsections are implemented in the *Module 3: LMI-based robust state-feedback control design.*

3.1 LMI-based state-feedback control design: quadratic stability

The robust state-feedback controller design to guarantee quadratic stability is based on the polytopic description of the uncertain plant. Considering (1), (2) and (7), the closed-loop uncertain system has the following form

$$\dot{x}(t) = (A_i - BK)x(t) = A_{c_i} x(t) \qquad i = 1,\ldots,N \tag{8}$$

The state-feedback control design algorithm is based on the following theorem.

Theorem 1. The polytopic system (8) is quadratically stabilizable if there exists a matrix $K \in R^{1\times n}$ and a symmetric positive definite matrix $P \in R^{n\times n}$ satisfying

$$(A_i - BK)^T P + P(A_i - BK) < 0, \quad i = 1,2,\ldots,N \tag{9}$$

The matrix inequality (9) can readily be transformed into LMIs [Boyd, Ghaoui, Feron and Balakrishnam, 1994] yielding the following control design algorithm to guarantee quadratic stability.

<u>Algorithm 1a (Quadratic stability)</u>:

Solve LMIs (10) for unknown symmetric positive definite matrix $Q \in R^{n\times n}$ and a full matrix $Y \in R^{1\times n}$.

$$\begin{aligned} & A_i Q + QA_i^T - BY - Y^T B^T < 0, \quad i = 1,2,\ldots,N \\ & Q > 0 \\ & Q = P^{-1} \\ & Y = KP^{-1} \end{aligned} \tag{10}$$

By including minimization of the trace of matrix *P* (equivalent to minimization of matrix (-*Q*) trace) to the inequalities (10) the, algorithm 1b is obtained:

<u>Algorithm 1b (Quadratic stability with minimization of trace(P))</u>:

Solve LMIs (11) for unknown symmetric positive definite matrix $Q \in R^{n\times n}$ and a full matrix $Y \in R^{1\times n}$.

$$\begin{aligned}
&min(-trace\,Q)\\
&A_iQ + QA_i^T - BY - Y^TB^T < 0, \quad i = 1,2,\ldots,N\\
&Q > 0 \qquad\qquad (11)\\
&Q = P^{-1}\\
&Y = KP^{-1}
\end{aligned}$$

The result of both the above algorithms, the state-feedback controller matrix K in the control law (2) is obtained according to

$$K = YQ^{-1} \qquad (12)$$

3.2 LMI-based state-feedback design: H_2 controller

Consider the continuous-time linear time invariant system

$$\begin{aligned}\dot{x} &= Ax + B_uu + B_ww\\ y &= Cx + D_uu + D_ww\end{aligned} \qquad (13)$$

where $x(t) \in R^n$, $u(t) \in R^r$, $y(t) \in R^l$, $w(t) \in R^l$ are state, control, output, and exogenous input vectors, respectively. We assume that the system (13) is strictly proper with $D_u = 0$ and consider a finite gain, i.e. $D_w = 0$.

System (13) with the state-feedback controller (2) yields the closed-loop system

$$\begin{aligned}\dot{x} &= (A - B_uK)x + B_ww\\ y &= Cx\end{aligned} \qquad (14)$$

The corresponding closed-loop transfer function is

$$G(s) = C(sI - A - B_uK)^{-1}B_w \qquad (15)$$

Consider the H_2 performance specification in the form

$$\|G(s)\|_2 \le \gamma \qquad (16)$$

H_2 norm is computed using the Lyapunov equation

$$(A - B_uK)P_c + P_c(A - B_uK)^T + B_wB_w^T = 0 \qquad (17)$$

where $P_c = \int_0^\infty e^{(A-B_uK)t}B_wB_w^Te^{(A-B_uK)^Tt}dt$ is the controllability Grammian; hence

$$\|G\|_2^2 = trace\left(CP_cC^T\right) \qquad (18)$$

For a polytopic uncertain system

$$(A,C) \in conv\{(A_1,C_1),\ldots,(A_N,C_N)\} \qquad (19)$$

inequality (16) holds if and only if there exists a matrix K such that

$$\begin{aligned}&trace\left(C_iQC_i^T\right) < \gamma\\ &(A_i - B_uK)Q + Q(A_i - B_uK)^T + B_wB_w^T < 0\end{aligned} \qquad (20)$$

The above inequality (20) can be rewritten as

$$trace\left(C_iQC_i^T\right) < trace W < \gamma \qquad (21)$$

for some matrix W such that

$$\begin{bmatrix} W & C_iQ\\ QC_i^T & Q\end{bmatrix} > 0 \qquad (22)$$

The resulting robust LMI-based state-feedback H_2 controller design algorithm is as follows.

Algorithm 2 (LMI–based robust H_2 controller design): Solve LMIs (23) for an unknown symmetric positive definite matrix $Q \in R^{n\times n}$, a full matrix $Y \in R^{1\times n}$ and a symmetric matrix $W \in R^{(n-1)\times(n-1)}$.

$$\begin{aligned}
&trace W < \gamma\\
&\begin{bmatrix} W & C_iQ\\ QC_i^T & Q\end{bmatrix} > 0, \quad i = 1,2,\ldots,N\\
&A_iQ + QA_i^T - B_uY - Y^TB_u^T + B_wB_w^T < 0 \qquad (23)\\
&Q > 0\\
&Q = P^{-1}\\
&Y = KP^{-1}
\end{aligned}$$

The resulting gain matrix K of the H_2 suboptimal state-feedback controller (2) obtained from the solution of (23) is given by (12).

3.3LMI-based state-feedback design: H_∞-controller

Consider a continuous-time linear t-invariant system (13) with the state-feedback controller (2). We assume that (13) is strictly proper ($D_u = 0$). The closed-loop system is

$$\begin{aligned}\dot{x} &= (A - B_uK)x + B_ww\\ y &= Cx + D_ww\end{aligned} \qquad (24)$$

Consider the H_∞ performance specification as follows

$$\|G(s)\|_\infty < \gamma \qquad (25)$$

where the H_∞ norm is defined as the worst-case gain

$$\|G\|_\infty = \sup_{\|x\|_2=1}\|Gx\|_2 = \sup_\omega\|G(j\omega)\| \qquad (26)$$

Contrary to the H_2 norm, computation of the H_∞ norm is iterative.

For $G(s) = C(sI - A)^{-1}B + D$, inequality (25) is equivalent to the existence of a symmetric positive definite matrix P satisfying

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} < 0 \qquad (27)$$

$$P > 0$$

Using the Schur complement, inequalities (27) can be expanded as follows

$$\begin{bmatrix} A^T P + PA & PB & C^T \\ B^T P & -\lambda I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \qquad (28)$$

$$P > 0$$

Consider the polytopic uncertainties

$$(A,C) \in conv\{(A_1, C_1), \ldots, (A_N, C_N)\} \qquad (29)$$

The next algorithm involves the H_∞ performance specification (25) for the uncertain plant (24) with polytopic uncertainties (29).

Algorithm 3 (LMI–based robust H_∞ controller design): Solve LMIs (30) for the unknown symmetric positive definite matrix $Q \in R^{n\times n}$, a full matrix $Y \in R^{1\times n}$, a symmetric matrix $W \in R^{(n-1)\times(n-1)}$ and the scalar $\gamma \in R$.

$$\min \gamma$$

$$\begin{bmatrix} A_i Q + QA_i^T - B_u Y - Y^T B_u^T & QC_i^T & B_w \\ C_i Q & -\gamma I & D_w \\ B_w^T & D_w^T & -\gamma I \end{bmatrix} < 0$$

$$Q > 0 \qquad (30)$$

$$Q = P^{-1}$$

$$Y = KP^{-1}$$

The resulting H_∞ optimal state-feedback gain matrix (2) is calculated according to (12).

4. Observer design

In most practical situation it is unusual to have all states of a dynamic plant measurable due to various reasons [Lewis, 1992]. In such a case, the plant states can be estimated based on available measured outputs using a dynamic observer. Then, the state estimates can be used for the feedback as if they were the actual states, i.e.the control law (2) modifies as follows

$$u(t) = -K\hat{x}(t) \qquad (31)$$

Module 4: Observer design

Using the measured input and output variables $u(t), y(t)$ of a plant the observer generates an estimate $\hat{x}(t)$ of the state variable $x(t)$ in (1) or (13). The state observer is a dynamic system described by

$$\dot{\hat{x}}(t) = A_R \hat{x}(t) + Bu(t) + Ly(t) \qquad (32)$$

where $A_R \in R^{n\times n}$, $B_R \in R^{n\times 1}$ are matrices of the state observer, $L \in R^{n\times 1}$ is the output injection matrix. To ensure that the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$ with the initial estimate $\hat{x}(0)$ vanishes with time for any $\tilde{x}(0)$, the output injection L has to be selected so that the observer matrix $A_R = A - LC$ is asymptotically stable. By appropriately choosing *L*, eigenvalues of A_R can be assigned to desired locations if and only if (C, A) is observable.

In the Module 4 of the developed software tool computation of the matrix *L* is realized using the Ackermann's formula. According to a general rule, to achieve a suitable accuracy in the state estimate the slowest observer pole should have a time constant 5-10 times faster than that of the fastest pole of the plant; however, location of the observer poles has to be chosen carefully to avoid too high gains which may be unfeasible in practice.

5 Anti-windup compensation

When PID control algorithms are used to control plants with saturated input, undesirable nonlinear effects such as windup may occur To avoid the actuator saturation, the back calculation anti windup scheme proposed by [Fertik and Ross, 1967] is used in the proposed software tool.

Module 5: Anti-windup compensation

When the controller output saturates, the back calculation does not reset the integral instantaneously but dynamically with a tracking time constant T_t which governs how quickly the integral term is being reset. Smaller tracking time constants reset integrator quicker which may seem to be an advantage at first; it is recommended to choose the tracking time constant between the integral and the derivative time constants [Åström and Hagglund, 1995; Åström and Hägglund, 2005].

6 Robust motion control design for a Modular Servo System laboratory plant

Plant description

The Modular Servo System (MSS) consists of the Inteco digital servomechanism and open-architecture software environment for real-time control experiments [INTECO, 2007]. The measurement system is based on the RTDAC4/USB acquisition board equipped with a D/A and A/D converters. I/O board communicates with the power interface unit. The whole logic necessary to activate and read the encoder signals and to generate the appropriate sequence of the PWM pulses to control the DC motor is configured in the Xilinx® chip of the RT-DAC/USB board. All functions of the board are accessed from the Modular Servo Toolbox, operating directly in the MATLAB Simulink environment.

MSS consists of the following modules arranged in the chain (Fig. 1): a DC motor with a generator, inertia load, encoder, magnetic brake and the gearbox with the output disk. In our experiments the backlash module was not applied. The servomechanism is connected to a computer where a control algorithm is based on measurements of the angular displacement and the angular velocity.

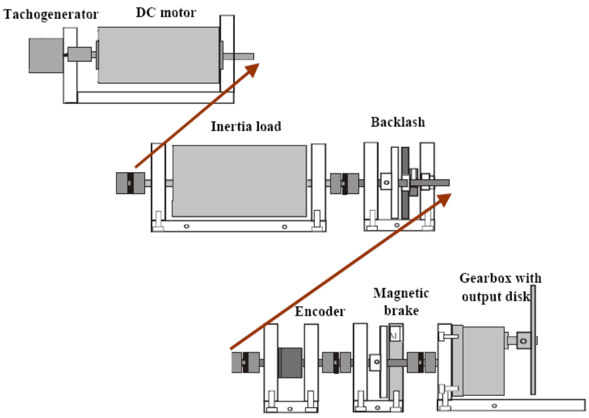

Figure 1. Modular Servo System (MSS)

Identification of the uncertain plant

Transfer functions for angular velocity have been obtained via identification in three working points; the results are summarized in Table 1.

The corresponding interval model in the state space is in form

$$\dot{x}(t) = Ax(t) + Bu(t) = [-a_0]x_1(t) + u(t)$$
$$y(t) = Cx(t) = [b_0]x(t) \qquad (33)$$

where

$b_0 \in \langle 259.6302, 318.6094 \rangle$, $a_0 \in \langle 1{,}4091, 1.8255 \rangle$

Table 1. Transfer functions for angular velocity in individual working points

Working point	Manipulated variable u [V]	Output variable y [rad/s]	Transfer function
1	0.4	47	$G_{p1}(s) = \frac{298.9195}{s+1.7089}$
2	0.6	80	$G_{p2}(s) = \frac{318.6094}{s+1.8255}$
3	0.8	120	$G_{p3}(s) = \frac{259.6302}{s+1.4091}$

If a PI controller of angular velocity is to be designed to reject the external input (reference/disturbance), the state space model has to be augmented to include its internal model. In case of setpoint tracking the augmented model reads as follows:

$$A_{aug} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, B_{aug} = \begin{bmatrix} B \\ 0 \end{bmatrix}, C_{aug} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \qquad (34)$$

In such a case the corresponding state-feedback design yields the gain matrix K composed of two parts:

$$K = [K_P \quad K_I] \qquad (35)$$

where K_P is the proportional and K_I the integral part of the state-feedback controller.

If a full state-feedback controller of angular velocity is to be designed based just on angular displacement measurements, the state vector is to be estimated. As angular displacement is an integral of angular velocity, nominal transfer function (mean value parameter model from the models in Tab. 1) with integrator was considered as the model of the controlled plant with the poles λ_1=0, λ_2=-1.5905. Desired poles of the state observer matrix have been chosen

$$\Lambda_1 = [-15 \quad -15] \qquad (36)$$

The resulting output injection matrix is

$$L_1 = \begin{bmatrix} 0.0996 \\ 0.6302 \end{bmatrix} \qquad (37)$$

Robust state feedback controller design

Using the created LMI-based robust state-feedback design tool, parameters of the angular velocity state controller to guarantee quadratic stability (QS) (*Algorithms 1a* and *1b*), as well as the H_2 and H_∞ controllers (*Algorithms 2* and *3*) were designed. The best results are presented in Table 2.

Table 2. PI controllers of angular displacement designed by LMI approaches

method	minimize	solver	$\begin{bmatrix} K_P \\ K_I \end{bmatrix}$
QS	no	SeDuMi	$\begin{bmatrix} 637.62 & 18.742 \\ & 3.5923 \end{bmatrix}$
H_2	-	LMI	$\begin{bmatrix} 62.852 & 7.1653 \\ & 0.14088 \end{bmatrix}$
H_∞	-	LMI	$\begin{bmatrix} 181.24 & 17.875 \\ & 1.0313 \end{bmatrix}$

As the manipulated variable is limited within the interval $\langle -1,1 \rangle$, the anti-windup compensation was activated with $T_t = 0.1$.

Remark: In the implementation of the state-feedback controller on the MSS, angular velocity is measured in degrees $([y_{st}] = \deg/s)$ and angular displacement in radians $([y_{rad}] = rad)$, hence this conversion has to be considered along with the transmission ratio (1:100) as follows

$$y_{st} = \frac{180}{100\pi} y_{rad} \tag{38}$$

Experimental results obtained by applying the three designed observer-based controllers to the MSS plant are shown in Fig. 2. Closed-loop step responses under the quadratic stability (QS) controller with and without minimization of trace P, and the H_2 and H_∞ controllers obtained by various solvers (LMI and SeDuMi) are compared. Figure 3 shows the respective control variable responses.

7 Conclusion

The paper presents an LMI-based software tool for robust motion control system design. Three basic state-feedback control design methods for uncertain continuous-time systems modelled using parametric uncertainties are implemented based on advanced optimization approaches (YALMIP toolbox with LMI and SeDuMi solvers). A practical application is provided illustrating application of the individual robust control design approaches for a laboratory plant. Implementation of these approaches to a real plant (observer-based angular velocity control of a Modular Servo System) illustrates effectiveness of the proposed methods.

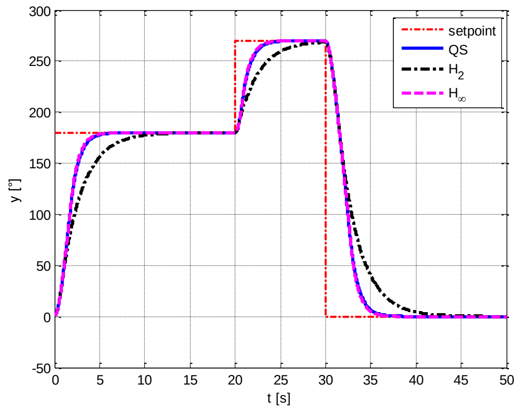

Figure 2. Closed-loop step responses of angular velocity under individual observer-based state-feedback controllers designed using the LMI and SeDuMi solvers.

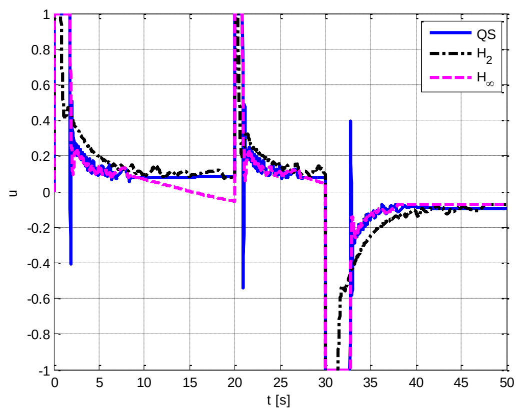

Figure 3. Control variable time responses under the individual designed controllers

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